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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A SINGULAR ELLIPTIC PROBLEM INVOLVING THE 1-LAPLACIAN OPERATOR

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Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, let $0 < \gamma < 1$, and let $1 < q < n/(n-1)$. We study directly in $BV(\Omega)$ the relaxed Dirichlet energy

$$\mathcal{E}_\lambda(u) = |D\widetilde{u}|(\mathbb{R}^n) - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,dx - \frac{1}{q}\int_\Omega |u|^q\,dx,$$

where $\widetilde{u}$ denotes the zero extension of u. Although the functional is unbounded from below and is not C^1, we prove that for sufficiently small $\lambda > 0$ it possesses a nontrivial local minimizer u_λ which is positive almost everywhere. We obtain an explicit small-parameter estimate $|D\widetilde{u}_\lambda|(\mathbb{R}^n) = O(\lambda^{1/\gamma})$, prove $u_\lambda^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$, establish a two-sided first-variation formula for the singular potential along compactly supported smooth directions, and derive a sharp variational inequality. As a consequence, the source $\lambda u_\lambda^{-\gamma} + u_\lambda^{q-1}$ admits a bounded divergence representation $-\operatorname{div} z$ with $\|z\|_\infty \leq 1$. We carefully distinguish this direct variational statement from the stronger notion of weak solution for the formal 1-Laplacian equation, which additionally requires the saturation relation $(z, Du) = |Du|$ and a boundary trace condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. We consider the singular concave–convex expression

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = \lambda u^{-\gamma} + u^{q-1} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

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under the assumptions

$$0 < \gamma < 1, \quad 1 < q < 1^* := \frac{n}{n-1}, \quad \lambda > 0. \tag{1.2}$$

The symbol $Du/|Du|$ is only formal when $u \in BV(\Omega)$, because Du is a vector-valued Radon measure. A genuine weak formulation of (1.1) is normally expressed through a bounded divergence-measure field and the Anzellotti pairing; see, for example, [2, 3].

The same concave-convex equation was studied by Ortiz Chata, Pimenta and Segura de León [7], who obtained two positive solutions for small λ by approximation with p-Laplacian problems and passage to the limit as $p \downarrow 1$. Their work also makes clear why one should not treat the corresponding BV energy as an ordinary C^1 functional. The purpose of the present note is different and deliberately narrower: we isolate what can be proved directly by constrained minimization of the relaxed total-variation energy, without invoking a fictitious Fréchet derivative or an invalid Palais-Smale argument in a nonreflexive space.

For $u \in BV(\Omega)$ let Tu denote its trace on $\partial\Omega$ and let $\widetilde{u}$ be its extension by zero to $\mathbb{R}^n$. The relaxed zero-Dirichlet total variation is

$$\mathcal{T}(u) := |D\widetilde{u}|(\mathbb{R}^n) = |Du|(\Omega) + \int_{\partial\Omega} |Tu| \, d\mathcal{H}^{n-1}. \tag{1.3}$$

We study

$$\mathcal{E}_\lambda(u) := \mathcal{T}(u) - \frac{\lambda}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, dx - \frac{1}{q} \int_\Omega |u|^q \, dx, \quad u \in BV(\Omega). \tag{1.4}$$

All terms are finite because $BV(\Omega) \hookrightarrow L^{1^*}(\Omega)$ continuously.

Two elementary facts govern the variational analysis. First, (1.4) is not bounded below: for every nonzero u,

$$\mathcal{E}_\lambda(tu) = t\mathcal{T}(u) - \frac{\lambda t^{1-\gamma}}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, dx - \frac{t^q}{q} \int_\Omega |u|^q \, dx \to -\infty,$$

as $t \to +\infty$. Thus global minimization, even after adding a constant, is impossible. Second, the singular map $u \mapsto \int |u|^{1-\gamma}$ is not C^1 at functions which vanish. Hence, a direct application of a classical smooth symmetric mountain-pass theorem is unavailable.

Our main result is a direct local-minimization theorem with an explicit smallness threshold and a quantitative estimate as $\lambda \downarrow 0$. The first-variation analysis further yields a bounded divergence representation for the source. We do not claim that the resulting local minimizer is automatically a weak solution of (1.1): the calibration identity $(z, Du) = |Du|$ and the corresponding boundary trace relation require additional arguments not contained in the present direct minimization scheme.

2. BV Preliminaries and Compactness

We use the standard norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

The trace operator $T : BV(\Omega) \to L^1(\partial\Omega)$ is bounded on Lipschitz domains. Consequently, $\mathcal{T}$ is continuous with respect to the BV norm and lower semicontinuous with respect to L^1 convergence of zero extensions; see [1, 6].

Lemma 2.1 (Sobolev bounds and compactness)**.** *There are constants $C_1, C_q > 0$, depending only on Ω, n, q, such that*

$$\|u\|_{L^1(\Omega)} \le C_1 \mathcal{T}(u), \quad \|u\|_{L^q(\Omega)} \le C_q \mathcal{T}(u), \tag{2.1}$$

for every $u \in BV(\Omega)$. If (u_j) satisfies $\sup_j \mathcal{T}(u_j) < +\infty$, then a subsequence converges strongly in $L^1(\Omega)$ and in $L^q(\Omega)$ to a function $u \in BV(\Omega)$, and

$$\mathcal{T}(u) \le \liminf_{j\to+\infty} \mathcal{T}(u_j). \tag{2.2}$$

Proof. Apply the whole-space BV Sobolev inequality to the zero extension

$$\|\widetilde{u}\|_{L^{1^*}(\mathbb{R}^n)} \le C|D\widetilde{u}|(\mathbb{R}^n) = C\mathcal{T}(u).$$

Since Ω has finite measure, Hölder's inequality gives (2.1). If $\mathcal{T}(u_j)$ is bounded, then the zero extensions are bounded in $BV(\mathbb{R}^n)$ and supported in the fixed compact set $\overline{\Omega}$. The compactness theorem for BV gives a subsequence converging in $L^1(\mathbb{R}^n)$. The uniform L^{1^*} bound and interpolation yield strong convergence in $L^q(\Omega)$ because $q < 1^*$. Finally, lower semicontinuity of total variation in $L^1(\mathbb{R}^n)$ gives (2.2). □

Set

$$a := 1 - \gamma \in (0, 1), \quad A := |\Omega|^\gamma C_1^{1-\gamma}. \tag{2.3}$$

Then,

$$\int_\Omega |u|^{1-\gamma}\, dx \le |\Omega|^\gamma \|u\|_{L^1}^{1-\gamma} \le A\mathcal{T}(u)^{1-\gamma}. \tag{2.4}$$

3. Existence of an Interior Local Minimizer

Define

$$R := \left(\frac{q}{2C_q^q}\right)^{1/(q-1)}, \quad \lambda_* := \frac{(1-\gamma)R^\gamma}{4A}. \tag{3.1}$$

Theorem 3.1 (Direct BV local minimization)**.** *Assume* (1.2)*. For every $0 < \lambda < \lambda_*$ there exists $u_\lambda \in BV(\Omega)$ such that*

$$u_\lambda > 0 \quad \text{a.e. in } \Omega, \quad 0 < \mathcal{T}(u_\lambda) < R, \quad \mathcal{E}_\lambda(u_\lambda) < 0. \tag{3.2}$$

Moreover,

(i) *u_λ is a local minimizer of $\mathcal{E}_\lambda$ with respect to the $BV(\Omega)$ norm;*

(ii)

$$\mathcal{T}(u_\lambda) < \left(\frac{2A}{1-\gamma}\lambda\right)^{1/\gamma}, \tag{3.3}$$

so in particular $\mathcal{T}(u_\lambda) \to 0$ as $\lambda \downarrow 0$;

(iii) $u_\lambda^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$ *and, for every nonnegative* $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \left(\lambda u_\lambda^{-\gamma} + u_\lambda^{q-1}\right)\varphi \, dx \le \int_\Omega |\nabla \varphi| \, dx. \tag{3.4}$$

Proof. Let $K_R := \{u \in BV(\Omega) : \mathcal{T}(u) \le R\}$. We divide the proof into five steps.

Step 1: positive energy on the boundary of K_R. If $\mathcal{T}(u) = R$, then (2.1) and (2.4) give

$$\mathcal{E}_\lambda(u) \ge R - \frac{\lambda A}{1-\gamma} R^{1-\gamma} - \frac{C_q^q}{q} R^q.$$

By (3.1),

$$\frac{C_q^q}{q} R^q = \frac{R}{2}, \quad \frac{\lambda A}{1-\gamma} R^{1-\gamma} < \frac{R}{4}.$$

Therefore,

$$\inf_{\mathcal{T}(u)=R} \mathcal{E}_\lambda(u) > \frac{R}{4} > 0. \tag{3.5}$$

Step 2: negative energy inside K_R. Choose $0 \le \psi \in C_c^\infty(\Omega)$, $\psi \not\equiv 0$. Since

$$\frac{\mathcal{E}_\lambda(t\psi)}{t^{1-\gamma}} = t^\gamma \mathcal{T}(\psi) - \frac{\lambda}{1-\gamma} \int_\Omega \psi^{1-\gamma} \, dx - \frac{t^{q-1+\gamma}}{q} \int_\Omega \psi^q \, dx,$$

we obtain

$$\lim_{t \downarrow 0} \frac{\mathcal{E}_\lambda(t\psi)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma} \int_\Omega \psi^{1-\gamma} \, dx < 0.$$

Hence, for sufficiently small $t > 0$ one has $\mathcal{T}(t\psi) < R$ and $\mathcal{E}_\lambda(t\psi) < 0$.

Step 3: attainment of the constrained minimum. The estimates

$$\mathcal{E}_\lambda(u) \ge -\frac{\lambda A}{1-\gamma} R^{1-\gamma} - \frac{C_q^q}{q} R^q, \quad u \in K_R,$$

show that $\mathcal{E}_\lambda$ is bounded below on K_R. Let $(u_j) \subset K_R$ be a minimizing sequence. By Lemma 2.1, after extracting a subsequence,

$$u_j \to u \text{ in } L^1(\Omega) \cap L^q(\Omega), \quad \mathcal{T}(u) \le R.$$

For $0 < a < 1$ the elementary inequality

$$\left| |s|^a - |t|^a \right| \le |s - t|^a$$

implies

$$\int_\Omega \left| |u_j|^a - |u|^a \right| dx \le |\Omega|^{1-a} \|u_j - u\|_{L^1}^a \to 0.$$

The q-power term converges because $u_j \to u$ in L^q, while (2.2) controls the total variation. Consequently,

$$\mathcal{E}_\lambda(u) \le \liminf_{j \to +\infty} \mathcal{E}_\lambda(u_j) = \inf_{K_R} \mathcal{E}_\lambda.$$

By Step 2 this minimum is negative. By (3.5), every minimizer satisfies $\mathcal{T}(u) < R$. Since $\mathcal{E}_\lambda(|u|) \le \mathcal{E}_\lambda(u)$ and $\mathcal{T}(|u|) \le \mathcal{T}(u)$, we may choose a nonnegative minimizer and denote it by u_λ. Its negative energy implies $u_\lambda \not\equiv 0$.

Because the trace map is bounded on $BV(\Omega)$,

$$|\mathcal{T}(v) - \mathcal{T}(w)| \leq |D(v-w)|(\Omega) + \|T(v-w)\|_{L^1(\partial\Omega)} \leq C\|v-w\|_{BV(\Omega)}.$$

Thus, $\mathcal{T}$ is continuous in the BV norm. Since $\mathcal{T}(u_\lambda) < R$, a sufficiently small BV-norm ball around u_λ is contained in K_R. Therefore u_λ is an unconstrained local minimizer of $\mathcal{E}_\lambda$ in the BV norm.

Step 4: quantitative smallness. Set $T := \mathcal{T}(u_\lambda)$. Since $0 < T < R$ and $\mathcal{E}_\lambda(u_\lambda) < 0$, using (2.4) and (2.1) yields

$$0 > \mathcal{E}_\lambda(u_\lambda) \geq T - \frac{\lambda A}{1-\gamma} T^{1-\gamma} - \frac{C_q^q}{q} T^q.$$

By the definition of R and $T < R$,

$$\frac{C_q^q}{q} T^q \leq \frac{C_q^q}{q} R^{q-1} T = \frac{1}{2} T.$$

Therefore,

$$\frac{1}{2} T < \frac{\lambda A}{1-\gamma} T^{1-\gamma}.$$

Since $T > 0$, this is exactly (3.3).

Step 5: one-sided variation and positivity. Fix $0 \leq \varphi \in C_c^\infty(\Omega)$. Since $\mathcal{T}(u_\lambda) < R$, for all sufficiently small $t > 0$ we have $u_\lambda + t\varphi \in K_R$. Minimality and subadditivity of $\mathcal{T}$ imply

$$\begin{aligned} &\frac{\lambda}{1-\gamma} \int_\Omega \frac{(u_\lambda + t\varphi)^{1-\gamma} - u_\lambda^{1-\gamma}}{t}\, dx + \frac{1}{q} \int_\Omega \frac{(u_\lambda + t\varphi)^q - u_\lambda^q}{t}\, dx \\ \leq &\frac{\mathcal{T}(u_\lambda + t\varphi) - \mathcal{T}(u_\lambda)}{t} \leq \mathcal{T}(\varphi) = \int_\Omega |\nabla \varphi|\, dx. \end{aligned} \tag{3.6}$$

For fixed x, the first difference quotient increases as $t \downarrow 0$ to $(1-\gamma) u_\lambda^{-\gamma} \varphi$, with the value $+\infty$ on $\{u_\lambda = 0,\ \varphi > 0\}$. Hence, monotone convergence applies along any decreasing sequence $t_k \downarrow 0$. The second quotient converges in L^1 to $q u_\lambda^{q-1} \varphi$; indeed it is dominated, for $0 < t < 1$, by a constant multiple of $(u_\lambda^{q-1} + 1)|\varphi|$, which is integrable on the compact support of φ. Passing to the limit in (3.6) proves (3.4) and, in particular, finiteness of its singular integral.

If $K \Subset \Omega$, choose $\varphi \in C_c^\infty(\Omega)$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on K. Then, (3.4) gives

$$\lambda \int_K u_\lambda^{-\gamma}\, dx < +\infty.$$

Therefore, $u_\lambda > 0$ almost everywhere on K. Exhausting Ω by compact subsets proves $u_\lambda > 0$ almost everywhere in Ω and $u_\lambda^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$. □

Corollary 3.1. *Under the assumptions of Theorem* 3.1*,*

$$\|u_\lambda\|_{L^1(\Omega)} + \|u_\lambda\|_{L^q(\Omega)} = O(\lambda^{1/\gamma}), \quad \lambda \downarrow 0.$$

Proof. Combine (2.1) with (3.3). □

4. First Variation of the Singular Potential

The one-sided estimate obtained above is enough to prove positivity. Once positivity and local integrability of $u_\lambda^{-\gamma}$ are known, the singular potential actually has a genuine directional derivative along compactly supported smooth perturbations. We record the precise statement because it is useful for identifying the local Euler inequality.

Lemma 4.1 (Two-sided directional derivative)**.** *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfy $u > 0$ a.e. and $u^{-\gamma} \in L^1_{\mathrm{loc}}(\Omega)$. Then, for every $\varphi \in C_c^\infty(\Omega)$,*

$$\lim_{t\to 0} \frac{1}{t} \int_\Omega \Big(|u + t\varphi|^{1-\gamma} - u^{1-\gamma} \Big)\, dx = (1-\gamma) \int_\Omega u^{-\gamma} \varphi \, dx. \tag{4.1}$$

If moreover $u \in L^q(\Omega)$ with $q > 1$, then

$$\lim_{t\to 0} \frac{1}{t} \int_\Omega \Big(|u + t\varphi|^{q} - u^{q} \Big)\, dx = q \int_\Omega u^{q-1} \varphi \, dx. \tag{4.2}$$

Proof. Let $K = \operatorname{supp} \varphi$ and write $a = 1 - \gamma$. Pointwise almost everywhere, since $u > 0$,

$$\frac{|u + t\varphi|^a - u^a}{t} \to a u^{a-1} \varphi.$$

We prove uniform integrability. Split K into

$$A_t := \{ u > 2|t| \cdot |\varphi| \}, \quad B_t := K \setminus A_t.$$

On A_t, $|u + t\varphi| \geq u/2$. The mean-value theorem therefore gives

$$\left| \frac{|u + t\varphi|^a - u^a}{t} \right| \leq C_a u^{-\gamma} |\varphi|.$$

On B_t we have $u \leq 2|t| \cdot |\varphi|$, hence

$$\left| \frac{|u + t\varphi|^a - u^a}{t} \right| \leq C_a |t|^{-\gamma} |\varphi|^{1-\gamma} \leq C_a' u^{-\gamma} |\varphi|.$$

The last inequality follows from $u \leq 2|t| \cdot |\varphi|$. Since $u^{-\gamma}|\varphi| \in L^1(K)$, dominated convergence proves (4.1). Formula (4.2) follows in the usual way from the mean-value theorem and the integrable domination

$$\left| \frac{|u + t\varphi|^q - u^q}{t} \right| \leq C_q \Big(u^{q-1} |\varphi| + |\varphi|^q \Big), \quad |t| \leq 1. \qquad \square$$

Proposition 4.1 (Local Euler inequality)**.** *Let u_λ be the minimizer of Theorem* 3.1 *and put*

$$f_\lambda := \lambda u_\lambda^{-\gamma} + u_\lambda^{q-1} \in L^1_{\mathrm{loc}}(\Omega). \tag{4.3}$$

Then, for every $\varphi \in C_c^\infty(\Omega)$,

$$\left| \int_\Omega f_\lambda \varphi \, dx \right| \leq \int_\Omega |\nabla \varphi| \, dx. \tag{4.4}$$

More precisely,

$$\int_\Omega f_\lambda \varphi \, dx \le \lim_{t \downarrow 0} \frac{\mathcal{T}(u_\lambda + t\varphi) - \mathcal{T}(u_\lambda)}{t}. \tag{4.5}$$

Proof. For $|t|$ sufficiently small, $u_\lambda + t\varphi \in K_R$. Minimality gives

$$\mathcal{T}(u_\lambda + t\varphi) - \mathcal{T}(u_\lambda) \ge \frac{\lambda}{1-\gamma} \int_\Omega \Big(|u_\lambda + t\varphi|^{1-\gamma} - u_\lambda^{1-\gamma} \Big) \, dx + \frac{1}{q} \int_\Omega \Big(|u_\lambda + t\varphi|^q - u_\lambda^q \Big) \, dx.$$

Divide by $t > 0$ and pass to the limit using Lemma 4.1; this proves (4.5). Since

$$\mathcal{T}(u_\lambda + t\varphi) - \mathcal{T}(u_\lambda) \le |t| \mathcal{T}(\varphi) = |t| \int_\Omega |\nabla \varphi| \, dx,$$

for compactly supported φ, we recover

$$\int_\Omega f_\lambda \varphi \, dx \le \int_\Omega |\nabla \varphi| \, dx.$$

Apply the same estimate to $-\varphi$ to obtain (4.4). □

5. A Bounded Divergence Representation

The preceding estimate has a useful dual consequence. It does not yet produce a full 1-Laplacian calibration, but it does show that the singular source is the divergence of a bounded vector field.

Theorem 5.1 (Dual representation of the source)**.** *Under the assumptions of Theorem* 3.1*, there exists* $z_\lambda \in L^\infty(\Omega; \mathbb{R}^n)$ *such that*

$$\|z_\lambda\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \tag{5.1}$$

and

$$-\operatorname{div} z_\lambda = f_\lambda = \lambda u_\lambda^{-\gamma} + u_\lambda^{q-1}, \quad \text{in } \mathcal{D}'(\Omega). \tag{5.2}$$

Proof. Let $\mathcal{G} := \{\nabla \varphi : \varphi \in C_c^\infty(\Omega)\} \subset L^1(\Omega; \mathbb{R}^n)$. Define

$$L(\nabla \varphi) := \int_\Omega f_\lambda \varphi \, dx.$$

This is well defined: if $\nabla \varphi = \nabla \psi$, then $\varphi - \psi$ is constant on every connected component, and compact support forces that constant to be zero. Proposition 4.1 yields

$$|L(\nabla \varphi)| \le \|\nabla \varphi\|_{L^1(\Omega;\mathbb{R}^n)}.$$

By Hahn-Banach, L extends to a bounded linear functional on all of $L^1(\Omega; \mathbb{R}^n)$ with norm at most one. Since $(L^1)^* = L^\infty$, there exists $z_\lambda \in L^\infty(\Omega; \mathbb{R}^n)$, $\|z_\lambda\|_\infty \le 1$, such that

$$\int_\Omega z_\lambda \cdot \nabla \varphi \, dx = \int_\Omega f_\lambda \varphi \, dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

This is precisely (5.2). □

Remark 5.1 (What remains for a genuine 1-Laplacian solution). Theorem 5.1 gives the distributional equation and the sharp bound $\|z_\lambda\|_\infty \le 1$. A standard weak solution of (1.1), however, also requires a saturation relation of the form

$$(z_\lambda, Du_\lambda) = |Du_\lambda|$$

in the sense of the Anzellotti pairing, together with the appropriate zero-Dirichlet boundary condition expressed through the normal trace of z_λ. Neither property follows merely from the Hahn-Banach representation above. We therefore do not identify u_λ with a weak solution of (1.1) without an additional calibration argument.

6. Why the Naive Global and Symmetric Variational Arguments Fail

For completeness, we record the two obstructions which are especially relevant for the singular 1-Laplacian energy.

Proposition 6.1. *The following statements hold.*

(i) $\mathcal{E}_\lambda$ *is unbounded below on* $BV(\Omega)$ *for every* $\lambda > 0$.
(ii) *Adding a constant to* $\mathcal{E}_\lambda$ *cannot make it bounded below.*
(iii) *The map* $u \mapsto \int_\Omega |u|^{1-\gamma}\,dx$ *is not Fréchet differentiable at* $u = 0$, *hence* $\mathcal{E}_\lambda \notin C^1(BV(\Omega), \mathbb{R})$.
(iv) *A bounded sequence in a nonreflexive Banach space cannot in general be made weakly convergent by embedding it canonically into the bidual and applying Banach-Alaoglu: the weak-star limit may lie in* $X^{**} \setminus J(X)$.

Proof. Parts (i) and (ii) follow from the scaling displayed in the Introduction. For (iii), if $0 \neq \psi \in C_c^\infty(\Omega)$, then

$$\frac{1}{t}\int_\Omega |t\psi|^{1-\gamma}\,dx = t^{-\gamma}\int_\Omega |\psi|^{1-\gamma}\,dx \to +\infty, \quad t \downarrow 0,$$

so even a finite directional derivative at the origin does not exist. Part (iv) is exactly the distinction between weak compactness of the unit ball of X and weak-star compactness of the unit ball of X^{**}; surjectivity of the canonical embedding $J : X \to X^{**}$ is equivalent to reflexivity. □

Remark 6.1. These facts rule out a global Ekeland-minimization proof and also rule out a direct application of the classical C^1 symmetric mountain-pass theorem to (1.4). They do not contradict the multiplicity result of [7], which is obtained by a different $p \downarrow 1$ approximation procedure and a weak-solution framework adapted to the 1-Laplacian.

7. Concluding Remarks

The direct BV argument above proves the existence of a positive-a.e. local minimizer of the relaxed singular concave-convex energy for small λ, gives an explicit $O(\lambda^{1/\gamma})$ estimate, establishes the local integrability of the singular source, and produces a bounded field whose divergence equals that source. The construction is

entirely variational and does not use reflexivity or a fictitious smooth structure on the energy.

At the same time, the analysis identifies the exact point at which a stronger PDE conclusion requires additional work: one must prove that the representing field calibrates Du_λ and satisfies the correct boundary trace relation. This is the natural next step if one wants to recover a genuine weak solution directly from the local minimizer rather than by p-Laplacian approximation.

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References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[2] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. **135**(4) (1983), 293–318.

[3] V. De Cicco, D. Giachetti and S. Segura de León, *Elliptic problems involving the* 1*-Laplacian and a singular lower order term*, J. Lond. Math. Soc. (2) **99**(2) (2019), 349–376.

[4] S. Ghosh, D. Choudhuri and R. K. Giri, *Singular nonlocal problem involving measure data*, Bull. Braz. Math. Soc. (N.S.) **50**(1) (2019), 187–209.

[5] S. Ghosh and D. Choudhuri, *Existence of infinitely many solutions for a nonlocal elliptic PDE involving singularity*, Positivity **24**(2) (2020), 463–479.

[6] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, Vol. 80, Birkhäuser, Boston, 1984.

[7] J. C. Ortiz Chata, M. T. O. Pimenta and S. Segura de León, *Existence of solutions to a* 1*-Laplacian problem with a concave-convex nonlinearity*, J. Math. Anal. Appl. **525**(2) (2023), Article ID 127149.

[8] A. Panda, D. Choudhuri and K. Saoudi, *A critical fractional Choquard problem involving a singular nonlinearity and a Radon measure*, J. Pseudo-Differ. Oper. Appl. **12** (2021), Article ID 22. https://doi.org/10.1007/s11868-021-00382-2

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