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SOME ESTIMATES OF THE MODULUS OF CONTINUITY FOR FUNCTIONS BELONGING TO THE CLASSES H_{ω_p}, $p \in [1, +\infty]$

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Abstract. In this paper are given conditions when a function, the product of functions, and the convolution of two functions of different spaces belong to the classes H_{ω_p}, $p \in [1, +\infty]$. The results are obtained by the estimation of the modulus of continuity for the functions of different spaces belonging to the classes H_{ω_p}, $p \in [1, +\infty]$.

1. Introduction and Auxiliary Facts

The main problem in approximation theory is to determine the approximating properties of a function based on its axiomatic and constructive characteristics. Functions with the same a priori properties are grouped into classes, particularly classes of periodic functions classified by properties such as the modulus of smoothness, differentiability, upper bound, Fourier coefficients, and others. This ensures that results established for a given class apply to every function within it. This approach also allows for the formulation of new problems for entire classes of functions. In numerical methods, the given function is typically replaced by a function from these previously defined and studied classes, which are more suitable for analysis and computation [3, 5]. The modulus of continuity is a positive value that measures the continuity of a function and can be used to study the order of approximation of a function by polynomials, Fourier sums, or other methods.

In this paper, we present results on estimating the modulus of continuity for functions from various spaces belonging to the class H_{ω_p}, where $p \in [1, +\infty]$. We also

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provide estimates for the modulus of continuity of the product and convolution of two functions from this class. We will now introduce some notations and state some known facts from [2, 4, 8–10].

Denote by C the space of all 2π-periodic functions f, continuous on the entire axis with the norm $\|f\|_C = \max_x |f(x)|$.

Denote by M the space of 2π-periodic essentially bounded functions f with the norm $\|f\|_M = \operatorname{esssup}_x |f(x)|$.

We denote by L_p, $p \in [1, +\infty)$, the set of 2π-periodic functions f, such that f is measurable on $[0, 2\pi]$ for p a fixed number, $p \in [1, +\infty)$,

$$\|f\|_{L_p} = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}} < +\infty,$$

and for $p = +\infty$, $\|f\|_\infty = \|f\|_M = \operatorname{esssup}_{x\in[0,2\pi]} |f(x)| < +\infty$. In what follows we write $\|f\|_p$ instead of $\|f\|_{L_p}$.

For $p \in [1, +\infty]$ we write by $\omega_k(f, t)_p$ the modulus of smoothness of the power k, of the function $f \in L_p[0, 2\pi]$ given by

$$\omega_k(f, t)_p = \sup_{h\in[0,t]} \|\Delta_h^k f\|_p, \quad \text{where } \Delta_h^k f(x) = \sum_{m=0}^{k} (-1)^{k-m} \begin{pmatrix} k \\ m \end{pmatrix} f(x + mh).$$

We write by $\omega(f, t)_p$ instead of $\omega_1(f, t)_p$, to represent the L_p modulus of continuity of the function $f \in L_p[0, 2\pi]$. It is given by

$$\omega(f, t)_p = \begin{cases} \sup_{h\in[0,t]} \left(\int_0^{2\pi} |f(x + h) - f(x)|^p \, dx\right)^{\frac{1}{p}}, & p \in [1, +\infty), \\ \sup_{h\in[0,t]} \max_{x\in[0,2\pi]} |f(x + h) - f(x)|, & p = +\infty. \end{cases}$$

We write $\omega(f, t)$ instead of $\omega(f, t)_\infty$ and it represents the modulus of continuity of a function $f \in C[0, 2\pi]$, i.e.,

$$\begin{aligned} \omega(f, t) &= \sup_{h\in[0,t]} \max_{x\in[0,2\pi]} |f(x + h) - f(x)| \\ &= \sup \{|f(x_1) - f(x_2)|, |x_1 - x_2| \le t, x_1, x_2 \in [0, 2\pi]\}. \end{aligned}$$

We say that the function $f \in C[0, 2\pi]$ belongs to the class H_ω if its modulus of continuity satisfies the condition: $\omega(f, t) \le \omega(t)$, $t \in [0, 2\pi)$.

The class H_{ω_p} is characterized by the fact that for all functions $f \in L_p[0, 2\pi]$ holds the relation: $\omega(f, t)_p \le \omega(t)_p$, $t \in [0, 2\pi]$, i.e.,

$$\|f(x + t_1) - f(x + t_2)\|_p \le \omega_1(|t_1 - t_2|)_p = \omega(t)_p, \quad |t_1 - t_2| < t.$$

Definition 1.1 ([3])**.** Let f be an element of $L(0, 2\pi)$ and let $a_m = a_m(f)$ and $b_m = b_m(f)$, $m = 1, 2, \dots$ be the Fourier coefficients of f, i.e., the Fourier series of f is defined as follows

$$S[f] = \frac{a_0}{2} + \sum_{m=1}^{+\infty} (a_m \cos mx + b_m \sin mx).$$

The nth-order partial sum $S_n(f)$ of the series $S[f]$ is the trigonometric polynomial called the nth-order Fourier sum for the function f.

Remark 1.1 ([3]). For any p, q, for $1 < p < q < +\infty$ it is clear that $C = M = L_\infty \subset L_q \subset L_p \subset L$.

Definition 1.2 ([4]). The function f is called the convolution of two functions $\varphi, \psi \in L\left(0, 2\pi\right)$ if it can be represented in the form $f(x) = \frac{1}{\pi}\int_0^{2\pi} \varphi(x-u)\psi(u)du$ and it is denoted by $f = \varphi * \psi$.

Remark 1.2. The relation $\varphi * \psi = \psi * \varphi$ is valid.

Proposition 1.1 ([1, 2] Hölder inequality). *Let f and g be two measurable functions in interval $(0, 2\pi)$. Let $1 < p < +\infty$ and $q \in \mathbb{R}^+$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\int_0^{2\pi} |f(x)g(x)|\, dx \leq \left\{\int_0^{2\pi} |f(x)|^p\, dx\right\}^{\frac{1}{p}} \left\{\int_0^{2\pi} |g(x)|^q\, dx\right\}^{\frac{1}{q}}.$$

Proposition 1.2 ([6]). *Let $0 < p < q < +\infty$ and $a_k \geq 0$, for all $k = 1, 2, \ldots, n$. Then, the inequality*

$$\left(\sum_{k=1}^{n} a_k^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}$$

holds.

2. Main Results

In this section we will give and prove our results.

Theorem 2.1. *Let $f = \varphi * \psi$ such that $\varphi \in L_p$, $\psi \in L_q$, $1 < p < +\infty$ and $q \in \mathbb{R}^+$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\omega_1(\varphi, t)_p \leq \pi\frac{\omega_1(t)_C}{\|\psi\|_q}$, then $f \in H_\omega$.*

Proof. Using the properties of convolution, properties of modulus of continuity, Hölder inequality and the conditions in the given theorem for $|h| \leq t$, we get

$$\begin{aligned}
\omega_1(f,t)_1 &= \sup_{|h|<t} \|\Delta_h^1 \varphi * \psi\|_1 = \sup_{|h|<t} \int_0^{2\pi} |\Delta_h^1(\varphi * \psi)|dx \\
&\leq \sup_{|h|<t} \left(\int_0^{2\pi} \left(\Delta_h^1\left(\varphi * \psi\right)(x)\right)^2 dx\right)^{\frac{1}{2}} \left(\int_0^{2\pi} dx\right)^{\frac{1}{2}} \\
&\leq \sqrt{2\pi} \sup_{|h|<t} \max_{y\in\mathbb{R}} \left(\int_0^{2\pi} \left(\Delta_h^1\left(\varphi * \psi\right)(y)\right)^2 dy\right)^{\frac{1}{2}} \\
&= \sqrt{2\pi}\omega_1\left(\varphi * \psi, t\right)_C \left(\int_0^{2\pi} dy\right)^{\frac{1}{2}} = 2\pi \sup_{|h|<t} \left|\left(\varphi * \psi\right)(x+h) - \left(\varphi * \psi\right)(x)\right|
\end{aligned}$$

$$
\begin{aligned}
&=2\pi \sup_{|h|<t}\left(\frac{1}{\pi}\int_0^{2\pi}|(\varphi(x+h-u)-\varphi(x-u))\,\psi(u)|\,du\right)\\
&\leq 2\sup_{|h|<t}\left(\frac{1}{\pi}\int_0^{2\pi}|\varphi(x+h-u)-\varphi(x-u)|^p\,du\right)^{\frac{1}{p}}\left(\int_0^{2\pi}|\psi\,(u)|^q\,du\right)^{\frac{1}{q}}\\
&=2\omega_1\,(\varphi,t)_p\,\|\psi\|_q \leq 2\pi\frac{\omega_1\,(t)_C}{\|\psi\|_q}\|\psi\|_q = 2\pi\omega_1\,(t)_C\,.
\end{aligned}
$$

So, we get $\omega_1\,(\varphi*\psi,t)_C \leq \omega_1\,(t)_C$, consequently $f\in H_\omega$. ☐

Corollary 2.1. *Let $f=\varphi*\psi$ such that $\varphi\in L_p$, $\psi\in L_q$, $1<p<+\infty$ and $q\in\mathbb{R}^+$, where $\frac{1}{p}+\frac{1}{q}=1$. If $\omega_1\,(\varphi,t)_p\leq\frac{\omega_1(t)_1}{2\|\psi\|_q}$, then $f\in H_{\omega_1}$.*

The proof can be inserted from the proof of Theorem 2.1, where we get

$$
\omega_1\,(f,t)_1 \leq 2\omega_1\,(\varphi,t)_p\,\|\psi\|_q \leq 2\frac{\omega_1\,(t)_1}{2\|\psi\|_q}\|\psi\|_q \leq \omega_1\,(t)_1\,.
$$

Theorem 2.2. *Let $f=\varphi*\psi$ such that function φ or ψ belongs to the space M. If $\omega_1(\psi,t)_1\leq\pi\frac{\omega_1(t)_C}{\|\varphi\|_M}$, for $\varphi\in M$ or $\omega_1(\varphi,t)_1\leq\pi\frac{\omega_1(t)_C}{\|\psi\|_M}$ for $\psi\in M$, then $f\in H_\omega$.*

Proof. Assuming that $\varphi\in M$ and $|h|\leq t$ we get

$$
\begin{aligned}
\omega_1(f,t)_C &= \sup_{|h|<t}\|\Delta_h^1\varphi*\psi\|_C = \sup_{|h|<t}|(\varphi*\psi)\,(x+h)-(\varphi*\psi)\,(x)\,|\\
&=\sup_{|h|<t}\left(\frac{1}{\pi}\int_0^{2\pi}|\,(\psi\,(x+h-u)-\psi\,(x-u))\,\varphi\,(u)\,|du\right)\\
&\leq \sup_{|h|<t}\operatorname{esssup}_{u\in[0,2\pi]}\left(\frac{1}{\pi}\int_0^{2\pi}|\,(\psi\,(x+h-u)-\psi\,(x-u))\,\varphi\,(u)\,|du\right)\\
&=\sup_{|h|<t}\|\varphi\|_M\frac{1}{\pi}\int_0^{2\pi}|(\psi(\tau+h)-\psi\,(\tau))|\,du = \frac{1}{\pi}\|\varphi\|_M\omega_1\,(\psi,t)_1\\
&\leq\frac{1}{\pi}\|\varphi\|_M\pi\frac{\omega_1\,(t)_C}{\|\varphi\|_M} = \omega_1\,(t)_C\,.
\end{aligned}
$$

So, we get $\omega_1\,(f,t)_C\leq\omega_1\,(t)_C$, consequently $f\in H_\omega$. The case of $\psi\in M$ can be elaborated similarly. ☐

Theorem 2.3. *Let $f\in L_p$ and $\omega_1\,(f,t)_p$ be the modulus of continuity of the function f. If*

$$
\omega_1\,(f,t)_p \leq \left(\frac{\omega_1\,(t)_p}{2^{\frac{p+1}{p^2}}\pi^{\frac{1}{p^2}}\|f\|_C^{\frac{1}{p}}}\right)^{\frac{p}{p-1}},
$$

then $f\in H_{\omega_p}$.

Proof. Using the properties of convolution, properties of modulus of continuity, and Hölder inequality for $|h| \leq t$, we get

$$
\begin{aligned}
(\omega_1(f,t)_p)^p &= \sup_{|h|<t} \int_0^{2\pi} |f(x+h) - f(x)|^p dx \\
&\leq \sup_{|h|<t} \int_0^{2\pi} |f(x+h) - f(x)|^{p-1} |f(x+h) - f(x)| \, dx \\
&\leq \sup_{|h|<t} \max_{y \in \mathbb{R}} \int_0^{2\pi} |f(x+h) - f(x)|^{p-1} (|f(y+h)| + |f(y)|) \, dx \leq \\
&\leq \sup_{|h|<t} 2\|f\|_C \left(\int_0^{2\pi} \left(|f(x+h) - f(x)|^{p-1} \right)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_0^{2\pi} 1^p dx \right)^{\frac{1}{p}} \\
&= 2\|f\|_C (2\pi)^{\frac{1}{p}} \left(\omega_1(f,t)_p \right)^{p-1}.
\end{aligned}
$$

From the conditions in the stated theorem we get

$$
\omega_1(f,t)_p \leq 2^{\frac{p+1}{p^2}} \pi^{\frac{1}{p^2}} \|f\|_C^{\frac{1}{p}} \left(\omega_1(f,t)_p \right)^{\frac{p-1}{p}} \leq 2^{\frac{p+1}{p^2}} \pi^{\frac{1}{p^2}} \|f\|_C^{\frac{1}{p}} \frac{\omega_1(t)_p}{2^{\frac{p+1}{p^2}} \pi^{\frac{1}{p^2}} \|f\|_C^{\frac{1}{p}}} \leq \omega_1(t)_p .
$$

Consequently, $f \in H_{\omega_p}$. □

In the following theorem we give the generalization of results obtained in [7].

Theorem 2.4. *Let $f \in L_p$ and $g \in C$. If $\omega(g,t) \leq A \frac{\omega_1(t)_1}{\|f\|_p}$ and $\omega_1(f,t)_p \leq B \frac{\omega_1(t)_1}{\|g\|}$, where A and B are positive constants such that $A + B \leq \frac{\sqrt[p]{2\pi}}{2\pi}$, then $f, g \in H_{\omega_1}$.*

Proof. Using Hölder's inequality for $|h| \leq t$, we see that

$$
\begin{aligned}
\omega_1(fg,t)_1 &= \sup_{|h|<t} \|\Delta_h^1 fg\|_1 = \sup_{|h|<t} \int_0^{2\pi} \Big| f(x+h)\, g(x+h) - f(x)\, g(x) \Big| dx \\
&= \sup_{|h|<t} \int_0^{2\pi} \Big| f(x+h)\, g(x+h) + f(x+h)\, g(x) \\
&\quad - f(x+h)\, g(x) - f(x)\, g(x) \Big| dx \\
&\leq \sup_{|h|<t} \int_0^{2\pi} \Big(|f(x+h)| (|g(x+h) - g(x)|) + |g(x)| \cdot |f(x+h) - f(x)| \Big) dx \\
&\leq \sup_{|h|<t} \int_0^{2\pi} \Big(|f(x+h)| \cdot \Big| \Delta_h^1 g(x) \Big| \Big) dx + \sup_{|h|<t} \int_{-\pi}^{\pi} \Big(|g(x)| \cdot \Big| \Delta_h^1 f(x) \Big| \Big) dx \\
&\leq \sup_{|h|<t} \left(\int_0^{2\pi} \Big(|f(x+h)| \cdot \Big| \Delta_h^1 g(x) \Big| \Big)^p dx \right)^{\frac{1}{p}} \left(\int_0^{2\pi} 1^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&\quad + \sup_{|h|<t} \left(\int_0^{2\pi} \Big(|g(x)| \cdot \Big| \Delta_h^1 f(x) \Big| \Big)^p dx \right)^{\frac{1}{p}} \left(\int_0^{2\pi} 1^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (2\pi)^{\frac{p-1}{p}} \Bigg(\sup_{|h|<t} \max_{y\in\mathbb{R}} \left|\Delta_h^1 g\left(y\right)\right| \left(\int_0^{2\pi} \left|f\left(y\right)\right|^p dy \right)^{\frac{1}{p}} \\
&\quad + \sup_{|h|<t} \max_{y\in\mathbb{R}} \left|g\left(y\right)\right| \left(\int_0^{2\pi} \left|\Delta_h^1 f\left(x\right)\right|^p dx \right)^{\frac{1}{p}} \Bigg) \\
&= (2\pi)^{\frac{p-1}{p}} \left(\|f\|_p \,\omega\left(g,t\right) + \|g\| \,\omega_1\left(f,t\right)_p \right) \\
&\leq (2\pi)^{\frac{p-1}{p}} \left(\|f\|_p \, A \frac{\omega_1\left(t\right)_1}{\|f\|_p} + \|g\| \, B \frac{\omega_1\left(t\right)_1}{\|g\|} \right) \leq \omega_1\left(t\right)_1 .
\end{aligned}
$$

It follows that $\omega_1\left(fg,t\right)_1 \leq \omega_1\left(t\right)_1$. Consequently, $fg \in H_{\omega_1}$. □

Theorem 2.5. *Let $f \in L_p$ and $\omega_k\left(f,t\right)_p$ the modulus of smoothness of the power k of the function f. If $\omega_1\left(f,t\right)_p \leq \frac{\omega_k(t)_p}{k2^{k-1}}$, then $f \in H_{\omega_p^k}$.*

Proof. Using the properties of the smoothness of continuity, the given conditions of the stated theorem, and Proposition 1.2, for $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, and $|h|\leq t$, we get the following result

$$
\begin{aligned}
\omega_k\left(f,t\right)_p &= \sup_{h\in[0,t]} \left\|\Delta_h^k f\right\|_p = \sup_{h\in[0,t]} \left(\int_0^{2\pi} \left|\Delta_h^k f\left(x\right)\right|^p dx \right)^{\frac{1}{p}} \\
&= \sup_{h\in[0,t]} \left(\int_0^{2\pi} \left(\sum_{j=0}^{k-1} \begin{pmatrix} k-1 \\ j \end{pmatrix} \left|\Delta_h^1 f\left(x+jh\right)\right| \right)^p dx \right)^{\frac{1}{p}} \\
&\leq \sup_{h\in[0,t]} \left(\int_0^{2\pi} \left(\sum_{j=0}^{k-1} \begin{pmatrix} k-1 \\ j \end{pmatrix}^q \right)^{\frac{p}{q}} \sum_{j=0}^{k-1} \left|\Delta_h^1 f\left(x+jh\right)\right|^p dx \right)^{\frac{1}{p}} dx \\
&\leq k2^{k-1} \sup_{h\in[0,t]} \left(\int_0^{2\pi} \left|\Delta_h^1 f\left(x+jh\right)\right|^p dx \right)^{\frac{1}{p}} = k2^{k-1}\omega_1\left(f,t\right)_p \\
&\leq k2^{k-1} \frac{\omega_k\left(t\right)_p}{k2^{k-1}} \leq \omega_k\left(t\right)_p .
\end{aligned}
$$

Finally, $\omega_k\left(f,t\right)_p \leq \omega_k\left(t\right)_p$, i.e., $f \in H_{\omega_p^k}$. □

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