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DERIVATIONS OF BE-ALGEBRAS FROM HYPER BE-ALGEBRAS

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Abstract. In this paper, we investigate some results in (BE-algebras) dual BCK-algebras and hyper (BE-algebras) dual K-algebras. We show that by a set X, we can construct a hyper (BE-algebra) dual K-algebra. By concept of (BE-algebras) dual BCK-algebras and fundamental relation on hyper (BE-algebras) dual K-algebras the notion of fundamental (BE-algebras) dual BCK-algebras is introduced. We prove that any (BE-algebra) dual BCK-algebra is a fundamental (BE-algebra) dual BCK-algebra, in practical, any infinite set converts to fundamental (BE-algebra) dual BCK-algebra of itself.

1. Introduction

In 1966, Imai and Iseki [2, 3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra and investigated several properties [4].

The hyper algebraic structure theory was introduced in 1934 [5], by F. Marty at the 8th congress of Scandinavian mathematicians. Hyperstructures have many applications to several sectors of both pure and applied sciences. A. Radfar et al. introduced the notion of hyper BE–algebra, dual hyper K-algebra and investigate some properties [6]. Furthermore, they showed that under some condition hyper BE-algebras are equivalent to dual hyper K-algebras. Fundamental relations are one of the main tools in algebraic hyperstructures theory. In [1], M. Hamidi et al. introduced the concept of fundamental relation on hyper BE-algebras and hyper K-algebras.

Key words and phrases. (Hyper BE-algebra) dual hyper K-algebra, fundamental (BE-algebra) dual BCK-algebra.

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Furthermore, they showed that quotient of any dual hyper K-algebra on a regular relation is a hyper BE-algebra and this quotient on any good strongly regular relation is a dual BCK-algebra. They introduced "δ" as a relation on (hyper BE-algebras) dual hyper K-algebras and "δ^*" as a transitive closure of "δ" such that is the smallest equivalence relation that contains "δ".

In this paper, we need a relation for connecting category of (BE-algebras) dual BCK-algebras and category of (hyper BE-algebras) dual hyper K-algebras. For this, firstly we show that by any nonempty set can construct a hyper BE-algebra and dual hyper K-algebra. We introduce the notion of fundamental (BE-algebra) dual BCK-algebra via fundamental relation δ^* on (hyper BE-algebra) dual hyper K-algebra. We show that any nonempty set converts to a (BE-algebra) dual BCK-algebra such that is isomorphic to a fundamental (BE-algebra) dual BCK-algebra and especially any infinite (BE-algebra) dual BCK-algebra is isomorphic to fundamental (BE-algebra) dual BCK-algebra of itself. Moreover, we find some relations between category of (BE-algebras) dual BCK-algebras and (hyper BE-algebras) dual hyper K-algebras.

2. Preliminaries

Definition 2.1. [4, 7] An algebra $(X;*,1)$ of type $(2,0)$ is called a BE-algebra if following axioms hold:
(BE1) $x*x=1$;
(BE2) $x*1=1$;
(BE3) $1*x=x$;
(BE4) $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$.
We introduce a relation "$\leq$"on X by $x\leq y$ if and only if $x*y=1$. The BE–algebra $(X;*,1)$ is said to be commutative, if for all $x,y\in X$, $(x*y)*y=(y*x)*x$.

Definition 2.2. [7] An algebra $(X;*,1)$ of type $(2,0)$ is called a dual BCK-algebra if
(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(dBCK1) $x*y=y*x=1\Rightarrow x=y$;
(dBCK2) $(x*y)*((y*z)*(x*z))=1$;
(dBCK3) $x*((x*y)*y)=1$.

The dual BCK-algebra $(X;*,1)$ is said to be commutative, if for all $x,y\in X$, $(x*y)*y=(y*x)*x$.

Theorem 2.1. [7] *By every nonempty set, can construct a commutative dual BCK-algebra.*

Definition 2.3. [1, 6] Let H be a nonempty set and $\circ : H\times H\to P^*(H)$ be a hyperoperation. Then $(H;\circ,1)$ is called a hyper BE–algebra, if it satisfies the following axioms:
(HBE_1) $x<1$ and $x<x$;

(HBE_2) $x \circ (y \circ z) = y \circ (x \circ z)$;
(HBE_3) $x \in 1 \circ x$;
(HBE_4) $1 < x$ implies $x = 1$, for all $x, y, z \in H$.

$(H; \circ, 1)$ is called a dual hyper K-algebra if satisfies (HBE_1), (HBE_2) and the following axioms:
(DHK_1) $x \circ y < (y \circ z) \circ (x \circ z)$;
(DHK_4) $x < y$ and $y < x$ imply that $x = y$, for all $x, y, z \in H$,
where the relation "$<$" is defined by $x < y$ if and only if $1 \in x \circ y$. For any two nonempty subsets A and B of H, we define $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$ and $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$. The dual hyper K-algebra $(X; \circ, 1)$ is called a weak commutative dual hyper K-algebra, if for any $x, y \in X$, $((x \circ y) \circ y) \bigcap ((y \circ x) \circ x) \neq \emptyset$.

Theorem 2.2. [6] *Let H be a hyper BE-algebra. Then*
(i) $A \circ (B \circ C) = B \circ (A \circ C)$*;*
(ii) $A < A$*;*
(iii) $1 < A$ *implies* $1 \in A$*;*
(iv) $x < y \circ x$*;*
(v) $x < y \circ z$ *implies* $y < x \circ z$*;*
(vi) $x < (x \circ y) \circ y$*;*
(vii) $z \in x \circ y$ *implies* $x < z \circ y$*;*
(viii) $y \in 1 \circ x$ *implies* $y < x$*, for all* $x, y, z \in H$ *and* $A, B, C \subseteq H$*.*

Theorem 2.3. [1] *Let $(X; \circ, 1)$ be a (hyper BE-algebra) dual hyper K-algebra and R be an equivalence relation on X. Then, R is a regular relation on X if and only if $(X/R; *, \overline{1})$ is a hyper BE-algebra, where for any $\overline{x}, \overline{y} \in X/R$*

$$\overline{x} * \overline{y} = \{\overline{z} | z \in x \circ y\}$$

and a binary relation "$<$" on X/R by

$$\overline{x} < \overline{y} \Leftrightarrow \overline{1} \in \overline{x} * \overline{y}.$$

Theorem 2.4. [1] *Let $(X; \circ, 1)$ be a (hyper BE-algebra) dual hyper K-algebra and R be an equivalence relation on X. If R is a strongly regular relation on X, then $(X/R; *, \overline{1})$ is a BE-algebra.*

3. Constructing of (BE-algebras) dual BCK-algebras and (hyper BE-algebras) dual hyper K-algebras

In this section, we construct a commutative (BE-algebra) dual BCK-algebra and a weak commutative (hyper BE-algebra) dual hyper K-algebra from arbitrary set. We show that the sets with the same cardinal number convert to isomorphic BE-algebras, dual BCK-algebras, hyper BE-algebras and dual hyper K-algebras.

Theorem 3.1. *Let* $(X, *, 1)$ *be a* $(BE$*-algebra) dual* BCK*-algebra. If* Y *is a set and* $|X| = |Y|$*, then there exists a binary operation "*$*'$*" and* $1'$ *on* Y*, such that* $(Y, *', 1')$ *is a* $(BE$*-algebra) dual* BCK*-algebra and* $(X, *, 1) \cong (Y, *', 1')$*.*

Proof. Since $|X| = |Y|$, then there exists a bijection $\varphi : X \to Y$. For any $y_1, y_2 \in Y$, we define a binary operation "$*'$" on Y as follows:

$$y_1 *' y_2 = \varphi(x_1 * x_2),$$

where $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and $x_1, x_2 \in X$. It is easy to show that $*'$ is well-defined. Let $y_1 = y'_1$ and $y_2 = y'_2$. Since φ is a bijection, then there exist unique elements $x_1, x_2, x'_1, x'_2 \in X$ such that $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$, $y'_1 = \varphi(x'_1)$, $y'_2 = \varphi(x'_2)$ and so $x_1 = x'_1$, $x_2 = x'_2$. Hence, $y_1 *' y_2 = \varphi(x_1 * x_2) = \varphi(x'_1 * x'_2) = y'_1 *' y'_2$. We can show that $(Y, *', 1')$ is a $(BE$-algebra) dual BCK-algebra. Let $x_1, x_2 \in X$. Then

$$\varphi(x_1 * x_2) = \varphi(x_1) *' \varphi(x_2)$$

and $\varphi(1) = \varphi(x * x) = \varphi(x) *' \varphi(x) = 1'$. Therefore, φ is homomorphism and then is isomorphism. □

Theorem 3.2. *Let* $(X, \circ, 1)$ *be a (hyper* BE*-algebra) dual hyper* K*-algebra. If* Y *is a set and* $|X| = |Y|$*, then there exists a binary hyperoperation "*$\circ'$*" and* $1'$ *on* Y*, such that* $(Y, \circ', 1')$ *is a (hyper* BE*-algebra) dual hyper* K*-algebra and* $(X, \circ, 1) \simeq (Y, \circ', 1')$*.*

Proof. Since $|X| = |Y|$, then there exists a bijection $\varphi : X \to Y$. For any $y_1, y_2 \in Y$, we define a binary hyperoperation "$\circ'$" on Y as follows:

$$y_1 \circ' y_2 = \varphi(x_1 \circ x_2),$$

where $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and $x_1, x_2 \in X$. It is easy to show that $\circ'$ is well-defined. Now, we define $1' = \varphi(1)$. We can show that $(Y, \circ', 1')$ is a (hyper BE-algebra) dual hyper K-algebra.

Let $x_1, x_2 \in X$. Then

$$\varphi(x_1 \circ x_2) = \varphi(x_1) \circ' \varphi(x_2).$$

Therefore, φ is homomorphism and then is isomorphism. □

Theorem 3.3. *Let* $(X, \star, 1)$ *and* $(Y, \star', 1')$ *be dual* BCK*-algebras. Then there exists a binary hyperoperation "*$\circ$*" and "*$1''$*" on* $X \times Y$*, such that* $(X \times Y, \circ, 1'')$ *is a dual hyper* K*-algebra.*

Proof. Let $(X, \star, 1)$ and $(Y, \star', 1')$ be two dual BCK-algebras. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, we define the binary hyperoperation "$\circ$" on $X \times Y$ as follows:

$$(x_1, y_1) \circ (x_2, y_2) = \{(x_1 \star x_2, y_2), (x_1 \star x_2, y_1 \star' y_2)\}$$

and $1'' = (1, 1')$. First, we show that the hyperoperation "$\circ$" is well defined.

Let $(x_1, y_1) = (x'_1, y'_1)$ and $(x_2, y_2) = (x'_2, y'_2)$. Then,

$$\begin{aligned}(x_1, y_1) \circ (x_2, y_2) =& \{(x_1 \star x_2, y_2), (x_1 \star x_2, y_1 \star' y_2)\} \\ =& \{(x'_1 \star x'_2, y'_2), (x'_1 \star x'_2, y'_1 \star' y'_2)\}\end{aligned} \tag{3.1}$$

$$=(x_1', y_1') \circ (x_2', y_2').$$

Now, we define a binary relation "$<''$" on $X \times Y$ as follows:

$$(x, y) <'' (z, w) \text{ if and only if } (1, 1') \in (x, y) \circ (z, w).$$

We show that for any $(x, y), (z, w) \in X \times Y$,

$$(x, y) <'' (z, w) \text{ if and only if } \; x < z \; \text{ and } \; y <' w. \tag{3.2}$$

For this, let $(x, y) <'' (z, w)$. Then $(1, 1') \in (x, y) \circ (z, w) = \{(x \star z, w), (x \star z, y \star' w)\}$ and so $(1, 1') = (x \star z, w)$ or $(1, 1') = (x \star z, y \star' w)$. If $(1, 1') = (x \star z, w)$, then $x < z$ and $y <' w = 1'$. If $(1, 1') = (x \star z, y \star' w)$, then $x < z$ and $y <' w$. Therefore, in any cases, we have, $x < z$ and $y <' w$.

Conversely, let $x < z$ and $y <' w$. Then $x \star z = 1$ and $y \star' w = 1'$. Hence $(1, 1') \in (x, y) \circ (z, w)$. Therefore, $(x, y) <'' (z, w)$.

Now, we will show that $(X \times Y, \circ, 1'')$ is a dual hyper K-algebra. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Then we have the following.

(HBE1) By (3.2), $(x_1, y_1) <'' (x_1, y_1)$ and $(x_1, y_1) <'' (1, 1')$.

(HBE2)

$$\begin{aligned}
&(x_1, y_1) \circ ((x_2, y_2)) \circ (x_3, y_3)) \\
=&(x_1, y_1) \circ \{(x_2 \star x_3, y_3), (x_2 \star x_3, y_2 \star' y_3)\} \\
=&\{(x_1 \star (x_2 \star x_3), y_3), (x_1 \star (x_2 \star x_3), y_1 \star' y_3), (x_1 \star (x_2 \star x_3), y_2 \star' y_3), \\
&(x_1 \star (x_2 \star x_3), y_1 \star' (y_2 \star' y_3))\} \\
=&\{(x_2 \star (x_1 \star x_3), y_3), (x_2 \star (x_1 \star x_3), y_1 \star' y_3), (x_2 \star (x_1 \star x_3), y_2 \star' y_3), \\
&(x_2 \star (x_1 \star x_3), y_2 \star' (y_1 \star' y_3))\} \\
=&(x_2, y_2) \circ \{(x_1 \star x_3, y_3), (x_1 \star x_3, y_1 \star' y_3)\} \\
=&(x_2, y_2) \circ ((x_1, y_1) \circ (x_3, y_3)).
\end{aligned}$$

(HBE3) $(1, 1') \circ (x, y) = \{(1 \star x, y), (1 \star x, 1' \star' y)\} = \{(x, y)\}$.

(HBE4) If $(1, 1') <'' (x, y)$, then by (3.2), $(x, y) = (1, 1')$.

(DHK3)

$$\begin{aligned}
&((x_2, y_2) \circ (x_3, y_3)) \circ ((x_1, y_1) \circ (x_3, y_3)) \\
=&\{(x_2 \star x_3, y_3), (x_2 \star x_3, y_2 \star' y_3)\} \circ \{(x_1 \star x_3, y_3), (x_1 \star x_3, y_1 \star' y_3)\} \\
=&\{((x_2 \star x_3) \star (x_1 \star x_3), y_3), ((x_2 \star x_3) \star (x_1 \star x_3), y_3 \star' y_3), \\
&((x_2 \star x_3) \star (x_1 \star x_3), y_1 \star' y_3)), ((x_2 \star x_3) \star (x_1 \star x_3), y_3 \star' (y_1 \star' y_3))), \\
&((x_2 \star x_3) \star (x_1 \star x_3), (y_2 \star' y_3) \star' y_3)), ((x_2 \star x_3) \star (x_1 \star x_3), (y_2 \star' y_3) \star' (y_1 \star' y_3)\}.
\end{aligned}$$

Since $(x_1, y_1) \circ (x_2, y_2) = \{(x_1 \star x_2, y_2), (x_1 \star x_2, y_1 \star' y_2)\}$, we get

$$(x_1, y_1) \circ (x_2, y_2) <'' ((x_2, y_2) \circ (x_3, y_3)) \circ ((x_1, y_1) \circ (x_3, y_3)).$$

(DHK4) Let $(x_1, y_1) <'' (x_2, y_2)$ and $(x_2, y_2) <'' (x_1, y_1)$. Then by (3.2), $(x_1, y_1) = (x_2, y_2)$. Therefore, $(X \times Y, \circ, 1'')$ is a dual hyper K-algebra. $\square$

Corollary 3.1. *Let $(X, \star, 1)$ and $(Y, \star', 1')$ be BE-algebras. Then there exists a binary hyperoperation "$\circ$" and "$1''$" on $X \times Y$, such that $(X \times Y, \circ, 1'')$ is a hyper BE-algebra.*

Example 3.1. Consider the dual BCK-algebras $(\{1,2,3,4\}, \star, 1)$ and $(\{a,b\}, \star', a)$ as follows:

$\star$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	2	3	1

,

$\star'$	a	b
a	a	b
b	a	a

.

Now for any x, y, w we set $w_x = (w, x)$, $\overline{w}_x = \{(w, x)\}$, $\overline{w}_{x,y} = \{(w, x), (w, y)\}$ and define a hyperoperation "$\circ$" on $\{1,2,3,4\} \times \{a,b\}$ as follows:

$\circ$	1_a	1_b	2_a	2_b	3_a	3_b	4_a	4_b
1_a	$\overline{1}_a$	$\overline{1}_b$	$\overline{2}_a$	$\overline{2}_b$	$\overline{3}_a$	$\overline{3}_b$	$\overline{4}_a$	$\overline{4}_b$
1_b	$\overline{1}_a$	$\overline{1}_{a,b}$	$\overline{2}_a$	$\overline{2}_{a,b}$	$\overline{3}_a$	$\overline{3}_{a,b}$	$\overline{4}_a$	$\overline{4}_{a,b}$
2_a	$\overline{1}_a$	$\overline{1}_b$	$\overline{1}_a$	$\overline{1}_b$	$\overline{3}_a$	$\overline{3}_b$	$\overline{4}_a$	$\overline{4}_b$
2_b	$\overline{1}_a$	$\overline{1}_{a,b}$	$\overline{1}_a$	$\overline{1}_{a,b}$	$\overline{3}_a$	$\overline{3}_{a,b}$	$\overline{4}_a$	$\overline{4}_{a,b}$
3_a	$\overline{1}_a$	$\overline{1}_b$	$\overline{2}_a$	$\overline{2}_b$	$\overline{1}_a$	$\overline{1}_b$	$\overline{4}_a$	$\overline{4}_b$
3_b	$\overline{1}_a$	$\overline{1}_{a,b}$	$\overline{2}_a$	$\overline{2}_{a,b}$	$\overline{1}_a$	$\overline{1}_{a,b}$	$\overline{4}_a$	$\overline{4}_{a,b}$
4_a	$\overline{1}_a$	$\overline{1}_b$	$\overline{2}_a$	$\overline{2}_b$	$\overline{3}_a$	$\overline{3}_b$	$\overline{1}_a$	$\overline{1}_b$
4_b	$\overline{1}_a$	$\overline{1}_{a,b}$	$\overline{2}_a$	$\overline{2}_{a,b}$	$\overline{3}_a$	$\overline{3}_{a,b}$	$\overline{1}_a$	$\overline{1}_{a,b}$

.

Then it is easy to see that $(\{1,2,3,4\} \times \{a,b\}, \circ, 1_a)$ is a dual hyper K-algebra.

Corollary 3.2. *Let $(X, \star, 1)$ be a (BE-algebra) dual BCK-algebra. Then for any arbitrary distinct elements a, b:*
(i) *there exists a binary hyperoperation "$\circ$", and constant "$1'$" on $X \times \{a,b\}$, such that $(X \times \{a,b\}, \circ, 1')$ is a (hyper BE-algebra) dual hyper K-algebra;*
(ii) *if X is infinite, then there exists a binary hyperoperation "$\circ$", a constant "1" on X, $\circ'$ and $1'$ on $X \times \{a,b\}$ such that $(X, \circ, 1)$ and $(X \times \{a,b\}, \circ', 1')$ are (hyper BE-algebra) dual hyper K-algebras and $(X \times \{a,b\}, \circ', 1') \cong (X, \circ, 1)$.*

Theorem 3.4. *By every nonempty set, we can construct a weak commutative dual hyper K-algebra.*

Proof. Let X be an arbitrary nonempty set. If $|X| \leq 2$, is clear. But if $|X| > 2$, for any $x, y \in X$, we define a binary operation "$\circ$" on X as follows:

$$x \circ y = \begin{cases} \{x_0, p\}, & \text{if } x = y, \\ \{y\}, & \text{otherwise,} \end{cases}$$

and

$$x < y \text{ if and only if } x_0 \in x \circ y,$$

where x_0 and p are fixed elements of X. Now, we show that $(X, \circ, x_0)$ is a dual hyper K-algebra.

(HBE1) Let $x \in X$. Then by definition of $\circ$, $x_0 \in x \circ x$ and $x_0 \in x \circ x_0$ and so $x < x_0$ and $x < x$.
(HBE2) Let $x, y, z \in X$. We consider the following cases.
Case 1: x, y, z are distinct. Then, $x \circ (y \circ z) = x \circ \{z\} = \{z\} = y \circ \{z\} = y \circ (x \circ z)$.
Case 2: $x = y \neq z$. Then, $x \circ (y \circ z) = x \circ \{z\} = \{z\} = y \circ \{z\} = y \circ (x \circ z)$.
Case 3: $x = z \neq y$. Then, $x \circ (y \circ z) = x \circ \{z\} = \{x_0, p\} = y \circ \{x_0, p\} = y \circ (x \circ z)$.
Case 4: $y = z \neq x$. Then, $x \circ (y \circ z) = x \circ \{x_0, p\} = \{x_0, p\} = y \circ \{z\} = y \circ (x \circ z)$.
Case 5: $x = y = z$. Then, $x \circ (y \circ z) = x \circ \{x_0, p\} = \{x_0, p\} = y \circ \{x_0, p\} = y \circ (x \circ z)$.
(DHK3) Let $x, y, z \in X$. We have the following cases.
Case 1: x, y, z are distinct. Then, $x \circ y = \{y\} < \{x_0, p\} = \{z\} \circ \{z\} = (y \circ z) \circ (x \circ z)$.
Case 2: $x = y \neq z$. Then, $x \circ y = \{x_0, p\} < \{x_0, p\} = \{z\} \circ \{z\} = (y \circ z) \circ (x \circ z)$.
Case 3: $x = z \neq y$. Then, $x \circ y = \{y\} < \{x_0, p\} = \{z\} \circ \{x_0, p\} = (y \circ z) \circ (x \circ z)$.
Case 4: $y = z \neq x$. Then, $x \circ y = \{y\} < \{z\} = \{x_0, p\} \circ \{z\} = (y \circ z) \circ (x \circ z)$.
Case 5: $x = y = z$. Then, $x \circ y = \{x_0, p\} < \{x_0, p\} = \{x_0, p\} \circ \{x_0, p\} = (y \circ z) \circ (x \circ z)$.
(DHK4) Let $x, y \in X$. Then by definition of $\circ$, $x_0 \in x \circ y = x \circ y$ implies that $x = y$. Moreover, for any $x, y \in X, x_0 \in (x \circ y) \circ y \cap (y \circ x) \circ x \neq \emptyset$.

Therefore, $(X, \circ, x_0)$ is a weak commutative dual K-algebra. $\square$

Example 3.2. Let $X = \{1, a, b, c, d, e\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b	c	d	e
1	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
a	$\{1,e\}$	$\{1,e\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
b	$\{1,e\}$	$\{a\}$	$\{1,e\}$	$\{c\}$	$\{d\}$	$\{e\}$
c	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{1,e\}$	$\{d\}$	$\{e\}$
d	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1,e\}$	$\{e\}$
e	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{1,e\}$

.

Then $(X; \circ, 1)$ is a weak commutative dual hyper K–algebra.

Corollary 3.3. *By every nonempty set, we can construct a weak commutative hyper BE-algebra.*

4. Fundamental (BE-algebras) dual BCK-algebras

In this section, by using the notion of fundamental relation, we define the concept of fundamental (BE-algebra) dual BCK-algebra and we prove that any (BE-algebra) dual BCK-algebra is a fundamental (BE-algebra) dual BCK-algebra.

Now, in the following, we apply the concept of δ^* relation on (hyper BE–algebra) dual hyper K–algebra which is studied in [1]. Let $(X; \circ, 1)$ be a (hyper BE-algebra) dual hyper K-algebra and A be a subset of X. $\mathcal{L}(A)$ will denote the set of all finite combinations of elements A with $\circ$ and $\bigodot\limits_{i=1}^{n} a_i = a_1 \circ a_2 \circ \ldots a_n$.

Definition 4.1. Let $(X;\circ,1)$ be a (hyper BE–algebra) dual hyper K–algebra. Consider

$$\delta_1=\{(x,x)|x\in X\}$$

and for every integer $n\geq 1$, δ_n, define as follows:

$$x\delta_n y\Leftrightarrow \exists(a_1,a_2,\dots,a_n)\in X^n,\exists u\in\mathcal{L}(a_1,a_2,\dots,a_n)\ \{x,y\}\subseteq u.$$

Obviously, for every $n\geq 1$, the relations δ_n are symmetric, the relation $\delta=\bigcup\limits_{n\geq 1}\delta_n$ is a reflexive and symmetric relation. Let δ^* be the transitive closure of δ (the smallest transitive relation such that contains δ).

Example 4.1. Let $X=\{1,a,b,c\}$. Then $(X,\circ,1)$ is a weak dual hyper K-algebra as follows:

$\circ$	1	a	b	c
1	$\{1,a\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1,a\}$	$\{1,a\}$	$\{b\}$	$\{c\}$
b	$\{1,a\}$	$\{a\}$	$\{1,a\}$	$\{c\}$
c	$\{1,a\}$	$\{a\}$	$\{b\}$	$\{1,a\}$

.

Now, $\delta=\{(1,1),(a,a),(b,b),(c,c),(1,a),(a,1)\}$ and clearly $\delta^*=\delta$.

Theorem 4.1. [1] *Let $(X;\circ,1)$ be a (hyper BE-algebra) weak commutative dual hyper K-algebra. Then δ^* is a (strongly regular) good strongly regular on X.*

Theorem 4.2. [1] *Let $(X;\circ,1)$ be a (hyper BE-algebra) weak commutative dual hyper K-algebra. Then $(X/\delta^*;*,\overline{1})$ is a (BE-algebra) commutative dual BCK-algebra.*

Lemma 4.1. *Let $(X,\circ_X,1_X)$ and $(Y,\circ_Y,1_Y)$ be (hyper BE-algebras) dual hyper K-algebras. Then*
(i) *there exists a hyperoperation "$\circ_{X\times Y}$" and "$1_{X\times Y}$" on $X\times Y$ such that $(X\times Y,\circ_{X\times Y}$, $1_{X\times Y})$ is a (hyper BE-algebras) dual hyper K-algebra;*
(ii) *for any $(x,y),(x',y')\in X\times Y$, we have $(x,y)\delta^*_{X\times Y}(x',y')$ if and only if $x\ \delta^*_X\ x'$ and $y\ \delta^*_Y\ y'$.*

Proof. (i) For any $(x,y),(x',y')\in X\times Y$, define $\circ_{X\times Y}$ on $X\times Y$ as follows:

$$(x,y)\circ_{X\times Y}(x',y')=(x\circ_X x')\times(y\circ_Y y')$$

and $1_{X\times Y}=(1_X,1_Y)$. It is easy to verify that $(X\times Y,\circ_{X\times Y},1_{X\times Y})$ is a (hyper BE-algebra) dual hyper K-algebra.
(ii) Let $(x,x')\in X^2$ and $(y,y')\in Y^2$. Then $x\delta^*_X x'$ and $y\delta^*_Y y'$ if and only if there exist $u\in\mathcal{L}(X)$ and $v\in\mathcal{L}(Y)$ such that $\{x,x'\}\subseteq u$ and $\{y,y'\}\subseteq v$, if and only if $\{(x,y),(x',y')\}\in u\times v$ if and only if $(x,y)\delta^*_{X\times Y}(x',y')$. $\square$

Example 4.2. Let $X=\{1,2,3\}$ and $Y=\{a,b\}$. Then $(X;\circ_1,1)$ and $(Y;\circ_2,1)$ are dual hyper K-algebras by the following tables:

$\circ_1$	1	2	3
1	$\{1,3\}$	$\{2\}$	$\{3\}$
2	$\{1,3\}$	$\{1,3\}$	$\{3\}$
3	$\{1,3\}$	$\{2\}$	$\{1,3\}$

,

$\circ_2$	a	b
a	$\{a,b\}$	$\{b\}$
b	$\{a,b\}$	$\{a,b\}$

.

Now we define a hyperoperation "$\circ$" on $\{1,2,3\}\times\{a,b\}$ as follows:

$\circ$	$(1,a)$	$(1,b)$	$(2,a)$	$(2,b)$	$(3,a)$	$(3,b)$
$(1,a)$	$A\times B$	$A\times\{b\}$	$\{2\}\times B$	$\{(2,b)\}$	$\{3\}\times B$	$\{(3,b)\}$
$(1,b)$	$A\times B$	$A\times\{a,b\}$	$\{2\}\times B$	$\{2\}\times B$	$\{3\}\times B$	$\{3\}\times B$
$(2,a)$	$A\times B$	$A\times\{b\}$	$A\times B$	$A\times\{b\}$	$\{3\}\times B$	$\{(3,b)\}$
$(2,b)$	$A\times B$	$A\times B$	$A\times B$	$A\times B$	$\{3\}\times B$	$\{3\}\times B$
$(3,a)$	$A\times B$	$A\times\{b\}$	$\{2\}\times B$	$\{(2,b)\}$	$A\times B$	$A\times\{b\}$
$(3,b)$	$A\times B$	$A\times B$	$\{2\}\times B$	$\{2\}\times B$	$A\times B$	$A\times B$

,

where $A=\{1,3\}$ and $B=\{a,b\}$. Clearly $(\{1,2,3\}\times\{a,b\},\circ,(1,a))$ is a dual hyper K-algebra.

Theorem 4.3. *Let* $(X,\circ_X,1_X)$ *and* $(Y,\circ_Y,1_Y)$ *be* (*hyper* BE*-algebras*) *dual hyper* K*-algebras. Then*

$$\frac{(X\times Y,\circ_{X\times Y},1_{X\times Y})}{\delta^*_{X\times Y}}\cong\frac{(X,\circ_X,1_X)}{\delta^*_X}\times\frac{(Y,\circ_Y,1_Y)}{\delta^*_Y}.$$

Proof. Define the mapping:

$$\varphi:\left(\frac{(X\times Y,\circ_{X\times Y})}{\delta^*_{X\times Y}},*\right)\to\left(\frac{(X,\circ_X)}{\delta^*_X}\times\frac{(K,\circ_Y)}{\delta^*_Y},*\right),$$

by $\varphi(\delta^*_{X\times Y}(x,y))=(\delta^*_X(x),\delta^*_Y(y))$. Firstly, we show that φ is well defined and one to one. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then, by Lemma 4.1, $\delta^*_{X\times Y}(x_1,y_1)=\delta^*_{X\times Y}(x_2,y_2)$ if and only if $\delta^*_X(x_1)=\delta^*_X(x_2)$ and $\delta^*_Y(y_1)=\delta^*_Y(y_2)$ if and only if $\varphi(\delta^*_{X\times Y}(x_1,y_1))=\varphi(\delta^*_{X\times Y}(x_2,y_2))$. Secondly, we show that φ is homomorphism. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then, by Lemma 4.1, for any $c\in\delta^*_X(x_1)\circ_X\delta^*_X(x_2)$ and $d\in\delta^*_Y(y_1)\circ_Y\delta^*_Y(y_2)$ we have

$$\begin{aligned}\varphi(\delta^*_{X\times Y}(x_1,y_1)*(\delta^*_{X\times Y}(x_2,y_2))=&\varphi(\delta^*_{X\times Y}(c,d))=(\delta^*_X(c),\delta^*_Y(d))\\=&(\delta^*_X(x_1),\delta^*_Y(y_1))*(\delta^*_X(x_2),\delta^*_Y(y_2))\\=&\varphi(\delta^*_{X\times Y}(x_1,y_1))*\varphi(\delta^*_{X\times Y}(x_2,y_2))\end{aligned}$$

and $\varphi(\delta^*_{X\times Y}(1_X,1_Y))=(\delta^*_X(1_X),\delta^*_Y(1_Y))$. Clearly, φ is a bijection and hence, φ is an isomorphism. $\square$

Theorem 4.4. *Let* $(X,\circ,1)$, $(Y,\circ',1')$ *be* (*hyper* BE*-algebras*) *dual hyper* K*-algebras and* $f:(X,\circ,1)\to(Y,\circ',1')$ *be a homomorphism. Then the following statements are satisfied:*

(i) *for any* $x,y\in X$, $x\delta^*y$ *implies that* $f(x)\delta^*f(y)$*;*

(ii) *if f is an injective, then for any $x, y \in X$, $f(x)\delta^* f(y)$ implies that $x\delta^* y$;*
(iii) *if f is a bijection, then for any $x, y \in X$, $x\delta^* y$ if and only if $f(x)\delta^* f(y)$;*
(iv) *if f is a bijection, then for any $x \in X$, $f(\delta^*(x)) = \delta^*(f(x))$.*

Proof. (i) Let $x, y \in X$. Since $x\delta^* y$, then there exists $u \in \mathcal{L}(X)$, such that $\{x, y\} \subseteq u$. Now, for a homomorphism $f : (X, \circ, 1) \to (Y, \circ', 1')$ we have $\{f(x), f(y)\} = f\{x, y\} \subseteq f(u) \in \mathcal{L}(Y)$. Therefore, $f(x)\delta^* f(y)$.
(ii) For $x, y \in X$, since $f(x)\delta^* f(y)$, there exists $v \in \mathcal{L}(Y)$, such that $\{f(x), f(y)\} \subseteq v$. Since $f : (X, \circ, 1) \to (Y, \circ', 1')$ is injective, we get,

$$\{x, y\} = \{f^{-1}(f(x)), f^{-1}(f(y))\} = f^{-1}\{f(x), f(y)\} \subseteq f^{-1}(v) \in \mathcal{L}(X).$$

Therefore, $x\delta^* y$.
(iii) By (i) and (ii), the proof is straightforward.
(iv) Let $x \in X$. Then we have $f(\delta^*(x)) = \bigcup\limits_{y\in\delta^*(x)} f(y) = \bigcup\limits_{y\delta^* x} f(y)$. By (iii), for any $x, y \in X$, $x\delta^* y$ if and only if $f(x)\delta^* f(y)$. Therefore,

$$\begin{aligned} f(\delta^*(x)) &= \bigcup_{y\in\delta^*(x)} f(y) = \bigcup_{y\delta^* x} f(y) \\ &= \bigcup_{f(y)\delta^* f(x)} f(y) = \bigcup_{f(y)\in\delta^*(f(x))} f(y) \\ &= \delta^*(f(x)). \end{aligned}$$

□

Example 4.3. Let $X = \{1, a, b, c, d\}$ and $Y = \{x, y, z, t\}$. Then $(X; \circ_1, 1)$ and $(Y; \circ_2, 1)$ are hyper BE-algebras by the following tables:

$\circ_1$	1	a	b	c	d
1	$\{1, d\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
a	$\{1, d\}$	$\{1, d\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{1, d\}$	$\{a\}$	$\{1, d\}$	$\{c\}$	$\{d\}$
c	$\{1, d\}$	$\{a\}$	$\{b\}$	$\{1, d\}$	$\{d\}$
d	$\{1, d\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1, d\}$

,

$\circ_2$	x	a	b	c
x	$\{x\}$	$\{y\}$	$\{z\}$	$\{t\}$
y	$\{x\}$	$\{x\}$	$\{z\}$	$\{t\}$
z	$\{x\}$	$\{y\}$	$\{x\}$	$\{t\}$
t	$\{x\}$	$\{y\}$	$\{z\}$	$\{x\}$

.

Now, we define a map $f : (X; \circ_1, 1) \to (Y; \circ_2, x)$ by $f(c) = z$, $f(d) = t$ and $f(1) = f(a) = f(b) = x$. Clearly f is a homomorphism, but is not injective and $f(b) \in \delta^*(f(1))$, but $b \notin \delta^*(1)$.

Theorem 4.5. *Let $(X, \circ, 1)$ be a (hyper BE-algebra) dual hyper K-algebra. If Y is a set and $|X| = |Y|$, then there exists a binary hyperoperation "$\circ'$" "$1'$" on Y, such that $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$.*

Proof. Since $|X| = |Y|$, then by Theorem 3.2, there exists a binary hyperoperation "$\circ'$", and $1'$ on Y such that $(Y, \circ', 1')$ is a (hyper BE-algebras) dual hyper K-algebra. Moreover, there exists an isomorphism $f : (X, \circ, 1) \to (Y, \circ', 1')$, such that $f(1) = 1'$. Now, we define the map $\varphi : \left(\frac{(X,\circ,1)}{\delta^*}, *\right) \to \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$ by $\varphi(\delta^*(x)) = \delta^*(f(x))$. First, we show that for any $x_1, x_2 \in X$, $\varphi(\delta^*(x_1) * \delta^*(x_2)) = \varphi(\delta^*(x_1)) * \varphi(\delta^*(x_2))$. For any $x_1, x_2 \in X$ we have

$$\begin{aligned}\varphi(\delta^*(x_1) * \delta^*(x_2)) &= \varphi(\delta^*(x_1 \circ x_2)) = \delta^*(f(x_1 \circ x_2)) \\ &= \delta^*(f(x_1) \circ' f(x_2)) = \delta^*(f(x_1)) * \delta^*(f(x_2)) \\ &= \varphi(\delta^*(x_1)) * \varphi(\delta^*(x_2)). \qquad (4.1)\end{aligned}$$

Since f is bijection, then φ is a bijection. Now, we show that φ is well-defined. Let $y_1, y_2 \in Y$. Then there exist unique elements $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now, by Equation (4.1) and Theorem 4.4, $\varphi(\delta^*(x_1)) = \varphi(\delta^*(x_2))$ if and only if $\delta^*(f(x_1)) = \delta^*(f(x_2))$ if and only if $\delta^*(x_1) = \delta^*(x_2)$. Therefore, φ is well-defined and one to one and by Equation (4.1), is a homomorphism. Hence φ is an isomorphism. Therefore, $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$. $\square$

Corollary 4.1. *Let* $(X, \circ, 1)$ *and* $(Y, \circ', 1')$ *be isomorphic* (*hyper* BE*-algebras*) *dual hyper* K*-algebra. Then* $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$.

Definition 4.2. A (BE-algebra) dual BCK-algebra $(X, \star, 1)$, is called a (*fundamental* BE*-algebra*) *fundamental dual* BCK*-algebra, if there exists a nontrivial* (*hyper* BE*-algebra*) *dual hyper* K*-algebra* $(Y, \circ', 1')$, *such that* $\left(\frac{(Y,\circ',1')}{\delta^*}, *\right) \cong (X, \star, 1)$. *In other words, it is equal to the fundamental of nontrivial* (*hyper* BE*-algebra*) *dual hyper* K*-algebra up to isomorphic.*

Example 4.4. Let $X = \{1, 2, 3\}$. Then $(X, \star, 1)$ is a BE-algebra as follows:

$\star$	1	2	3
1	1	2	3
2	1	1	3
3	1	2	1

We construct a dual hyper K-algebra $(Y = \{a, b, c, d\}, \circ, a)$ as follows:

$\circ$	a	b	c	d
a	$\{a, b\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$
c	$\{b, a\}$	$\{b\}$	$\{a, b\}$	$\{d\}$
d	$\{a, b\}$	$\{b\}$	$\{c\}$	$\{a, b\}$

Clearly, $\delta^*(a) = \{a, b\}$, $\delta^*(c) = \{c\}$ and $\delta^*(d) = \{d\}$ and so $\frac{Y}{\delta^*} = \{\{a, b\}, \{c\}, \{d\}\}$. Now, we have

$*$	$\delta^*(a)$	$\delta^*(c)$	$\delta^*(d)$
$\delta^*(a)$	$\delta^*(a)$	$\delta^*(c)$	$\delta^*(d)$
$\delta^*(c)$	$\delta^*(a)$	$\delta^*(a)$	$\delta^*(d)$
$\delta^*(d)$	$\delta^*(a)$	$\delta^*(c)$	$\delta^*(a)$

Clearly, $\left(\frac{Y}{\delta^*}, *, \delta^*(a)\right)$ is a dual BCK-algebra and $(X, \star, 1) \cong \left(\frac{Y}{\delta^*}, *, \delta^*(a)\right)$. Therefore, $(X, \star, 1)$ is a fundamental dual BCK-algebra.

Remark 4.1. We know that on any (BE-algebra) dual BCK-algebra $(X, \star, 1)$, if define a binary hyperoperation $''\circ''$ as $x \circ y = \{x \star y\}$ such that is a singleton, then $(X, \circ, 1)$ is a *trivial* (hyper BE-algebra) dual hyper K-algebra. Therefore, its fundamental (BE-algebra) dual BCK-algebra is isomorphic to $(X, \star, 1)$. In the following, we define nontrivial (hyper BE-algebra) dual hyper K-algebra such that its fundamental (BE-algebra) dual BCK-algebra, be isomorphic to given (BE-algebra) dual BCK-algebra $(X, \star, 1)$.

Theorem 4.6. *Every dual BCK-algebra is isomorphic to a fundamental dual BCK-algebra.*

Proof. Let $(X, \star, 1)$ be a dual BCK-algebra. Then by Theorem 3.3, for any dual BCK-algebra $(Y, \star', 1')$, $(X \times Y, \circ, (1, 1'))$ is a dual hyper K-algebra. First, we show that for any $(a, b) \in X \times Y$, $\delta^*(a, b) = \{(a, x) \mid x \in Y\}$. For this let, $u = \bigodot_{i=1}^{n}(x_i, y_i) \in \mathcal{L}(X \times Y)$, where $(x_i, y_i) \in X \times Y$. We have

$$u = \bigodot_{i=1}^{n}(x_i, y_i) = \left\{(\bigodot_{i=1}^{n} x_i, y_i), |x_i \in X, y_i \in Y\right\}.$$

Then we obtain $u = \{(a, y_i)|a \in X \text{ is fixed and } y_i \in Y\}$. Hence, for any $(a, b), (c, d) \in X \times Y$, $(a, b)\delta^*(c, d)$ if and only if $a = c$. Now, we define the map

$$\varphi : \left(\frac{(X \times Y, \circ, (1, 1'))}{\delta^*}, *\right) \to (X, \star, 1)$$

by $\varphi(\delta^*(x, y)) = x$. It is clear that $\delta^*(x, y) = \delta^*(x', y')$ if and only if $x = x'$ if and only if $\varphi(\delta^*(x, y)) = \varphi(\delta^*(x', y'))$. Then, φ is well defined and one to one. In follow, we show that φ is a homomorphism. For this we have,

$$\varphi(\delta^*(x, y) * \delta^*(x', y')) = \varphi(\delta^*(x \star x', y)) = x \star x' = \varphi(\delta^*(x, y)) \star \varphi(\delta^*(x', y')).$$

Clearly, φ is onto. Therefore, φ is an isomorphism. $\square$

Remark 4.2. (i) The (hyper BE-algebras) dual hyper K-algebra $(X \times Y, \circ, 1)$ is called the *associated (hyper BE-algebras) dual hyper K-algebra* to X via Y (or shortly associated (hyper BE-algebras) dual hyper K-algebra) and denote by X_Y.
(ii) The mapping $\varphi : X \to X_Y$ by $\varphi(x) = (x, 1)$ is an embedding.

Theorem 4.7. *Let $(X, *, 1_X)$ and $(Y, *', 1_Y)$ be isomorphic dual BCK-algebras. Then, for any dual BCK-algebra $(Z, *'', 1_Z)$, X_Z and Y_Z are isomorphic dual hyper K-algebras.*

Proof. Let $f : (X, *, 1_X) \to (Y, *', 1_Y)$ be an isomorphism. Define map $\theta : (X_Z, \circ, 1) \to (Y_Z, \circ, 1)$ by $\theta(x, y) = (f(x), y)$, where $x \in X$, $y \in Y$. Clearly θ is a bijection, now we show that θ is a homomorphism. Let $(x_1, y), (x_2, y') \in X_Z$. Then,

$$\begin{aligned}\theta((x_1, y) \circ (x_2, y')) =& \theta(\{(x_1 * x_2, y), (x_1 * x_2, y')\}) \\ =& \{\theta(x_1 * x_2, y), \theta(x_1 * x_2, y')\} = \{(f(x_1 * x_2), y), (f(x_1 * x_2), y')\}\end{aligned}$$

$$=\{(f(x_1) *' f(x_2), y), (f(x_1) *' f(x_2), y')\}$$
$$=(f(x_1), y) \circ (f(x_2), y'))$$
$$=\theta((x_1, y)) \circ \theta((x_2, y')).$$

Therefore, θ is an isomorphism and $(X_Z, \circ, 1) \cong (Y_Z, \circ, 1)$. □

Corollary 4.2. *Every BE-algebra is isomorphic to a fundamental BE-algebra.*

Example 4.5. Let $X = \{1, 2, 3\}$. Then $(X, \star, 1)$ is a BE-algebra as follows:

$\star$	1	2	3
1	1	2	3
2	1	1	3
3	1	2	1

.

By Example 4.4, we saw that $(X, \star, 1)$ is a fundamental dual BCK-algebra. Now, by Theorem 4.6, we construct a new dual hyper K-algebra. Consider the dual BCK-algebras $(\{1, 2, 3\}, \star, 1)$ and $(\{a, b\}, \star', a)$ as follows:

$\star$	1	2	3
1	1	2	3
2	1	1	3
3	1	2	1

,

$\star'$	a	b
a	a	b
b	a	a

.

Now we define a hyperoperation "$\circ$" on $\{1, 2, 3\} \times \{a, b\}$ as follows:

$\circ$	$(1, a)$	$(1, b)$	$(2, a)$	$(2, b)$	$(3, a)$	$(3, b)$
$(1, a)$	$\{(1, a)\}$	A	$\{(2, a)\}$	B	$\{(3, a)\}$	C
$(1, b)$	A	$\{(1, b)\}$	B	$\{(2, b)\}$	C	$\{(3, b)\}$
$(2, a)$	$\{(2, a)\}$	B	$\{(3, a)\}$	C	$\{(3, a)\}$	C
$(2, b)$	B	$\{(2, b)\}$	C	$\{(3, b)\}$	C	$\{(3, b)\}$
$(3, a)$	$\{(3, a)\}$	C	$\{(3, a)\}$	C	$\{(3, a)\}$	C
$(3, b)$	C	$\{(3, b)\}$	C	$\{(3, b)\}$	C	$\{(2, b)\}$

,

where $A = \{(1, a), (1, b)\}, B = \{(2, a), (2, b)\}$ and $C = \{(3, a), (3, b)\}$. Then it is easy to see that $(Z = \{1, 2, 3\} \times \{a, b\}, \circ, (1, a))$ is a dual hyper K-algebra. We have $\delta^*((1, a)) = \{(1, a), (1, b)\}$, $\delta^*((2, a)) = \{(2, a), (2, b)\}$ and $\delta^*((3, a)) = \{(3, a), (3, b)\}$. Hence $\frac{Z}{\delta^*} = \{\delta^*((1, a)), \delta^*((2, a)), \delta^*((3, a))\}$ and obtain the following table:

$*$	$\delta^*(1, a)$	$\delta^*(2, a)$	$\delta^*(3, a)$
$\delta^*(1, a)$	$\delta^*(1, a)$	$\delta^*(2, a)$	$\delta^*(3, a)$
$\delta^*(2, a)$	$\delta^*(1, a)$	$\delta^*(1, a)$	$\delta^*(3, a)$
$\delta^*(3, a)$	$\delta^*(1, a)$	$\delta^*(2, a)$	$\delta^*(1, a)$

.

Clearly, $\left(\frac{Z}{\delta^*}, *, \delta^*((0, a))\right)$ is a dual BCK-algebra and $(X, \star, 1) \cong \left(\frac{Z}{\delta^*}, *, \delta^*((0, a))\right)$. Therefore, $(X, \star, 1)$ is a fundamental dual BCK-algebra.

Corollary 4.3. *By every nonempty set, we can construct a fundamental (BE-algebra) dual BCK-algebra.*

Proof. By Theorem 2.1, there exists a binary operation "$\star$" and "x_0" such that $(X, \star, x_0)$ is a dual BCK-algebra. By Theorem 4.6, $(X, \star, x_0)$ is a fundamental dual BCK-algebra. □

Theorem 4.8. *Let $(X, \star, 1)$ be any finite (BE-algebra) dual BCK-algebra. Then for any binary hyperoperation "$\circ$", and constant "$1'$" on X, such that $(X, \circ, 1')$ is a (hyper BE-algebra) dual hyper K-algebra, there is not any isomorphic between $(X, \star, 1)$ and $\left(\frac{(X,\circ,1')}{\delta^*}, *\right)$, that is $(X, \star, 1) \ncong \left(\frac{(X,\circ,1')}{\delta^*}, *\right)$.*

Proof. Let $(X, \star, 1)$ be a finite (BE-algebra) dual BCK-algebra, $|X| = n$, "$\circ$" be a hyperoperation, and "$1'$" constant on X, such that $(X, \circ, 1')$ be a (hyper BE-algebra) dual hyper K-algebra. Then there exist $x, y \in X$ such that $|x \circ y| \geq 2$. Hence, there are $m, n \in x \circ y$ such that $\delta^*(m) = \delta^*(n)$. Since $\frac{X}{\delta^*} = \{\delta^*(x) \mid x \in X\}$, then, $|\frac{X}{\delta^*}| < n = |X|$. Therefore, $\left(\frac{(X,\circ,1')}{\delta^*}, *\right) \ncong (X, \star, 1)$. □

Theorem 4.9. *Let $(X, \star, 1)$ be a finite (BE-algebra) dual BCK-algebra such that $|X| \geq 2$. Then, there exists a hyperoperation "$\circ$" on X such that $(X, \circ, 1)$ is a dual hyper K-algebra and $|(X, \star, 1)| = |((X, \circ, 1)/\delta^*, \delta^*(1), *)| + 1$.*

Proof. Let $p \in X$ be a fixed element of X. Then by Theorem 3.4, $(X, \circ, 1)$ is a dual hyper K-algebra as follows:

$$x \circ y = \begin{cases} \{1, p\}, & \text{if } x = y, \\ \{y\}, & \text{otherwise,} \end{cases}$$

where $x, y \in X$. Clearly $\delta^*(1) = \{1, p\}$ and for any $x \in X, \delta^*(x) = \{x\}$. Therefore, $|(X, \star, 1)| = |((X, \circ, 1)/\delta^*, \delta^*(1), *)| + 1$. □

Remark 4.3. Now, in the follow we try to show that for any infinite set X, there exists an operation "$\star$", constant 1 and a hyperoperation "$\circ$" on X, such that $(X, \star, 1)$ is an (BE-algebra) dual BCK-algebra and $(X, \circ, 1)$ is a (hyper BE-algebra) dual hyper K-algebra. Moreover, $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong (X, \star, 1)$.

Theorem 4.10. *Let X be an infinite set. Then there exists an operation "$\star$", constant "1" and a binary hyperoperation "$\circ$" on X such that $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong (X, \star, 1)$. That is, X is a fundamental dual BCK-algebra of itself.*

Proof. Let X be an infinite set. Then for any arbitrary set $\{a, b\}$, by Corollary 3.2, there exists a binary hyperoperation "$\circ$", a constant "1" on X, $\circ'$ and $1'$ on $X \times \{a, b\}$ such that $(X, \circ, 1)$ and $(X \times \{a, b\}, \circ', 1')$ are (hyper BE-algebra) dual hyper K-algebra and $(X \times \{a, b\}, \circ', 1') \cong (X, \circ, 1)$ and so

$$\frac{(X \times \{a, b\}, \circ', (1, a))}{\delta^*} \cong \frac{(X, \circ, 1)}{\delta^*}. \tag{4.2}$$

First, we show that for any $(m,t) \in X \times \{a,b\}$, $\delta^*(m,t) = \{(m,a),(m,b)\}$. For this let $u = \bigodot_{i=1}^{n}(m_i,n_i) \in \mathcal{L}(X \times \{a,b\})$, where $(m_i,n_i) \in X \times \{a,b\}$. We have

$$u = \bigodot_{i=1}^{n}(m_i,n_i) = \left\{(\bigodot_{i=1}^{n} m_i, a), (\bigodot_{i=1}^{n} m_i, b)\right\}.$$

Now, we obtain $u = \{(m,a),(m,b) | m \in X \text{ is fixed}\}$. Hence, for any $(m,t),(n,s) \in X \times \{a,b\}$,

$$(m,t)\delta^*(n,s) \Leftrightarrow m = n.$$

Moreover, by Theorem 2.1, there exists an operation $\star$ and constant 1 such that $(X,\star,1)$ is a dual BCK-algebra.

Now, we define the map $\varphi : \left(\frac{(X\times\{a,b\},\circ',(1,a))}{\delta^*}, *\right) \to (X,\star,1)$ by $\varphi(\delta^*(m,t)) = m$. It is clear that $\delta^*(m,t) = \delta^*(m',s)$ if and only if $m = m'$ if and only if $\varphi(\delta^*(m,t)) = \varphi(\delta^*(m',s))$. Then, φ is well defined and one to one.

Now, we show that φ is a homomorphism. For this we have,

$$\varphi(\delta^*(m,t) * \delta^*(m',s)) = \varphi(\delta^*(m \star m',t)) = m \star m' = \varphi(\delta^*(m,t)) \star \varphi(\delta^*(m',s)).$$

Clearly, φ is onto. Hence, φ is an isomorphism and so

$$\left(\frac{(X \times \{a,b\},\circ',(1,a))}{\delta^*}, *\right) \cong (X,\star,1). \tag{4.3}$$

Therefore, by (4.2) and (4.3), we have

$$(X,\star,1) \cong \frac{(X \times \{a,b\},\circ',(1,a))}{\delta^*} \cong \frac{(X,\circ,1)}{\delta^*}.$$

□

Corollary 4.4. *Any infinite set is isomorphic to fundamental BE-algebra of itself.*

5. Categorical Relations on Dual Hyper K-algebras and Dual BCK-algebras

In this section we need to connect (BE-algebras) dual BCK-algebras and (hyper BE-algebras) dual hyper K-algebras, so apply the results of previous sections. First, briefly introduce the category of dual hyper K-algebras (dual BCK-algebras). Category $\mathcal{DH}_k$ ($\mathcal{D}_{bck}$), objects: $(X,\circ,1),(Y,\circ',1'),\ldots$ that are dual hyper K-algebras ($(X,\star,1),(Y,\star',1'),\ldots$ that are dual BCK-algebras). Arrows: $f,g,\ldots$ that are homomorphisms. For two categories $\mathcal{DH}_k$ and $\mathcal{D}_{bck}$, define a categorical morphism as follows:

$$F : \mathcal{DH}_k \to \mathcal{D}_{bck} \text{ by } F(X) = \left(\frac{X}{\delta^*}, *, \delta^*(1)\right), \tag{5.1}$$

where $(X, \circ, 1)$ is a dual hyper K-algebra and for any homomorphism $f : (X, \circ, 1) \to (Y, \circ', 1')$, we define

$$F(f) : \left(\frac{X}{\delta^*}, *, \delta^*(1)\right) \to \left(\frac{Y}{\delta^*}, *, \delta^*(1')\right) \text{ by } F(f) = \delta^*(f). \tag{5.2}$$

By Corollary 4.1, F is well-defined and we have the next result.

Theorem 5.1. *F is a functor of $\mathcal{DH}_k$ to $\mathcal{D}_{bck}$.*

Proof. For any object $(X, \circ, 1)$ of $\mathcal{DH}_k$, by (5.1), $F(X) = \left(\frac{X}{\delta^*}, *, \delta^*(1)\right)$ is a dual BCK-algebra and then $F(X)$ is an object in $\mathcal{D}_{bck}$. Now, we show that for any morphism $f : (X, \circ, 1) \to (Y, \circ', 1'), Ff$ is a morphism in $\mathcal{D}_{bck}$. Let $\delta^*(x), \delta^*(y) \in \frac{X}{\delta^*}$. Then, by (5.2),

$$\begin{aligned} Ff(\delta^*(x) * \delta^*(y)) =& Ff(\delta^*(x \circ y)) = \delta^*(f(x \circ y)) \\ =& \delta^*(f(x) \circ' f(y)) = Ff(\delta^*(x)) * Ff(\delta^*(y)). \end{aligned}$$

Hence, if $g : (X, \circ, 1) \to (Y, \circ', 1')$ and $f : (Y, \circ', 1') \to (Z, \circ'', 1'')$ are morphisms in $\mathcal{DH}_k$, then $F(g) : \left(\frac{X}{\delta^*}, *, \delta^*(1)\right) \to \left(\frac{Y}{\delta^*}, *, \delta^*(1')\right)$ by $F(g) = \delta^*(g)$ and $F(f) : (\frac{Y}{\delta^*}, *, \delta^*(1')) \to (\frac{Z}{\delta^*}, *, \delta^*(1''))$ by $F(f) = \delta^*(f)$ are morphisms in $\mathcal{D}_{bck}$. Now,

$$F(f) \circ F(g) = F(f)(F(g)) = F(f)(\delta^*(g)) = \delta^*(f \circ g) = F(f \circ g).$$

Moreover, for $1 : X \to X$, $F(1) = \delta^*(1)$ and then for any $x \in X$,

$$F(1)(x) = \delta^*(1(x)) = \delta^*(x) = 1_{FX}(x).$$

Therefore, F is a functor of $\mathcal{DH}_k$ to $\mathcal{D}_{bck}$. □

Remark 5.1. If X is a dual BCK-algebra and

$$\mathcal{B}(X) = \{S \in \mathcal{D}_{bck} | \text{ X is a fundamental dual } BCK\text{-algebra of S}\},$$

by Theorem 4.6, $\mathcal{B}(X) \neq \emptyset$.

Now, for $\mathcal{DH}_k$ and $\mathcal{D}_{bck}$ and any dual BCK-algebras $(X, *, 1)$ and $S = \{a, b\}$, define a categorical morphism, as follows:

$$U : \mathcal{D}_{bck} \to \mathcal{DH}_k \text{ by } U(X) = X_S \tag{5.3}$$

and for any dual BCK-algebra homomorphism $f : (X, *, 1) \to (Y, *', 1')$ define

$$U(f) : X_S \to Y_S \text{ by } U(f) = (f, 1). \tag{5.4}$$

By Theorem 4.7, U is well defined and now, we have the next result.

Theorem 5.2. *U is a faithful functor of $\mathcal{D}_{bck}$ to $\mathcal{DH}_k$.*

Proof. For any object $(X, *, 1)$ of $\mathcal{D}_{bck}$, by Theorem 4.6 and 5.3, $U(X) = X_S$ is a dual hyper K-algebra and then $U(X)$ is an object in $\mathcal{DH}_k$. Now, we show that for any morphism $f : (X, \circ, 1) \to (Y, \circ', 1'), Uf$ is a morphism in $\mathcal{DH}_k$. Let $(x_1, y_1), (x_2, y_2) \in X_S$. Now, by (5.4),

$$Uf((x_1, y_1) \circ'' (x_2, y_2)) =(f, 1)((x_1, y_1) \circ'' (x_2, y_2))$$

$$=(f(x_1), y_1) \circ'' (f(x_2), y_2) = Uf((x_1, y_1)) \circ'' Uf((x_2, y_2)).$$

Hence, if $g : (X, *, 1) \to (Y, *', 1')$ and $f : (Y, *', 1') \to (Z, *'', 1'')$ are morphisms in $\mathcal{D}_{bck}$, then $U(g) : (X_S, \circ'', 1'') \to (Y_S, \circ'', 1'')$ by $U(g) = (g, 1)$ and $U(f) : (Y_S, \circ'', 1) \to (Z_S, \circ'', 1)$ by $U(f) = (f, 1)$ are morphisms in $\mathcal{DH}_k$. Now,

$$\begin{aligned}(U(f) \circ U(g))(x, y) =& U(f)(U(g))(x, y) = U(f)(g(x), y)\\ =& (f(g(x)), y) = (f \circ g, 1)(x, y) = U(f \circ g)(x, y).\end{aligned}$$

Moreover, for $1 : X \to X$, $U(1) = (1, 1)$ and then for any $(x, y) \in X_S$,

$$U(1)(x, y) = (1, 1)(x, y) = (x, y) = 1_{FX}(x, y).$$

Therefore, U is a functor of $\mathcal{D}_{bck}$ to $\mathcal{DH}_k$. Now, let $(X, *, 1)$ and $(Y, *, 1')$ be objects in $\mathcal{D}_{bck}$, $f_1, f_2 : X \to Y$ be parallel arrows of $\mathcal{D}_{bck}$ and $U(f_1) = U(f_2)$. Then, for any $(x, y) \in X_S$, $U(f_1)(x, y) = U(f_2)(x, y)$ implies that $f_1(x) = f_2(x)$ and so $f_1 = f_2$. Therefore, U is a faithful functor. □

Theorem 5.3. *On Objects of $\mathcal{D}_{bck}$, $F \circ U = 1$.*

Proof. For any object $(X, *, 1)$ in $\mathcal{D}_{bck}$ by Theorem 4.6, (5.1) and (5.3)

$$(F \circ U)(X, *, 1) = F(X_S, \circ, 1) = \left(\frac{(X_S, \circ, 1)}{\delta^*}, *\right) \cong (X, *, 1).$$ □

Theorem 5.4. *For functors $1, F \circ U : \mathcal{D}_{bck} \to \mathcal{D}_{bck}$ there exists a transformation $\tau : 1 \to F \circ U$ such that is natural.*

Proof. For two functors 1 (identity) and $F \circ U$ of $\mathcal{D}_{bck}$ to $\mathcal{D}_{bck}$, define a map $\tau : 1 \to F \circ U$ as follows:

$$\tau : 1(X) \to (F \circ U)(X) \text{ by } \tau(x) = \delta^*(x, 1). \tag{5.5}$$

Now, for any dual BCK-algebra homomorphism $f : X \to X'$, consider the following diagram:

$$\begin{tikzcd} 1(X) \arrow[r, "\tau_X"] \arrow[d, "1(f)"'] & (F \circ U)(X) \arrow[d, "F\circ U(f)."] \\ 1(X') \arrow[r, "\tau_{X'}"] & (F \circ U)(X') \end{tikzcd}$$

For any $r \in X$ by (5.4) and (5.5), we have

$$\begin{aligned}((F \circ U)(f) \circ \tau)x =& (F \circ U)f(\tau(x))\\ =& (F \circ U)(f)(\delta^*(x, 1))\\ =& (\delta^*(f(x)), 1) = \tau_{X'}(f(x))\\ =& \tau_{X'}(1(f)x) = (\tau_{X'} \circ 1(f))x.\end{aligned}$$

Therefore, $\tau : 1 \to (F \circ U)$ is a natural transformation. □

Theorem 5.5. *For two functors* $1, U \circ F : \mathcal{DH}_k \to \mathcal{DH}_k$ *there exists a transformation* $\upsilon : 1 \to U \circ F$ *such that is natural.*

Proof. For two functors 1 (identity) and $U \circ F$ of $\mathcal{DH}_k$ to $\mathcal{DH}_k$, define a map $\upsilon : 1 \to U \circ F$ as follows:

$$\upsilon : 1(X) \to (U \circ F)(X) \text{ by } \upsilon(x) = (\delta^*(x), 1). \tag{5.6}$$

Now, for morphism homomorphism $f : X \to X'$, consider the following diagram:

$$\begin{tikzcd}
1(X) \arrow[r, "\nu_X"] \arrow[d, "1(f)"'] & (U \circ F)(X) \arrow[d, "U\circ F(f)."] \\
1(X') \arrow[r, "\nu_{X'}"] & (U \circ F)(X')
\end{tikzcd}$$

For any $r \in X$ by (5.2) and (5.6), we have

$$\begin{aligned}
((U \circ F)(f) \circ \upsilon)x =& (U \circ F)f(\upsilon(x) = (U \circ F)f((\delta^*(x), 1)) \\
=& (\delta^*(f(x)), 1) \\
=& \upsilon_{X'}(f(x)) = \upsilon_{X'}(1(f)x) = (\upsilon_{X'} \circ 1(f))x.
\end{aligned}$$

Therefore, $\upsilon : 1 \to (U \circ F)$ is a natural transformation. □

6. Conclusion

In the present paper, on any set, we construct a (BE-algebra) dual BCK-algebra, (hyper BE-algebra) dual hyper K-algebra. We introduce the notion of fundamental (BE-algebra) dual BCK-algebra via the fundamental relation δ^* and investigated some of their useful properties. Moreover,
(i) by every nonempty set, we can construct a (hyper BE-algebra) dual hyper K-algebra;
(ii) by every nonempty set, we can construct a (BE-algebra) dual BCK-algebra such that is isomorphic to a fundamental (BE-algebra) dual BCK-algebra;
(iii) By every nonempty set, we can construct a (BE-algebra) dual BCK-algebra such that is isomorphic to fundamental (BE-algebra) dual BCK-algebra of itself;
(iv) We define functors between these categories and construct a natural transformation between their combinations and identity functor.

In our future work, we should be get more results in (hyper BE-algebras) dual hyper K–algebras and its application.

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