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SOME RESULTS RELATED TO THE CONTINUOUS WEINSTEIN WAVELET TRANSFORM

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Abstract. The aim of this paper is to investigate a variant of the Heisenberg-type uncertainty principle, Pitt's inequality, the Benkner-type uncertainty principle and logarithmic Sobolev inequalities for the Weinstein wavelet transform.

1. Introduction

The Weinstein operator is the elliptic partial differential operator Δ_W considered in the upper half space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times [0, +\infty)$

$$\begin{aligned} \Delta_W &= \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_d} \cdot \frac{\partial}{\partial x_d}, \quad \alpha > -\frac{1}{2}, \\ \Delta_W &= \Delta_{d-1} + \ell_\alpha, \end{aligned}$$

where Δ_{d-1} is the Laplacian operator on $\mathbb{R}^{d-1}$ and ℓ_α is the Bessel operator with respect to the variable x_d defined by

$$\ell_\alpha = \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha+1}{x_d} \cdot \frac{\partial}{\partial x_d}.$$

The harmonic analysis associated with the Weinstein operator is studied by Nahia and Ben Salem [3,4]. In particular, the authors introduced and studied the generalized Fourier transform associated with the Weinstein operator.

The theory of wavelet and continuous wavelet transform has been extended to hypergroups, in particular, to Chébli-Triméche hypergroup see [22]. Recently, there have

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been many studies on wavelet transforms see [12, 13, 17]. The authors studied uncertainty inequalities for the continuous Weinstein wavelet transform and the deformed wavelet transform.

In this paper, we introduce some new uncertaint inequalities for the continuous Weinstein wavelet transform.

Let us now to be more precise and describe our results. To do so, we need to introduce some notation.

- For $1 \leq p < +\infty$, we denote by $L^p_\alpha(\mathbb{R}^d_+)$ the Lebesgue space consisting of measurable functions f on $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$ equipped with the norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^d_+)} = \left(\int_{\mathbb{R}^d_+} |f(x', x_d)|^p d\mu_\alpha(x', x_d) \right)^{1/p},$$

  where

$$\begin{aligned} d\mu_\alpha(x) = d\mu_\alpha(x', x_d) &= \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha + \frac{d-1}{2}} \Gamma(\alpha+1)} dx' dx_d \\ &= \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha + \frac{d-1}{2}} \Gamma(\alpha+1)} dx_1 \cdots dx_d. \end{aligned}$$

- For $f \in L^1_\alpha(\mathbb{R}^d_+)$, the Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi', \xi_d) = \int_{\mathbb{R}^d_+} f(x', x_d) e^{-i\langle x', \xi' \rangle} j_\alpha(x_d \xi_d) d\mu_\alpha(x', x_d),$$

  where j_α is the spherical Bessel function:

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha+k+1)} \left(\frac{z}{2} \right)^{2k}, \quad z \in \mathbb{C}.$$

- The Weinstein wavelet on $\mathbb{R}^d_+$ is a measurable function h on $\mathbb{R}^d_+$ satisfying, for almost all $\xi \in \mathbb{R}^d_+$

$$0 < C_h = \int_0^{+\infty} |\mathcal{F}_W(h)(t)|^2 \frac{dt}{t} < +\infty.$$

- We denote by $L^p_{\omega_\alpha}$, $1 \leq p < +\infty$ the space of measurable functions f on $\mathbb{R}^d_+ \times [0, +\infty] = \mathbb{R}^{d+1}_{++}$, $d\omega_\alpha(x, b) = d\mu_\alpha(x) \frac{1}{b^{2\alpha+d+2}} db$ such that

$$\|f\|_{L^p_{\omega_\alpha}(\mathbb{R}^{d+1}_{++})} = \left(\int_{\mathbb{R}^{d+1}_{++}} |f(x, b)|^p d\omega_\alpha(x, b) \right)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < +\infty.$$

- Let h be a Weinstein wavelet on $\mathbb{R}^d_+$, we define the Weinstein continuous wavelet transform as follows

$$\mathcal{S}^h_W(f)(b, y) = \int_{\mathbb{R}^d_+} f(y) \overline{h_{b,y}(x)} d\mu_\alpha(x), \quad (b, y) \in \mathbb{R}^{d+1}_{++},$$

  where $h_{b,y}(x)$ is a determined kernel.

- We define for $s \in \mathbb{R}^d$, $1 < p \leq 2$, the $\mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+)$ space by
$$\mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+) = \left\{ f \in L^2_\alpha(\mathbb{R}^d_+) : |\xi|^s \mathcal{F}_W(f) \in L^2_\alpha(\mathbb{R}^d_+) \right\}.$$

Recently, there are many results for the wavelet transform see [12, 17] and [13], that drove us to concentrate on some uncertainties principles and some results for the Weinstein wavelet transform.

Our first result is the Heisenberg-type uncertainty principle for the wavelet transform.

Theorem 1.1. *Let $s, t > 1$. Then, for all $f \in L^2_\alpha(\mathbb{R}^d_+)$, we have*
$$\||x|^s \mathcal{S}^h_W(f)(b,x)\|^{\frac{t}{s+t}}_{L^2_{\omega_\alpha}\mathbb{R}^{d+1}_{++}} \||\xi|^t \mathcal{F}_W(f)(\xi)\|^{\frac{s}{s+t}}_{L^2_\alpha \mathbb{R}^d_+} \geq C_{\alpha,d}(s,t)\mathcal{C}_h^{\frac{t}{2(s+t)}} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)},$$
where $C_{\alpha,d}(s,t) = (\alpha + \frac{d+1}{2})^{\frac{ts}{s+t}}$.

The second result is the Pitt's inequality for the Weinstein wavelet transform.

Theorem 1.2. *For $0 \leq s < \alpha + \frac{d+1}{2}$ and f in Schwartz space on $\mathbb{R}^d_+$, the Pitt's-type inequality for the wavelet transform is given by*
$$\||\xi|^s \mathcal{F}_W(f)\|_{L^2_\alpha(\mathbb{R}^d_+)} \leq \mathcal{C}_h^{-1} C(\alpha, s, d) \||x|^s \mathcal{S}^h_W(f)(b,x)\|_{L^2_\alpha(\mathbb{R}^{d+1}_{++})}.$$

The third is the Benkner-type uncertainty principle for the Weintein wavelet transform.

Theorem 1.3. *Let $f \in \mathrm{S}(\mathbb{R}^d_+)$, the following logarithmic uncertainty principle inequality for the Weinstein wavelet transform holds*
$$\int_{\mathbb{R}^{d+1}_{++}} \ln(|x|) |\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(b,x) + \mathcal{C}_h \int_{\mathbb{R}^d_+} \ln(|\xi|) |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi)$$
$$\geq \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \mathcal{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

Finally, the logarithmic Sobolev inequalities for the Weinstein wavelet transform.

Theorem 1.4. *Let $s > \frac{2\alpha+d+1}{2q}$, $1 < p \leq 2$, and for all $f \in \mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+)$ there exists a positive constant $\varepsilon(\alpha, d, s, h, p)$, such that*
$$\|\mathcal{S}^h_W(f)\|^2_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)} \leq \varepsilon(\alpha, d, s, h, p, q) \left(\|f\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)} + \||\xi|^s \mathcal{F}_W\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)} \right).$$

Theorem 1.5. *Let $h \in L^2_\alpha(\mathbb{R}^d_+)$ be a Weinstein wavelet and for all $f \in \mathrm{H}^{2,1}_\alpha$, there exists a positive constant $\mathcal{C}(\alpha, d)$, such that*
$$\int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}^h_W(f)(b,x)|^2 \ln \left(\frac{|\mathcal{S}^h_W(f)(b,x)|}{\|\mathcal{S}^h_W(f)(b,\cdot)\|_{L^2_\alpha(\mathbb{R}^d_+)}} \right) d\omega_\alpha(x)$$
$$\leq \left(\alpha + \frac{d+1}{2} \right) \mathrm{C}_h \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(\xi)|^2 \ln|\xi| d\mu_\alpha(x) - \mathcal{C}(\alpha, d) \mathrm{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

Theorem 1.6. *For all* $f \in L^2_\alpha(\mathbb{R}^d_+) \bigcap \mathrm{H}^{2,1}_\alpha$, *there exists a positive constant such that*

$$\int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}^h_W(f)(b,x)|^2 \ln(1+|x|^2) d\omega_\alpha(b,x) + \mathrm{C}_h \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(x)|^2 \ln(1+|x|^2) d\mu_\alpha(x)$$
$$\geq \mathcal{C}(\alpha,d)\mathcal{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

The structure of the paper is as follows. In the next section we introduce some further notations as well as some preliminary results. The section 3 is devoted to prove some result for the Weinstein wavelet transform.

2. Preliminary

2.1. **Generalities.** In this section, we fix some notation. We will denote by $|x|$ and $\langle x, y\rangle$ the usual norm and scalar product on $\mathbb{R}^d$. The unit sphere of $\mathbb{R}^d$ is denoted by $\mathbb{S}^{d-1}$.

If we denote by $\mathbb{S}^{d-1}_+ = \mathbb{S}^{d-1} \cap \mathbb{R}^d_+$, the area of the sphere $\mathbb{S}^{d-1}_+$ is

$$A_{d,\alpha} := \int_{\mathbb{S}^{d-1}_+} x_d^{2\alpha+1} d\sigma_d(x) = \frac{\pi^{\frac{d-1}{2}}\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{d+1}{2}\right)},$$

where σ_d is the normalized surface measure on $\mathbb{S}^{d-1}_+$.

For a radial function $f \in L^1_\alpha(\mathbb{R}^d_+)$ the function $\widetilde{f}$ defined on $\mathbb{R}_+$ such that $f(x) = \widetilde{f}(\|x\|)$, for all $x \in \mathbb{R}^d_+$ is integrable with respect to the measure $r^{2\alpha+d}dr$. More precisely. For a radial function $f \in L^1_\alpha(\mathbb{R}^d_+)$ the function $\widetilde{f}$ defined on $\mathbb{R}_+$ such that $f(x) = \widetilde{f}(\|x\|)$, for all $x \in \mathbb{R}^d_+$ is integrable with respect to the measure $r^{2\alpha+d}dr$.

More precisely,

$$\int_{\mathbb{R}^d_+} f(x) d\mu_\alpha(x) = a_\alpha \int_0^{+\infty} \widetilde{f}(r) r^{2\alpha+d} dr, \tag{2.1}$$

where $a_\alpha = \frac{A_{d,\alpha}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} = \frac{1}{2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+\frac{d+1}{2})}$.

For $r > 0$ we will denote by $B_r = \{x \in \mathbb{R}^d_+ : |x| < r\}$ the „ball“ in $\mathbb{R}^d_+$ of center 0 and radius r and the characteristic function of a set A will be denoted by χ_A, so that

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

2.2. **Harmonic analysis associated with the Weinstein operator.** We consider the Weinstein operator (also called Laplace-Bessel operator) (see [3, 4]), defined on $\mathbb{R}^{d-1} \times (0, +\infty)$ by

$$\Delta_W = \sum_{i=1}^d \frac{\partial}{\partial x_i^2} + \frac{2\alpha+1}{x_d} \cdot \frac{\partial}{\partial x_{d-1}}, \quad d \geq 2, \ \alpha > -1/2. \tag{2.2}$$

For $d > 2$, the operator Δ_W is the Laplace-Beltrami operator on the Riemanian space $\mathbb{R}^{d-1} \times (0, +\infty)$ equipped with the metric [3]

$$ds^2 = x_d^{4\alpha+2/(d-2)} \sum_{i=1}^{d} dx_i^2.$$

The Weinstein operator has several applications in pure and applied mathematics especially in fluid mechanics (see e.g. [7,23]). For $1 \le p \le +\infty$, we denote by $L^p_\alpha(\mathbb{R}^d_+)$ the Lebesgue space consisting of measurable functions f on $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$ equipped with the norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^d_+)} = \left(\int_{\mathbb{R}^d_+} \|f(x', x_d)\|^p \, d\mu_\alpha(x', x_d) \right)^{1/p}, \quad 1 \le p < +\infty,$$
$$\|f\|_{L^{+\infty}_\alpha(\mathbb{R}^d_+)} = \text{ess} \sup_{x \in \mathbb{R}^d_+} |f(x)| < +\infty,$$

where for $x = (x_1, \ldots, x_{d-1}, x_d) = (x', x_d)$ and

$$d\mu_\alpha(x) = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} dx' dx_d = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} dx_1 \cdots dx_d.$$

For $f \in L^1_\alpha(\mathbb{R}^d_+)$, the Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi', \xi_d) = \int_{\mathbb{R}^d_+} f(x', x_d) e^{-i\langle x', \xi' \rangle} j_\alpha(x_d \xi_d) d\mu_\alpha(x', x_d),$$

where j_α is the spherical Bessel function:

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha+k+1)} \left(\frac{z}{2} \right)^{2k}, \quad z \in \mathbb{C},$$

extends uniquely to an isometric isomorphism on $L^2_\alpha(\mathbb{R}^d_+)$, i.e.,

$$\|\mathcal{F}_W(f)_{L^2_\alpha(\mathbb{R}^d_+)} = \|f\|_{L^2_\alpha(\mathbb{R}^d_+)} \tag{2.3}$$

and we have

$$\mathcal{F}_W^{-1}(f)(\xi) = \mathcal{F}_W(f)(-\xi', \xi_d), \quad \xi = (\xi', \xi_d) \in \mathbb{R}^d_+.$$

Moreover if $f \in L^1_\alpha(\mathbb{R}^d_+)$, then

$$\|\mathcal{F}_W(f)\|_{L^\infty_\alpha(\mathbb{R}^d_+)} \le \|f\|_{L^1_\alpha(\mathbb{R}^d_+)}. \tag{2.4}$$

We recall the generalized translation operator $\mathcal{T}_x$, $x \in \mathbb{R}^d_+$, associated with the Weinstein operator Δ_W is defined for a continuous function f on $\mathbb{R}^d_+$, even with respect to the last variable by

$$\mathcal{T}_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\pi f\left(x' + y'; \sqrt{x_d^2 + y_d^2 + 2x_d^2 y_d^2 \cos\theta} \right) (\sin\theta)^{2\alpha} \, d\theta, \ y \in \mathbb{R}^d_+,$$

where $x' + y' = (x_1 + y_1, \ldots, x_{d-1} + y_{d-1})$.

For any function $f, g \in L^1_\alpha(\mathbb{R}^d_+)$, we define the convolution product associated with the Weinstein operator $f *_W g$ by

$$f *_W g(x) = \int_{\mathbb{R}^d_+} f(y)\mathcal{T}_{-x}(g)(y)d\mu_\alpha(y).$$

2.3. **The Weinstein wavelet transform.** A Weinstein wavelet on $\mathbb{R}^d_+$ is a measurable function h on $\mathbb{R}^d_+$ satisfying, for almost all $\xi \in \mathbb{R}^d_+$

$$0 < C_h = \int_0^{+\infty} |\mathcal{F}_W(h)(t)|^2 \frac{dt}{t} < +\infty. \tag{2.5}$$

Let $b > 0$, and let $h \in L^2_\alpha(\mathbb{R}^d_+)$, we define the dilation of h as follows:

$$h_b(y) = \frac{1}{b^{2\alpha+d+1}} h\left(\frac{y}{b}\right), \quad \text{for all } y \in \mathbb{R}^d_+.$$

It easy to see that $h_b \in L^2_\alpha(\mathbb{R}^d_+)$ and

$$\mathcal{F}_W(h_b)(\xi) = \mathcal{F}_W(h)(b\xi), \quad \text{for all } \xi \in \mathbb{R}^d_+,$$

$$0 < C_h = \int_0^{+\infty} |\mathcal{F}_W(h)(b\xi)|^2 \frac{db}{b} < +\infty. \tag{2.6}$$

For $y \in \mathbb{R}^d_+$, $b > 0$, $h_{b,y}$ is the Weinstein wavelet on $\mathbb{R}^d_+$ in $L^2_\alpha(\mathbb{R}^d_+)$, defined by

$$h_{b,y}(x) = b^{\alpha+\frac{d+1}{2}} \mathcal{T}_x h_b(-y', y_d), \quad \text{for all } y, x \in \mathbb{R}^d_+, \tag{2.7}$$

we note that

$$\|h_{b,y}\|_{L^2_\alpha(\mathbb{R}^d_+)} \leq \|h\|_{L^2_\alpha(\mathbb{R}^d_+)}, \quad \text{for all } b > 0, y \in \mathbb{R}^d_+.$$

Also, we denote by $L^p_{\omega_\alpha}$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d_+ \times [0, +\infty] = \mathbb{R}^{d+1}_{++}$, $d\omega_\alpha(x, b) = d\mu_\alpha(x)\frac{1}{b^{2\alpha+d+2}}db$ such that

$$\|f\|_{L^p_{\omega_\alpha}(\mathbb{R}^{d+1}_{++})} = \left(\int_{\mathbb{R}^{d+1}_{++}} |f(x, b)|^p d\omega_\alpha(x, b)\right)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < +\infty,$$

$$\|f\|_{L^{+\infty}_{\omega_\alpha}(\mathbb{R}^{d+1}_{++})} = \operatorname{esssup}_{x,y\in\mathbb{R}^{d+1}_{++}} |f(x, b)| < +\infty.$$

Let h be a Weinstein wavelet on $\mathbb{R}^d_+ \in L^2_\alpha(\mathbb{R}^d_+)$. We defined the Weinstein continuous wavelet

$$\mathcal{S}^h_W(f)(b, y) = \int_{\mathbb{R}^d_+} f(y)\overline{h_{b,y}(x)}d\mu_\alpha(x), \quad (b, y) \in \mathbb{R}^{d+1}_{++}, \tag{2.8}$$

and

$$\mathcal{S}^h_W(f)(b, y) = b^{\alpha+\frac{d+1}{2}} \langle f, \mathcal{T}_x h_b\rangle_{L^2_\alpha(\mathbb{R}^d_+)} = b^{\alpha+\frac{d+1}{2}} f *_W \overline{h_b}. \tag{2.9}$$

Then,

$$\mathcal{S}^h_W(f)(b, y) = b^{\alpha+\frac{d+1}{2}} \mathcal{F}^{-1}_W[\mathcal{F}_W(f)(\xi)\mathcal{F}_W(h)(\xi b)](y) \tag{2.10}$$

and

$$\|\mathcal{S}^h_W(f)\|_{L^2_{\omega_\alpha}(\mathbb{R}^{d+1}_{++})} \leq \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}\|h\|_{L^2_\alpha(\mathbb{R}^d_+)}. \tag{2.11}$$

Theorem 2.1 (Plancherel's formula for $\mathcal{S}_W^h$)**.** *Let h be a Weinstein wavelet. For all $f \in L^2_\alpha(\mathbb{R}^d_+)$*

$$\int_{\mathbb{R}^d_+} |f(x)|^2 d\mu_\alpha(x) = \mathcal{C}_h^{-1} \int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}_W^h(f)(b,y)|^2 d\omega_\alpha(b,y). \tag{2.12}$$

Corollary 2.1 (Parseval's formula for $\mathcal{S}_W^h$)**.** *Let h be a Weinstein wavelet. For all $f_1, f_2 \in L^2_\alpha(\mathbb{R}^d_+)$*

$$\int_{\mathbb{R}^d_+} f_1(x)\overline{f_2(x)} d\mu_\alpha(x) = \mathcal{C}_h^{-1} \int_{\mathbb{R}^{d+1}_{++}} \mathcal{S}_W^h(f_1)(b,y)\overline{\mathcal{S}_W^h(f_2)(b,y)} d\omega_\alpha(b,y). \tag{2.13}$$

Proposition 2.1. *For all $s \geq 0$ and $f \in L^2_\alpha(\mathbb{R}^d_+)$ we have*

$$\int_{\mathbb{R}^d_+} |\xi|^{2s} |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi) = \mathrm{C}_h^{-1} \int_{\mathbb{R}^{d+1}_{++}} |\xi|^{2s} |\mathcal{F}_W(\mathcal{S}_W^h(f))(\xi)|^2 d\omega_\alpha(b,\xi). \tag{2.14}$$

Proof. By (2.10) we have that

$$\begin{aligned} &\int_{\mathbb{R}^{d+1}_{++}} |\xi|^{2s} |\mathcal{F}_W(\mathcal{S}_W^h(f))(\xi)|^2 d\omega_\alpha(b,\xi) \\ =&\int_{\mathbb{R}^{d+1}_{++}} b^{2\alpha+d+1} |\xi|^{2s} |\mathcal{F}_W(f)(\xi)|^2 |\mathcal{F}_W(h)(\xi b)|^2 d\omega_\alpha(b,\xi). \end{aligned}$$

From Fubini's theorem and (2.6), the result follows. □

3. Some Results for the Weinstein Wavelet Transform

In this section, we will establish analogues of Heisenberg-type Uncertainty Principle for the Weinstein Wavelet Transform, Pitt's Inequality for the Weinstein Wavelet Transform, the Beckner-type uncertainty principle for the Weinstein wavelet transform and the logarithmic Sobolev inequalities for the Weinstein wavelet transform our proof is inspired by [13], who proved some results for the deformed wavelet transform and related uncertainty principles.

3.1. Heisenberg-type uncertainty principle for the Weinstein wavelet transform.

This is an extension of our study in [5] and there are many studies on the Heisenberg uncertainty principle inequality for wavelet transforms [13, 17]. In this section, we introduce the uncertainty inequality for the Weinstein wavelet transform. First, from our study [5, Corollary 3.5], we present the following theorem.

Theorem 3.1. *Let $s, t > 1$. Then, for all $f \in L^2_\alpha(\mathbb{R}^d_+)$, we have*

$$\| |x|^s \mathcal{S}_W^h(f)(b,x) \|^{\frac{t}{s+t}}_{L^2_{\omega_\alpha} \mathbb{R}^{d+1}_{++}} \| |\xi|^t \mathcal{F}_W(f)(\xi) \|^{\frac{s}{s+t}}_{L^2_\alpha \mathbb{R}^d_+} \geq C_{\alpha,d}(s,t) \mathcal{C}_h^{\frac{t}{2(s+t)}} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}, \tag{3.1}$$

where $C_{\alpha,d}(s,t) = (\alpha + \frac{d+1}{2})^{\frac{ts}{s+t}}$.

Proof. From [5, (3.10)], we have that for all $b > 0$

$$\left(\int_{\mathbb{R}^d_+} |\xi|^{2t} |\mathcal{F}_W[\mathcal{S}^h_W(f)(b,\cdot)](\xi)|^2 d\mu_\alpha(\xi)\right)^{\frac{s}{t+s}} \left(\int_{\mathbb{R}^d_+} |x|^{2s} |\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x)\right)^{\frac{t}{t+s}}$$
$$\geq (C_{\alpha,d}(s,t))^2 \int_{\mathbb{R}^d_+} |\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x).$$

Integrating both sides with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$, we obtain, by Hölder's inequality and Plancherel's formula,

$$\int_0^{+\infty} \left(\int_{\mathbb{R}^d_+} |\xi|^{2t} |\mathcal{F}_W[\mathcal{S}^h_W(f)(b,\cdot)](\xi)|^2 d\mu_\alpha(\xi)\right)^{\frac{s}{t+s}}$$
$$\times \left(\int_{\mathbb{R}^d_+} |x|^{2s} |\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x)\right)^{\frac{t}{t+s}} \frac{db}{b^{2\alpha+d+2}}$$
$$\geq (C_{\alpha,d}(s,t))^2 \int_0^{+\infty} \int_{\mathbb{R}^d_+} |\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x) \frac{db}{b^{2\alpha+d+2}}.$$

Thus, form (2.14), we deduce

$$\begin{aligned}\mathcal{C}_h^{\frac{s}{t+s}} \left(\int_{\mathbb{R}^d_+} |\xi|^{2t} |\mathcal{F}_W(f)(\xi)|^2 d\omega_\alpha(b,\xi)\right)^{\frac{s}{t+s}} & \left(\int_{\mathbb{R}^{d+1}_{++}} |x|^{2s} |\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(b,x)\right)^{\frac{t}{t+s}} \\ & \geq (C_{\alpha,d}(s,t))^2 \int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(x) \\ & \geq (C_{\alpha,d}(s,t))^2 \mathcal{C}_h \int_{\mathbb{R}^d_+} |f(x)|^2 d\mu_\alpha(x) \\ & \geq (C_{\alpha,d}(s,t))^2 \mathcal{C}_h \|f(x)\|^2_{\mathbb{R}^d_+}.\end{aligned}$$

□

3.2. **Pitt's inequality for the Weinstein wavelet transform.** The Pitt's inequality for the Weinstein transform is studied in [1], for all $f \in \mathrm{S}(\mathbb{R}^d_+)$, (the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d_+$, even with respect to the last variable) and $0 \leq s < \alpha + \frac{d+1}{2}$

$$\| |\xi|^{-s} \mathcal{F}_W(f)(\xi) \|_{L^2_\alpha(\mathbb{R}^d_+)} \leq C(\alpha,s,d) \| |x|^s f \|_{L^2_\alpha(\mathbb{R}^d_+)}, \tag{3.2}$$

where $C(\alpha,s,d) = 2^{-s} \frac{\Gamma\left(\frac{\alpha+\frac{d+1}{2}-s}{2}\right)}{\Gamma\left(\frac{\alpha+\frac{d+1}{2}+s}{2}\right)}$.

The main aim of this section is to formulate an analogue of Pitt's inequality (3.2) for the Weinstein wavelet transform.

Theorem 3.2. *For* $0 \le s < \alpha + \frac{d+1}{2}$ *and* $f \in \mathrm{S}(\mathbb{R}^d_+)$*, the Pitt's-type inequality for the wavelet transform is given by*

$$\| |\xi|^s \mathcal{F}_W(f) \|_{L^2_\alpha(\mathbb{R}^d_+)} \le \mathcal{C}_h^{-1} C(\alpha, s, d) \| |x|^s \mathcal{S}^h_W(f)(b, x) \|_{L^2_\alpha(\mathbb{R}^{d+1}_{++})}.$$

Proof. In (3.2), replace f with $\mathcal{S}^h_W(f)$ and for $b > 0$, we have

$$\int_{\mathbb{R}^d_+} |\xi|^s \mathcal{F}_W[\mathcal{S}^h_W(f)((b, \cdot)](\xi) d\mu_\alpha(x) \le C(\alpha, s, d) \int_{\mathbb{R}^{d+1}_{++}} |x|^s \mathcal{S}^h_W(f)((b, x) d\omega_\alpha(x).$$

By integrating both sides with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$, and using Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d_+} \int_0^{+\infty} |\xi|^{2s} |\mathcal{F}_W[\mathcal{S}^h_W(f)((b, \cdot)]|^2 \frac{db}{b^{2\alpha+d+2}} d\mu_\alpha(\xi) \\
\le & C(\alpha, s, d) \int_{\mathbb{R}^d_+} \int_0^{+\infty} |x|^{2s} |\mathcal{S}^h_W(f)((b, x)| \frac{db}{b^{2\alpha+d+2}} d\mu_\alpha(x).
\end{aligned}$$

Using (2.10), the last inequality is introduced as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d_+} \int_0^{+\infty} |\xi|^{2s} |\mathcal{F}_W(h_b)(\xi)|^2 |\mathcal{F}_W(f)(\xi)|^2 \frac{db}{b^{2\alpha+d+2}} d\mu_\alpha(\xi) \\
\le & C(\alpha, s, d) \int_{\mathbb{R}^{d+1}_{++}} |x|^{2s} |\mathcal{S}^h_W(f)((b, x)|^2 \frac{db}{b^{2\alpha+d+2}} d\omega_\alpha(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^d_+} |\xi|^{2s} \left(\int_0^{+\infty} |\mathcal{F}_W(h_b)(\xi)|^2 \frac{db}{b} \right) |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi) \\
\le & C(\alpha, s, d) \int_{\mathbb{R}^d_+} \int_0^{+\infty} |x|^{2s} |\mathcal{S}^h_W(f)((b, x)|^2 \frac{db}{b^{2\alpha+d+2}} d\omega_\alpha(x).
\end{aligned}$$

From (2.6), we obtain

$$\mathcal{C}_h \int_{\mathbb{R}^d_+} |\xi|^{2s} |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi) \le C(\alpha, s, d) \int_{\mathbb{R}^{d+1}_{++}} |x|^{2s} |\mathcal{S}^h_W(f)((b, x)|^2 \frac{db}{b^{2\alpha+d+2}} d\omega_\alpha(x).$$

This completes the proof. □

3.3. **Beckner-Type uncertainty principle for the Weinstein wavelet transform.** In this section, we study the Beckner-type uncertainty principle for the Weinstein wavelet transform, as stated in the following theorem.

Theorem 3.3. *Let* $f \in \mathrm{S}(\mathbb{R}^d_+)$*. The following logarithmic uncertainty principle inequality for the Weinstein wavelet transform holds*

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}_{++}} \ln(|x|) |\mathcal{S}^h_W(f)(b, x)|^2 d\omega_\alpha(b, x) + \mathcal{C}_h \int_{\mathbb{R}^d_+} \ln(|\xi|) |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi) \\
\ge & \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \mathcal{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.
\end{aligned}$$

Proof. From [1, Theorem 4.5], we observe

$$\int_{\mathbb{R}^d_+} \ln(|x|)|f(x)|^2 d\mu_\alpha(x) + \int_{\mathbb{R}^d_+} \ln(|\xi|)|\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi)$$
$$\geq \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

Here we replace f by $\mathcal{S}^h_W(b,\cdot)$, and we get

$$\int_{\mathbb{R}^d_+} \ln(|x|)|\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x) + \int_{\mathbb{R}^d_+} \ln(|\xi|)|\mathcal{F}_W[\mathcal{S}^h_W(f)(b,\cdot)](\xi)|^2 d\mu_\alpha(\xi)$$
$$\geq \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \int_{\mathbb{R}^d_+} |\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x).$$

Integrating both sides of the last inequality with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$, and using Fubini's theorem, we obtain

$$\int_{\mathbb{R}^{d+1}_{++}} \ln(|x|)|\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(b,x) + \int_{\mathbb{R}^{d+1}_{++}} \ln(|\xi|)|\mathcal{F}_W[\mathcal{S}^h_W(f)(b,\cdot)](\xi)|^2 d\omega_\alpha(b,\xi)$$
$$\geq \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}^h_W(f)(b,x)|^2 d\mu_\alpha(x).$$

Using (2.10), and Plancherel's formula, we obtain

$$\int_{\mathbb{R}^{d+1}_{++}} \ln(|x|)|\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(b,x) + \int_{\mathbb{R}^{d+1}_{++}} \ln(|\xi|)|\mathcal{F}_W(h)(b,\xi)|^2|\mathcal{F}_W(f)(\xi)|^2 d\omega_\alpha(b,\xi)$$
$$\geq \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \int_{\mathbb{R}^{d+1}_{++}} |\mathcal{F}_W(h)(b,\xi)|^2|\mathcal{F}_W(f)(\xi)|^2 d\omega_\alpha(b,x).$$

Consequently,

$$\int_{\mathbb{R}^{d+1}_{++}} \ln(|x|)|\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(b,x) + \mathcal{C}_h \int_{\mathbb{R}^d_+} \ln(|\xi|)|\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi)$$
$$\geq \left(\frac{\Gamma'(\frac{2\alpha+d+1}{4})}{\Gamma(\frac{2\alpha+d+1}{4})} + \ln(2) \right) \mathcal{C}_h \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi),$$

which proves the desired result. ☐

3.4. **Logarithmic Sobolev inequalities for the Weinstein wavelet transform.**

First, we introduce the Sobolev space that naturally arise in connection with the Weinstein transform.

Definition 3.1. Let $s \in \mathbb{R}^d_+$, $1 \leq p < +\infty$, and $T \in \mathrm{S}'(\mathbb{R}^d_+)$ be a distribution. We define:

$$\mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+) = \left\{ T \in \mathrm{S}(\mathbb{R}^d_+) : (1+|\xi|^2)^{\frac{ps}{2}} \mathcal{F}_W(T) \in L^p_\alpha(\mathbb{R}^d_+) \right\}. \tag{3.3}$$

Means that for $s>0$, $T\in \mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+)$ and $\mathcal{F}_W(T)$ is given by a function in $L^p_\alpha(\mathbb{R}^d_+)$, the norm on $\mathrm{H}^{p,s}_\alpha$ is defined by

$$\|T\|_{\mathrm{H}^{p,s}_\alpha}=\|(1+|x|^2)^{\frac{s}{2}}\|_{L^p_\alpha(\mathbb{R}^d_+)}.$$

It is clear that $\mathrm{S}(\mathbb{R}^d_+)\subset \mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+)$. Also, as is usually the case, it can be easily shown that $\mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+)$ is dense in $\mathrm{S}'(\mathbb{R}^d_+)$, see [2] more details.

Here, our interest is in the case $p=2$, for $T\in \mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+)$, it can be seen that T is necessary given by a function $f\in L^2_\alpha(\mathbb{R}^d_+)$ the space $\mathrm{H}^{2,s}_\alpha$ can be defined as

$$\mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+)=\left\{f\in L^2_\alpha(\mathbb{R}^d_+):|\xi|^s\mathcal{F}_W(f)\in L^2_\alpha(\mathbb{R}^d_+)\right\}. \tag{3.4}$$

In the following, we give a Sobolev embedding theorem.

Theorem 3.4. *Let $s>\frac{2\alpha+d+1}{2q}$, $1<p\le 2$, and for all $f\in \mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+)$ there exists a positive constant $\varepsilon(\alpha,d,s,h,p)$, such that*

$$\|\mathcal{S}^h_W(f)\|^2_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)}\le \varepsilon(\alpha,d,s,h,p,q)\left(\|f\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)}+\||\xi|^s\mathcal{F}_W\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)}\right). \tag{3.5}$$

Proof. From Plancherel's formula,we obtain

$$\|\mathcal{S}^h_W(f)\|^2_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)}\le \mathcal{C}_h\int_{\mathbb{R}^d_+}|\mathcal{F}_W(f)(\xi)|^2(1+|\xi|^2)^s(1+|\xi|^2)^{-s}d\mu_\alpha(\xi).$$

Using Hölder's inequality, we get

$$\|\mathcal{S}^h_W(f)\|^2_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)}\le\left(\int_{\mathbb{R}^d_+}|\mathcal{F}_W(f)(\xi)|^{2p}(1+|\xi|^2)^{sp}d\mu_\alpha(\xi)\right)^{1/p}\times\left(\int_{\mathbb{R}^d_+}(1+|\xi|^2)^{-sq}d\mu_\alpha(\xi)\right)^{1/q}.$$

By (2.1), we have

$$\int_{\mathbb{R}^d_+}(1+|\xi|^2)^{-sq}d\mu_\alpha(\xi)=\frac{\Gamma(sq-(\alpha+\frac{d+1}{2}))\Gamma(\alpha+\frac{d+1}{2})}{2^{\alpha+\frac{d+1}{2}}\Gamma(\alpha+\frac{d+1}{2})\Gamma(sq)}.$$

Using the fact that $(a+b)^s\le 2^s(a^s+b^s)$, we get

$$\|\mathcal{S}^h_W(f)\|^{2p}_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)}\le\mathcal{C}^p_h\left[\frac{\Gamma(sq-(\alpha+\frac{d+1}{2}))\Gamma(\alpha+\frac{d+1}{2})}{2^{\alpha+\frac{d+1}{2}}\Gamma(\alpha+\frac{d+1}{2})\Gamma(sq)}\right]^{p/q}\times 2^{sp}\left(\|f\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)}+\||\xi|^s\mathcal{F}_W\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)}\right).$$

Thus,

$$\varepsilon(\alpha,d,s,h,p,q)=\mathrm{C}^p_h\left[\frac{\Gamma(sq-(\alpha+\frac{d+1}{2}))\Gamma(\alpha+\frac{d+1}{2})}{2^{\alpha+\frac{d+1}{2}}.\Gamma(\alpha+\frac{d+1}{2})\Gamma(sq)}\right]^{1/q}2^{sp}$$

This completes the proof. □

Corollary 3.1. *Let* $s > \frac{2\alpha+d+1}{2q}$. *There exists a positive constant* $\epsilon(\alpha, d, s, h, p, q)$ *such that for* $f \in \mathrm{H}^{p,s}_{\alpha}(\mathbb{R}^d_+)$

$$\|\mathcal{S}^h_W(f)\|^2_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)} \leq \zeta(\alpha, d, s, h, p, q) \left(\|f\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)} \||\xi|^s \mathcal{F}_W\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)} \right). \tag{3.6}$$

Proof. Using the dilated $\mathcal{D}_\lambda(f)(x) = \lambda^{\alpha+\frac{d+1}{2}} f(\lambda x)$ to (3.5), we get

$$\begin{aligned}\|\mathcal{S}^h_W(f)\|^2_{L_{\omega_\alpha}(\mathbb{R}^d_+)} \leq & \varepsilon(\alpha, d, s, h, p, q) \Big(\lambda^{-(2\alpha+d+1)} \|f\|^p_{L^p_\alpha(\mathbb{R}^d_+)} \\ & + \lambda^{2sp-(2\alpha+d+1)} \||\xi|^s \mathcal{F}_W\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)} \Big).\end{aligned}$$

By minimizing the right-hand side of the last inequality, we obtain

$$\begin{aligned}\|\mathcal{S}^h_W(f)\|^2_{L_{\omega_\alpha}(\mathbb{R}^d_+)} \leq & \varepsilon(\alpha, d, s, h, p, q)^{1/2} \left(\frac{sp\left(ps - (\alpha + \frac{d+1}{2})\right)^{\frac{2\alpha+d+1-2sp}{2sp}}}{(\alpha + \frac{d+1}{2})^{\frac{2\alpha+d+1}{2ps}}} \right)^{1/2} \\ & \times \|f\|^{p-\frac{2\alpha+d+1}{2s}}_{L^{2p}_\alpha(\mathbb{R}^d_+)} \||\xi|^s \mathcal{F}_W\|^{\frac{2\alpha+d+1}{2s}}_{L^{2p}_\alpha(\mathbb{R}^d_+)}.\end{aligned}$$

This is the desired result. □

Now, to describe our next result, we recall Theorem 5.6 in [2]. There exists a positive constant $C(\alpha, d)$ such that for all $f \in \mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+)$

$$\begin{aligned}&\int_{\mathbb{R}^d_+} |f(x)|^2 \ln\left(\frac{|f(x)|}{\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}} \right) d\mu_\alpha(x) \\ &\leq \left(\alpha + \frac{d+1}{2} \right) \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(\xi)|^2 \ln|\xi| d\mu_\alpha(x) - \mathcal{C}(\alpha, d) \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.\end{aligned} \tag{3.7}$$

Now, with the same constant $\mathcal{C}(\alpha, d)$, we show the following theorem.

Theorem 3.5. *Let* $h \in L^2_\alpha(\mathbb{R}^d_+)$ *be a Weinstein wavelet and for all* $f \in \mathrm{H}^{2,s}_\alpha$, *there exists a positive constant* $\mathcal{C}(\alpha, d)$

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}^h_W(f)(b, x)|^2 \ln\left(\frac{|\mathcal{S}^h_W(f)(b, x)|}{\|\mathcal{S}^h_W(f)(b, \cdot)\|_{L^2_\alpha(\mathbb{R}^d_+)}} \right) d\omega_\alpha(x) \\ &\leq \left(\alpha + \frac{d+1}{2} \right) \mathrm{C}_h \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(\xi)|^2 \ln|\xi| d\mu_\alpha(x) - \mathcal{C}(\alpha, d) \mathrm{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.\end{aligned}$$

Proof. By replacing $\mathcal{S}_W^h(f)(b,x)$ and integrating both sides of (3.7) with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$ we get

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}_+^d}|\mathcal{S}_W^h(f)(b,x)|^2\ln\left(\frac{|\mathcal{S}_W^h(f)(b,x)|}{\|\mathcal{S}_W^h(f)(b,\cdot)\|_{L_\alpha^2(\mathbb{R}_+^d)}}\right)d\mu_\alpha(x)\frac{db}{b^{2\alpha+d+2}}\\ \leq&\left(\alpha+\frac{d+1}{2}\right)\int_0^{+\infty}\int_{\mathbb{R}_+^d}|\mathcal{F}_W(\mathcal{S}_W^h(f)(b,\cdot))(\xi)|^2\ln|\xi|d\mu_\alpha(x)\frac{db}{b^{2\alpha+d+2}}\\ &-\mathcal{C}(\alpha,d)\int_0^{+\infty}\|\mathcal{S}_W^h(f)(b,\cdot)\|_{L_\alpha^2(\mathbb{R}_+^d)}^2\frac{db}{b^{2\alpha+d+2}}.\end{aligned}$$

Now, by Fubini's theorem and Plancherel's formula, we get the required result. □

Finally, we give another version of logarithmic uncertainty for the Weinstein wavelet transform.

Theorem 3.6. *For all $f\in L_\alpha^2(\mathbb{R}_+^d)\bigcap \mathrm{H}_\alpha^{2,1}$, there exists a positive constant such that*

$$\begin{aligned}&\int_{\mathbb{R}_{++}^{d+1}}|\mathcal{S}_W^h(f)(b,x)|^2\ln(1+|x|^2)d\omega_\alpha(b,x)+\mathrm{C}_h\int_{\mathbb{R}_+^d}|\mathcal{F}_W(f)(x)|^2\ln(1+|x|^2)d\mu_\alpha(x)\\ &\geq\mathcal{C}(\alpha,d)\mathcal{C}_h\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2.\end{aligned}$$

Proof. In the same manner, replacing f with $\mathcal{S}_W^h(f)$ and integrating both sides of the inequality in [2, Theorem 5.8], we obtain

$$\begin{aligned}&\int_{\mathbb{R}_{++}^{d+1}}|\mathcal{S}_W^h(f)(b,x)|^2\ln(1+|x|^2)d\omega_\alpha(b,x)\\ &+\int_{\mathbb{R}_{++}^{d+1}}|\mathcal{F}_W(\mathcal{S}_W^h(f)(b,\cdot))(x)|^2\ln(1+|x|^2)d\mu_\alpha(x)\\ \geq&\mathcal{C}(\alpha,d)\|\mathcal{S}_W^h(f)(b,x)\|_{L_\alpha^2(\mathbb{R}_{++}^{d+1})}^2.\end{aligned}$$

Applying Proposition 2.1, Plancherel's formula, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_{++}^{d+1}}|\mathcal{S}_W^h(f)(b,x)|^2\ln(1+|x|^2)d\omega_\alpha(b,x)+\mathcal{C}_h\int_{\mathbb{R}_+^d}|\mathcal{F}_W(f)(x)|^2\ln(1+|x|^2)d\mu_\alpha(x)\\ &\geq\mathcal{C}(\alpha,d)\mathcal{C}_h\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2.\end{aligned}$$

□

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