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A COMPARISON OF CONSERVATION LAWS OF THE BOUSSINESQ SYSTEM

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Abstract. In this work we study the Boussinesq system, which is a natural model for the propagation of long waves on the surface of water with a small amplitude and is used to compute a complete set of local conservation laws of the model through the direct method. In this method, some local multipliers are found to construct the fluxes of the conservation law. These multipliers are used to find new conservation laws via another method such as Noether's theorem, Boyer's formulation, Homotopy operator method and Ibragimov's theorem. It is noteworthy that this paper reviews these methods to compare all obtained fluxes and local conservation laws.

1. Introduction

Russell's discovery of solitary waves in 1840 [6], raised a challenge: to devise a mathematical theory for water waves that would admit a wave solution that did not disperse with time. In 1870, the French hydrodynamicist Joseph Boussinesq proposed what he believed was a suitable model by assuming that the wave amplitude was small compared to the canal depth; he arrived at the *Boussinesq equation*,

$$u_{tt} = \Big(u - u^2 + u_{xx}\Big)_{xx}, \tag{1.1}$$

posed for $x \in \mathbb{R}$ and $t \geq 0$. This equation plays a vital part in fluid mechanics [34], and is completely integrable and admits multiple soliton solutions. It also arises in one-dimensional lattice waves (Zabusky [35]) and ion-acoustic solitons (Kako and Yajima [21]). In recent years, considerable attention has been given to new forms of Boussinesq equations (Madsen et al. [26]) dealing with water wave propagation and to

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modified Boussinesq equations [10, 31] in terms of a velocity potential on an arbitrary elevation and free surface displacement of water.

The $(2+1)$-dimensional Boussinesq equation

$$u_{tt} = \left(u - u^2 + u_{xx}\right)_{xx} + u_{yy}, \tag{1.2}$$

which describes the propagation of gravity waves on the surface of water, has been extensively studied by several authors [12, 20, 25, 33].

The $(3+1)$-dimensional Boussinesq equation given by

$$u_{tt} = \left(u - u^2 + u_{xx}\right)_{xx} + u_{yy} + u_{zz}, \tag{1.3}$$

has been extensively studied by several authors [7,8,13,29]. Also in [27,28] the authors obtained conservation laws for (1.2) and (1.3) respectively.

In this paper, we study the system of evolutionary equations

$$\begin{cases} u_t = v_{xx}, \\ v_t = u_{xx} - u - u^2, \end{cases} \tag{1.4}$$

which obviously is covering the Boussinesq equation (1.1).

The goal of this paper is to review and illustrate available methods of flux construction for equation (1.4). For this purpose five different methods are applied including Direct method, Noether's theorem, Boyer's generalization of Noether's theorem, Herman-Poole method and Ibragimov's theorem.

The paper is outlined as follows. Section 2 discusses the Lie symmetry analysis of the Boussinesq system (SNLB). Then in Section 3, the similarity method is applied to reduce the Boussinesq system into two ordinary differential equations. In Section 4.1, we present the direct construction method to find conservation laws. We briefly give the preliminaries concerning the Noether's theorem approach in Section 4.3 to find an exact Lagrangian of equation (1.4), which allows us to construct conservation laws by Noether's theorem. The next method [16] employs integral formulas, related to homotopy operators, to compute the fluxes in Section 4.4. In Section 4.5, we utilize the new conservation theorem which does not require the existence of a Lagrangian and is based on a concept of an adjoint equation for non-linear equations suggested recently by Ibragimov. Finally, a correspondence between the variational symmetry components and local conservation laws following from Noether's first theorem is outlined. Also, we present comparisons of these methods for computing conservation laws in the last section.

In the symmetry and conservation law computations the use of computer algebra systems such as Mathematica and Maple is needed.

2. Lie Symmetry Analysis of the System

In this section we obtain the Lie point symmetries of the Boussinesq system (1.4), which we exploit in Subsection 4.3.4.

First of all, let us consider a one-parameter Lie group of infinitesimal transformations:

$$t \mapsto t + \epsilon\tau(t,x,u,v),$$
$$x \mapsto x + \epsilon\xi(t,x,u,v),$$
$$u \mapsto u + \epsilon\phi(t,x,u,v),$$
$$v \mapsto v + \epsilon\psi(t,x,u,v),$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$X = \tau(t,x,u,v)\frac{\partial}{\partial t} + \xi(t,x,u,v)\frac{\partial}{\partial x} + \phi(t,x,u,v)\frac{\partial}{\partial u} + \psi(t,x,u,v)\frac{\partial}{\partial v}. \tag{2.1}$$

Applying the second prolongation $X^{(2)}$ to equation (1.4), we find the following system of Lie equations

$$\begin{aligned} &\phi^t - \psi^{xx} = 0,\\ &\psi^t - \phi^{xx} + \phi + 2u\phi = 0. \end{aligned} \tag{2.2}$$

Solving this system we obtain the following five Lie point symmetries:

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial v}, \quad X_4 = x\frac{\partial}{\partial v},\\ &X_5 = t\frac{\partial}{\partial t} + \left(\frac{x}{2}\right)\frac{\partial}{\partial x} - \left(u + \frac{1}{2}\right)\frac{\partial}{\partial u} + \left(\frac{t}{2} - v\right)\frac{\partial}{\partial v}. \end{aligned} \tag{2.3}$$

The commutator relations between these operators are given in Table 1.

Table 1. Lie algebra

$[\ ,\ ]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	$X_1 + \frac{1}{2}X_3$
X_2	0	0	X_3	0	$\frac{1}{2}X_2$
X_3	0	$-X_3$	0	0	$-\frac{3}{2}X_4$
X_4	0	0	0	0	$-X_3$
X_5	$-X_1 - \frac{1}{2}X_3$	$-\frac{1}{2}X_2$	$\frac{3}{2}X_4$	X_3	0

The table shows that they obviously construct a finite-dimensional Lie algebra of symmetries

$$G_1 : (\epsilon + t, x, u, v),$$
$$G_2 : (t, \epsilon + x, u, v),$$
$$G_3 : (t, x, u, \epsilon + v),$$
$$G_4 : (t, x, u, \epsilon x + v),$$
$$G_5 : \left(te^{\epsilon}, xe^{\frac{\epsilon}{2}}, -\frac{1}{2} + e^{-\epsilon}\left(u + \frac{1}{2}\right), \frac{1}{4}te^{\epsilon} + \left(-\frac{t}{4} + v\right)e^{-\epsilon}\right).$$

The corresponding action on solutions of the Boussinesq system says that if $u = f(t, x)$ and $v = g(t, x)$ are solutions, so are

$$u^{(1)} = f(t - \epsilon, x), \qquad v^{(1)} = g(t - \epsilon, x),$$
$$u^{(2)} = f(t, x - \epsilon), \qquad v^{(2)} = g(t, x - \epsilon),$$
$$u^{(3)} = f(t, x), \qquad v^{(3)} = g(t, x) + \epsilon,$$
$$u^{(4)} = f(t, x), \qquad v^{(4)} = g(t, x) + \epsilon x,$$
$$u^{(5)} = \left(f\left(e^{-\epsilon}t, e^{-\frac{\epsilon}{2}}x\right) + \frac{1}{2}\right)e^{\epsilon} - \frac{1}{2}, \qquad v^{(5)} = \left(g\left(e^{-\epsilon}t, e^{-\frac{\epsilon}{2}}x\right) - \frac{1}{4}t\right)e^{\epsilon} + \frac{1}{4}te^{-\epsilon}.$$

3. Classical Similarity Solutions

The first advantage of the symmetry group method is to construct new solutions from known solutions. To do this, the infinitesimals are considered and their corresponding invariants are determined. The Boussinesq system is expressed in the coordinates (t, x, u, v), in order to reduce this system allowing us to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (y, U, V) corresponding to an infinitesimal generator. So using the chain rule, the expression of the system in the new coordinates allows us to reduce the system. Since the original system of partial differential equations has two independent variables, then this system transforms into a system of ordinary differential equations after reduction. Here we will compute some invariant solutions with respect to symmetries. First we obtain the similarity variables for each term of the Lie algebra $\mathfrak{g}$, of symmetries; they can be obtained by integrating the characteristic equations. Then we use this method to reduce the system and find the invariant solutions.

3.1. **Time translation invariance X_1.** The classical similarity solution of (1.4) for this symmetry is obtained by integrating the group trajectories

$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1. \tag{3.1}$$

where ϵ is a parameter along the trajectories. Integration of (3.1) yields the invariant transformation

$$y = x, \quad U(y) = u(t, x), \quad V(y) = v(t, x), \tag{3.2}$$

thus the reduced system with respect to invariants (3.2) is

$$-V''(y) = 0, \quad -U''(y) + U(y) + U^2(y) = 0,$$

and the similarity solution for $v(t, x)$ is

$$v(t, x) = C_3\, x + C_4,$$

and $u(t, x)$ can be calculated in the following integrative form:

$$-\int^{u(x)} \frac{\pm 3}{\sqrt{6\,a^3 + 9\,a^2 + 9\,C_1}}da + x + C_2 = 0.$$

3.2. **Space translation invariance** X_2**.** The classical similarity solution of (1.4) is obtained by integrating the group trajectories

$$\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = 1. \tag{3.3}$$

Integration of (3.3) yields the invariant transformation

$$y = t, \quad U(y) = u(t,x), \quad V(y) = v(t,x), \tag{3.4}$$

thus the reduced system with respect to invariants (3.4) is

$$U'(y) = 0, \quad V'(y) + U(y) + U^2(y) = 0;$$

and the similarity solution is

$$u(t,x) = C_1, \quad v(t,x) = t\left(-C_1 - {C_1}^2\right) + C_2.$$

3.3. **Solution translation invariance** X_3**,** X_4**.** For these symmetries every translated solution with any constant is a similarity solution.

3.4. **Solution translation invariance** X_5**.** The classical similarity solution of (1.4) for the last symmetry is obtained by integrating the group trajectories

$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} = t, \quad \frac{\mathrm{d}x}{\mathrm{d}\epsilon} = \frac{x}{2}, \quad \frac{\mathrm{d}u}{\mathrm{d}\epsilon} = -u - \frac{1}{2}, \quad \frac{\mathrm{d}v}{\mathrm{d}\epsilon} = \frac{t}{2} - v. \tag{3.5}$$

Integration of (3.5) yields the invariant transformation

$$y = \frac{x}{\sqrt{t}}, \quad U(y) = u(t,x)\,t + \frac{t}{2}, \quad V(y) = -\frac{1}{4}\,(t - 4\,v(t,x))\,t, \tag{3.6}$$

thus the reduced system with respect to invariants (3.6) is

$$\begin{aligned} 2V''(y) + yU'(y) + 2\,U(y) &= 0, \\ 2\,U''(y) + yV'(y) - 2\,U^2(y) + 2\,V(y) &= 0. \end{aligned}$$

It easy to see that one has a particular solution

$$\begin{aligned} u(t,x) &= 1 + \frac{x^2}{2t} - \frac{x^3}{4t\sqrt{t}} + \frac{x^4}{6t^2} - \frac{x^5}{40t^2\sqrt{t}} + O\left(\frac{x^6}{t^3}\right), \\ v(t,x) &= \frac{x}{\sqrt{t}} - \frac{x^2}{2t} - \frac{x^4}{12t^2} + \frac{x^5}{32t^2\sqrt{t}} + O\left(\frac{x^6}{t^3}\right). \end{aligned}$$

4. Local Conservation Laws

A conservation law of a given DE system is a divergence expression that vanishes on all solutions of the DE system. In the study of systems of DEs, the concept of a conservation law plays an very important role in the analysis of essential properties of the solutions, particularly, investigation of existence, uniqueness and stability of solutions [22, 23]. They have also been used in the development of numerical methods [14, 24].

4.1. **Direct method.** A more general systematic method of constructing local conservation laws, called the direct method, was suggested in [1, 2]. When employing this method, one seeks a set of local multipliers (also called factors or characteristics) depending on independent and dependent variables of a given PDE system and derivatives of dependent variables up to some fixed order, such that a linear combination of the PDEs of the system taken with these multipliers yields a divergence expression. Families of multipliers that yield conservation laws are found from determining equations that follow from Euler differential operators. After finding sets of local conservation law multipliers, one needs to derive expressions for the corresponding conservation law fluxes.

Consider a system $\Delta\left(x, u^{(k)}\right)=0$ of N DEs of order k with $n-$independent variables $x=\left(x^1, \ldots, x^n\right)$ and $m-$dependent variables $u=\left(u^1, \ldots, u^m\right)$, given by

$$\Delta^\sigma[u]=\Delta^\sigma\left(x, u, \partial u, \ldots, \partial^k u\right)=0, \quad \sigma=1, \ldots, N. \tag{4.1}$$

A local conservation law of the system (4.1) is a divergence expression

$$\mathrm{D}_i \Phi^i[u]=\mathrm{D}_1 \Phi^1[u], \ldots, \mathrm{D}_n \Phi^n[u]=0, \tag{4.2}$$

holding on all solutions of the system (4.1). In (4.2), D_i are the total derivatives with respect to x_i and $\Phi^i[u]=\Phi^i\left(x, u, \partial u, \ldots, \partial^r u\right)$, $i=1, \ldots, n$, are the fluxes of conservation laws.

In general, for a given non-degenerate DE system (4.1), nontrivial local conservation laws arise from seeking scalar products that involve linear combinations of the equations of the DE system (4.1) with multipliers (factors) that yield nontrivial divergence expressions. In seeking such expressions, the dependent variables and each of their derivatives that appear in the DE system (4.1) or in the multipliers, are replaced by arbitrary functions. Such divergence expressions vanish on all solutions of the DE system (4.1) provided the multipliers are non-singular.

In particular a set of multipliers $\left\{\Lambda_\sigma[U]\right\}_{\sigma=1}^N=\left\{\Lambda_\sigma\left(x, U, \partial U, \ldots, \partial^\ell U\right)\right\}_{\sigma=1}^N$ yields a divergence expression for the DE system $\Delta\left(x, u^{(k)}\right)$ if the identity

$$\Lambda_\sigma[U] \Delta^\sigma[U] \equiv \mathrm{D}_i \Phi^i[U],$$

holds for arbitrary functions $U(x)$. Then on the solutions $U(x)=u(x)$ of the DE system (4.1), if $\Lambda_\sigma[u]$ is non-singular, one has the local conservation laws

$$\Lambda_\sigma[u] \Delta^\sigma[u]=\mathrm{D}_i \Phi^i[u]=0.$$

Definition 4.1. The Euler operator with respect to U^μ is the operator defined by

$$E_{U^\mu}=\frac{\delta}{\delta U^\mu}=\frac{\partial}{\partial U^\mu}+\sum_{s=1}^{\infty}(-1)^s \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s} \frac{\partial}{\partial U_{i_1 \cdots i_s}^\mu}. \tag{4.3}$$

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By direct calculation, one can show that the Euler operators (4.3) annihilate any divergence expression $\mathrm{D}_i\Phi^i\left(x,U,\partial U,\ldots,\partial^r U\right)$ for any r. In particular, the following identity holds for arbitrary $U\left(x\right)$:

$$E_{U^\mu}\left(\mathrm{D}_i\Phi^i\left(x,U,\partial U,\ldots,\partial^r U\right)\right)\equiv 0,\quad \mu=1,\ldots,m.$$

It is straightforward to show that the converse also holds. Namely, the only scalar expressions annihilated by Euler operators are divergence expressions. This establishes the following theorem.

Theorem 4.1. *The equations* $E_{U^\mu}F\left(x,U,\partial U,\ldots,\partial^s U\right)\equiv 0$, $\mu=1,\ldots,m$, *hold for arbitrary* $U\left(x\right)$ *if and only if* $F\left(x,U,\partial U,\ldots,\partial^s U\right)\equiv \mathrm{D}_i\Psi^i\left(x,U,\partial U,\ldots,\partial^{s-1}U\right)$ *for some functions* $\Psi^i\left(x,U,\partial U,\ldots,\partial^{s-1}U\right)$, $i=1,\ldots,n$.

From Theorem (4.1), the proof of the following theorem that connects local multipliers and local conservation laws is immediate.

Theorem 4.2. *A set of non-singular local multipliers* $\left\{\Lambda_\sigma\left(x,U,\partial U,\ldots,\partial^\ell U\right)\right\}_{\sigma=1}^N$ *yields a divergence expression for a DE system* (4.1) *if and only if the set of equations*

$$E_{U^\mu}\left(\Lambda_\sigma\left(x,U,\partial U,\ldots,\partial^\ell U\right)\Delta^\sigma\left(x,U,\partial U,\ldots,\partial^k U\right)\right)\equiv 0,\quad \mu=1,\ldots,m, \tag{4.4}$$

holds for arbitrary functions $U\left(x\right)$.

The set of equations (4.4) yields the set of linear determining equations to find all sets of local conservation law multipliers of a given DE system (4.1) by letting $\ell=1,2,\ldots$ in (4.4). Since the equations in (4.1) hold for arbitrary $U\left(x\right)$, it follows that they also hold for each derivative of $U\left(x\right)$ replaced by an arbitrary function. In particular, since derivatives of $U\left(x\right)$ of orders higher than ℓ can be replaced by arbitrary functions, it follows that the linear PDE system (4.1) splits into an over-determined linear system of determining equations whose solutions are the sets of local multipliers $\left\{\Lambda_\sigma\left(x,U,\partial U,\ldots,\partial^\ell U\right)\right\}_{\sigma=1}^N$ of the differential equation system $\Delta^\sigma\left(x,u^{(k)}\right)$.

The direct method to obtain local conservation laws is now illustrated through (1.4).

4.1.1. *Local conservation law multipliers of first order for DE system* (1.4). We seek all local conservation law multipliers of the form

$$\Lambda_1=\xi\left(t,x,U,V,U_t,V_t,U_x,V_x\right),\quad \Lambda_2=\varphi\left(t,x,U,V,U_t,V_t,U_x,V_x\right) \tag{4.5}$$

of the PDE system (1.4). In terms of the Euler operators

$$\begin{aligned} E_U&=\frac{\partial}{\partial U}-\mathrm{D}_x\frac{\partial}{\partial U_x}-\mathrm{D}_t\frac{\partial}{\partial U_t}+\mathrm{D}_x^2\frac{\partial}{\partial U_{xx}},\\ E_V&=\frac{\partial}{\partial V}-\mathrm{D}_x\frac{\partial}{\partial V_x}-\mathrm{D}_t\frac{\partial}{\partial V_t}+\mathrm{D}_x^2\frac{\partial}{\partial V_{xx}},\end{aligned} \tag{4.6}$$

the determining equations (4.4) for the multipliers (4.5) become

$$E_U[\xi(t,x,U,\ldots,V_x)(U_t-V_{xx})+\varphi(t,x,U,\ldots,V_x)\left(V_t-U_{xx}+U+U^2\right)]\equiv 0, \tag{4.7}$$

$$E_V[\xi(t,x,U,\ldots,V_x)(U_t-V_{xx})+\varphi(t,x,U,\ldots,V_x)\left(V_t-U_{xx}+U+U^2\right)]\equiv 0, \tag{4.8}$$

where $U(t,x)$ and $V(t,x)$ are arbitrary functions. The determining equations (4.7) and (4.8) yield the solutions

$$\begin{aligned}\Lambda_1&=\frac{1}{5}C_1(-7tV_t-5xV_x+2t-V)-C_4V_x+C_2x-C_3V_t+C_5,\\ \Lambda_2&=\frac{1}{5}C_1(7tU_t+5xU_x+4U+2)+C_3U_t+C_4U_x,\end{aligned} \tag{4.9}$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. As a result, one obtains linearly independent conservation laws, arising from the multipliers

$$\begin{aligned}(\xi_1,\varphi_1)&=(-V_t,U_t),\\ (\xi_2,\varphi_2)&=(-V_x,U_x),\\ (\xi_3,\varphi_3)&=(1,0),\\ (\xi_4,\varphi_4)&=(x,0),\\ (\xi_5,\varphi_5)&=\left(-\frac{7}{5}tV_t-xV_x+\frac{2}{5}t-\frac{1}{5}V,\frac{7}{5}tU_t+xU_x+\frac{4}{5}U+\frac{2}{5}\right).\end{aligned}$$

Each set (ξ,φ) determines a non-trivial local conservation law

$$D_x\Phi(t,x,U,V,U_t,V_t,U_x,V_x)+D_t\Psi(t,x,U,V,U_t,V_t,U_x,V_x)=0,$$

with the characteristic form

$$\begin{aligned}&\mathrm{D}_x\Phi(t,x,U,V,U_t,V_t,U_x,V_x)+\mathrm{D}_t\Psi(t,x,U,V,U_t,V_t,U_x,V_x)\\ &\equiv\xi(t,x,U,\ldots,V_x)(U_t-V_{xx})+\varphi(t,x,U,\ldots,V_x)\left(V_t-U_{xx}+U+U^2\right).\end{aligned} \tag{4.10}$$

The integration of the equations (4.10) for each set of multipliers yields the corresponding density-flux pairs.

$$\begin{aligned}\Phi^1&=\frac{1}{2}(u_tu_x-v_tv_x-uu_{tx}+vv_{tx}), & \Psi^1&=\frac{1}{2}\left(uu_{xx}-vv_{xx}-u^2-\frac{2}{3}u^3\right),\\ \Phi^2&=\frac{1}{6}\left(-3u^2-2u^3+3u_x^2-3v_x^2\right)-uv_t, & \Psi^2&=uv_x,\\ \Phi^3&=-v_x, & \Psi^3&=u,\\ \Phi^4&=v-xv_x, & \Psi^4&=xu.\end{aligned}$$

To calculate the conserved quantities Φ and Ψ corresponding to (ξ_5,φ_5), we need to invert the total divergence operator. This requires the integration (by parts) of an expression in multi-dimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. In Subsection 4.4 and Subsection 4.5, we will obtain these conserved quantities for (ξ_5,φ_5).

4.1.2. *Local conservation law multipliers of fourth order for DE system* (1.4). In [5], local multipliers of the form $\Lambda[U,V] = \Lambda(t,x,U,V,U_x,V_x,\ldots,U_{4x},V_{4x})$ of the Boussinesq system (1.4) given by

$$\begin{aligned}\Lambda_1[U,V] &= U_{4x} - (3U+1)U_{xx} - \frac{3}{2}U_x^2 - \frac{1}{2}V_x^2 + \frac{2}{3}U^3 + \frac{1}{2}U^2,\\ \Lambda_2[U,V] &= -V_{4x} + UV_{xx} + U_xV_x.\end{aligned} \tag{4.11}$$

The set multiplier (4.11) determines a non-trivial local conservation law with the characteristic form

$$\begin{aligned}&\mathrm{D}_x\Phi(t,x,U,V,\ldots,U_{4x},V_{4x}) + \mathrm{D}_t\Psi(t,x,U,V,\ldots,U_{4x},V_{4x})\\ \equiv&\Lambda_1(t,x,U,\ldots,V_{4x})(U_t - V_{xx}) + \Lambda_2(t,x,U,\ldots,V_{4x})\left(V_t - U_{xx} + U + U^2\right).\end{aligned}$$

One can determine the flux and density using integration by parts as follows:

$$\begin{aligned}\Psi =&\frac{1}{12}u\left(2u^2 + 2u^3 - 6{v_x}^2 - 6u_{xx} - 9uu_{xx} + 6u_{4x}\right) - \frac{1}{2}vv_{4x},\\ \Phi =&\frac{1}{2}u^2v_x + \frac{1}{3}u^3v_x - \frac{1}{2}{u_x}^2v_x + \frac{1}{6}{v_x}^3 + u_xv_{xx} + 2uu_xv_{xx} - v_{xx}u_{3x}\\ &- uv_{3x} - u^2v_{3x} + u_{xx}v_{3x} - \frac{1}{2}u_xu_t - \frac{3}{2}uu_xu_t + \frac{1}{2}u_{3x}u_t + uv_xv_t\\ &- \frac{1}{2}v_{3x}v_t + \frac{1}{2}uu_{tx} + \frac{3}{4}u^2u_{tx} - \frac{1}{2}u_{xx}u_{tx} + \frac{1}{2}v_{xx}v_{tx} + \frac{1}{2}u_xu_{txx}\\ &- \frac{1}{2}v_xv_{txx} - \frac{1}{2}uu_{t3x} + \frac{1}{24}vv_{t3x}.\end{aligned}$$

4.2. **Linearizing operators and adjoint equations.** Consider a given DE system $\Delta\left(x,u^{(k)}\right) = 0$ (4.1). The *linearizing operator* $\mathrm{L}[U]$ associated with the DE system (4.1) is given by

$$\mathrm{L}^\sigma_\mu[U]V^\mu = \left[\frac{\partial\Delta^\sigma[U]}{\partial U^\mu} + \frac{\partial\Delta^\sigma[U]}{\partial U^\mu_i}\mathrm{D}_i + \cdots + \frac{\partial\Delta^\sigma[U]}{\partial U^\mu_{i_1\ldots i_k}}\mathrm{D}_{i_1}\cdots\mathrm{D}_{i_k}\right]V^\mu, \quad \sigma = 1,\ldots,N, \tag{4.12}$$

in terms of an arbitrary function $V(x) = (V^1(x),\ldots,V^m(x))$. The *adjoint operator* $\mathrm{L}^*[U]$ associated with the DE system (4.1) is given by

$$\mathrm{L}^{*\sigma}_\mu[U]W_\sigma = \frac{\partial\Delta^\sigma[U]}{\partial U^\mu}W_\sigma - \mathrm{D}_i\left(\frac{\partial\Delta^\sigma[U]}{\partial U^\mu_i}W_\sigma\right) + \cdots + (-1)^k\mathrm{D}_{i_1}\cdots\mathrm{D}_{i_k}\left(\frac{\partial\Delta^\sigma[U]}{\partial U^\mu_{i_1\ldots i_k}}W_\sigma\right), \tag{4.13}$$

where $\mu = 1,\ldots,m$, in terms of an arbitrary function $W(x) = (W_1(x),\ldots,W_N(x))$.

4.2.1. *Self-adjoint DE systems.* An especially interesting situation arises when the linearizing operator (Fréchet derivative) $\mathrm{L}[U]$ of a given DE system (4.1) is self-adjoint.

Definition 4.2. Let $\mathrm{L}[U]$, with its components $\mathrm{L}^\sigma_\mu[U]$ given by (4.12), be the linearizing operator associated with a DE system (4.1). The adjoint operator of $\mathrm{L}[U]$ is $\mathrm{L}^*[U]$, with its components $L_\mu^{*\sigma}[U]$ given by (4.13). $L[U]$ is a self-adjoint operator if and only if $L[U] \equiv L^*[U]$, i.e., $L_\mu^\sigma[U] \equiv L_\mu^{*\sigma}[U]$, $\sigma, \mu = 1, \dots, m$.

Most importantly, one can show that a given DE system, as written, has a variational formulation (Lagrangian) if and only if its associated linearizing operator is self-adjoint [32].

If the linearizing operator associated with a given DE system is self-adjoint, then each set of local conservation law multipliers yields a local symmetry of the given DE system. In particular, one has the following theorem.

Theorem 4.3. *Consider a given DE system* (4.1) *with $N = m$. Suppose its associated linearizing operator* $L[U]$*, with components* (4.12)*, is self-adjoint. Suppose* $\left\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^\ell U)\right\}_{\sigma=1}^{m}$ *is a set of local conservation law multipliers of the DE system* (4.1)*. Let $\eta^\sigma(x, u, \partial u, \dots, \partial^\ell u) = \Lambda_\sigma(x, u, \partial u, \dots, \partial^\ell u)$, $\sigma = 1, \dots, m$, where $U(x) = u(x)$ is any solution of the DE system* (4.1)*. Then*

$$\eta^\sigma(x, u, \partial u, \dots, \partial^\ell u)\frac{\partial}{\partial u^\sigma}$$

is a local symmetry of the DE system (4.1)*.*

4.2.2. *Self-adjointness of SNLB.* Here we note that the system (1.1) does not admit a Lagrangian. Nevertheless, the linearized PDE system for (1.4) is given by

$$\begin{aligned} D_t\tilde{v}^1 - {D_x}^2\tilde{v}^2 &= 0, \\ D_t\tilde{v}^2 - {D_x}^2\tilde{v}^1 + (1+2u)\tilde{v}^1 &= 0, \end{aligned}$$

holding for any solution of PDE system (1.4), whereas the adjoint linearized system is given by

$$\begin{aligned} -D_t\tilde{w}^1 - {D_x}^2\tilde{w}^2 + (1+2u)\tilde{w}^2 &= 0, \\ -D_t\tilde{w}^2 - {D_x}^2\tilde{w}^1 &= 0, \end{aligned}$$

holding for any solution of PDE system (1.4). In general, for substitution $\tilde{w}^1 = \tilde{v}^1$ and $\tilde{w}^2 = \tilde{v}^2$, for the solutions $u = u(t, x)$ and $v = v(t, x)$ of the NLB system (1.4), the PDE system (1.4) is not self-adjoint. But if we set $\tilde{w}^1 = \tilde{v}^2$ and $\tilde{w}^2 = \tilde{v}^1$ the adjoint linearized system coincides with the orginal linearized PDE system. Further, equations (1.4) are the Euler-Lagrange equations determined by the action $\int \mathcal{L} dt dx$ with Lagrangian [11]:

$$\mathcal{L} = \frac{1}{2}(u_x^2 + v_x^2 + vu_t - uv_t) - \frac{1}{3}u^3 - \frac{1}{2}u^2. \tag{4.14}$$

4.3. **Noether's Theorem.** There are many methods of constructing conservation laws for DEs. One of the most well-known systematic methods is due to Emmy Noether (1918), who demonstrated that for self-adjoint (variational) PDE systems, conservation laws arise from variational symmetries, i.e., symmetries that preserve the action integral [30]. However, the applicability of Noether's theorem is severely limited, since the majority of PDE systems arising in applications are not self-adjoint [4].

4.3.1. *Preliminaries on the Noether's symmetry.* Consider a functional $\mathcal{J}[U]$ in terms of n independent variables $x = (x^1, \dots, x^n)$ and m arbitrary functions $U = (U^1(x), \dots, U^m(x))$ and their derivatives up to order k, defined on a domain ΩÂŋ,

$$\mathcal{J}[U] = \int_\Omega \mathcal{L}[U]\,dx = \int_\Omega \mathcal{L}\left(x, U, \partial U, \dots, \partial^k U\right) dx.$$

The function $\mathcal{L}[U] = \mathcal{L}\left(x, U, \partial U, \dots, \partial^k U\right)$ is called a Lagrangian and the functional $\mathcal{J}[U]$ is called an action integral.

Suppose $\mathcal{A}$ is the universal space of all differential functions of finite orders, clearly it is a vector space and forms an algebra. Consider The Lie-Backlund operator or generalized operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\sigma \frac{\partial}{\partial U^\sigma} + \sum_{s\geq 1} \zeta^\sigma_{i_1, i_2, \dots, i_s}, \tag{4.15}$$

where $\xi^i, \eta^\sigma \in \mathcal{A}$ and the additional coefficients are determined uniquely by the prolongation formulas

$$\zeta^\sigma_i = \mathrm{D}_i\left(\eta^\sigma\right) - U^\sigma_j \mathrm{D}_i\left(\xi^j\right),$$
$$\zeta^\sigma_{i_1, i_2, \dots, i_s} = \mathrm{D}_{i_s}\left(\zeta^\sigma_{i_1, i_2, \dots, i_{s-1}}\right) - U^\sigma_{j i_1, i_2, \dots, i_{s-1}} \mathrm{D}_{i_s}\left(\xi^j\right), \quad s > 1,$$

which D_i is the total derivatives with respect to x_i. A vector $C = (C^1, C^2, \dots, C^n)$, $C^i \in \mathcal{A}$, $i = 1, 2, \dots, n$, associated with (4.1) is called a conserved vector if $\mathrm{D}_i C^i = 0$ holds for all solutions of (4.1).

4.3.2. *Noether's formulation of Noether's theorem.*

Definition 4.3. A Lie-Backlund operator X, of the form (4.15), is called a Noether operator corresponding to a Lagrangian $\mathcal{L}[U]$ of equation (4.1) if

$$\frac{\delta \mathcal{L}}{\delta U^\sigma} = 0, \quad \sigma = 1, \dots, m \tag{4.16}$$

and

$$X(\mathcal{L}) + \mathcal{L}\mathrm{D}_i\left(\xi^i\right) = \mathrm{D}_i\left(B^i\right), \tag{4.17}$$

for some gauge functions $B^i[U] = B^i(x, U, \partial U, \dots, \partial^r U) \in \mathcal{A}$, $i = 1, \dots, n$.

The effectiveness of an available Noether operator resides in the following theorem.

Theorem 4.4 (Noether [30])**.** *If X, as given in* (4.15)*, is a Noether point symmetry generator corresponding to a given Lagrangian $\mathcal{L}[U]$, then the vector $C = (C^1, \dots, C^n)$, defined by*

$$C^i = N^i(\mathcal{L}) - B^i, \quad i = 1, \dots, n, \tag{4.18}$$

is a conserved vector associated with the operator X, and the Noether operator associated with X is

$$N^i = \xi^i + W^\sigma \frac{\delta}{\delta U_i^\sigma} + \sum_{s\geq 1} \mathrm{D}_{i_1}\cdots \mathrm{D}_{i_s}\left(W^\sigma\right) \frac{\delta}{\delta U^\sigma_{ji_1,i_2,\ldots,i_s}}, \quad i=1,\ldots,n,$$

where $W^\sigma = \hat{\eta}^\sigma = \eta^\sigma - \xi^i U_i^\sigma$, $\sigma = 1,\ldots,m$, is the Lie characteristics function.

4.3.3. *Boyer's formulation of Noether's theorem.* Boyer [9] extended Noether's theorem in order to construct conservation laws arising from invariance under generalized symmetries [32], i.e., symmetries with infinitesimals depending on higher-order derivatives by generalizing Noether's definition of invariance of an action integral $\mathcal{J}$ (see [5]). In particular, under the following definition, an action integral $\mathcal{J}$ is invariant under a one-parameter higher-order transformation if its integrand $\mathcal{L}[U]$ is invariant to within a divergence under such a transformation.

Definition 4.4. Let

$$\hat{X} = \hat{\eta}^\alpha(x,U,\partial U,\ldots,\partial^s U)\frac{\partial}{\partial U^\alpha} \tag{4.19}$$

be the infinitesimal generator of a one-parameter higher-order local transformation with its extension $\hat{X}^\infty$ given by

$$\hat{X}^\infty = \hat{\eta}^\mu \frac{\partial}{\partial U^\mu} + \hat{\eta}_i^{(1)\mu}\frac{\partial}{\partial U_i^\mu} + \cdots + \hat{\eta}^{(p)\mu}_{i_1\ldots i_p}\frac{\partial}{\partial U^\mu_{i_1\ldots i_p}}, \tag{4.20}$$

which means

$$\hat{\eta}^{(1)\mu} = \mathrm{D}_i\hat{\eta}^\mu, \quad \hat{\eta}^{(p)\mu}_{i_1\ldots i_p} = \mathrm{D}_{i_p}\hat{\eta}^{(p-1)\mu}_{i_1\ldots i_{p-1}}, \quad \mu=1,\ldots,m,\ i,i_j=1,\ldots,n,\ p=2,3,\ldots.$$

Let $\hat{\eta}^\alpha\,[U] = \hat{\eta}^\alpha(x,U,\partial U,\ldots,\partial^s U)$. The above transformation is a *local symmetry* of $\mathcal{J}$ if and only if

$$\hat{X}^\infty \mathcal{L}\,[U] = \mathrm{D}_i A^i\,[U] \tag{4.21}$$

holds for some set of functions $A^i\,[U] = A^i(x,U,\partial U,\ldots,\partial^r U)$, $i=1,\ldots,n$.

Definition 4.5. A local transformation with infinitesimal generator (4.19) that is, a local symmetry of $\mathcal{J}$, is called a variational symmetry of $\mathcal{J}$.

The following theorem generalizes Noether's formulation of her theorem.

Theorem 4.5 (Boyer's generalization of Noether's theorem)**.** *Suppose a given DE system $\Delta(x,U,\partial U,\ldots,\partial^k U)=0$ (4.1) is derivable from a variational principle, i.e., a system of the Euler-Lagrange equations whose solutions are extrema of an action integral $\mathcal{J}\,[x,U]$ with Lagrangian $\mathcal{L}\,[x,U]$. Suppose a local transformation with infinitesimal generator* (4.19) *yields a variational symmetry of $\mathcal{J}$. Let $R^l\,[U,w]$ be defined by*

$$R^i\,[U,w] = w^\sigma\left(\frac{\partial\mathcal{L}[U]}{\partial U_i^\sigma} + \cdots + (-1)^{k-1}\mathrm{D}_{j_1}\cdots\mathrm{D}_{j_{k-1}}\frac{\partial\mathcal{L}[U]}{\partial U^\sigma_{ij_1\ldots j_{k-1}}}\right)$$

$$+ w^{\sigma}_{j_1}\left(\frac{\partial \mathcal{L}[U]}{\partial U^{\sigma}_{ij_1}} + \cdots + (-1)^{k-2}\mathrm{D}_{j_2}\cdots \mathrm{D}_{j_{k-1}}\frac{\partial \mathcal{L}[U]}{\partial U^{\sigma}_{ij_1\cdots j_{k-1}}}\right)$$

$$+ \cdots + w^{\sigma}_{j_1\cdots j_{k-1}}\frac{\partial \mathcal{L}[U]}{\partial U^{\sigma}_{ij_1\cdots j_{k-1}}}, \quad w^{\sigma}(x) = \hat{\eta}^{\sigma}[U], \tag{4.22}$$

for arbitrary functions $U(x), w(x)$. Then

(a) *the identity*

$$\hat{\eta}^{\mu} E_{U^{\mu}}(\mathcal{L}[U]) \equiv \mathrm{D}_i(A^i[U] - R^i[U, \hat{\eta}[U]]) \tag{4.23}$$

holds for arbitrary functions $U(x)$, i.e., $\hat{\eta}^{\mu}[U]^{m}_{\mu=1}$ is a set of local multipliers of the Euler-Lagrange system (4.16)*;*

(b) *the local conservation law*

$$\mathrm{D}_i(R^i[u, \hat{\eta}[u]] - A^i[u]) = 0 \tag{4.24}$$

holds for any solution $u = \Theta(x)$ of the Euler-Lagrange system (4.16)*.*

4.3.4. *Construction of conservation laws for equations* (1.4) *via Noether's formulation.* In what follows, we apply Noether's theorem on the SNLB and obtain its conservation laws. Recall that PDE system (1.4) is a variational system arising from a classical variational principle, with the Lagrangian having the form [11]:

$$\mathcal{L} = \frac{1}{2}(U_x^2 + V_x^2 + VU_t - UV_t) - \frac{1}{3}U^3 - \frac{1}{2}U^2. \tag{4.25}$$

Recall that (1.4) admits the following five Lie point symmetry generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial v}, \quad X_4 = x\frac{\partial}{\partial v}, \tag{4.26}$$

$$X_5 = t\frac{\partial}{\partial t} + \left(\frac{x}{2}\right)\frac{\partial}{\partial x} - \left(u + \frac{1}{2}\right)\frac{\partial}{\partial u} + \left(\frac{t}{2} - v\right)\frac{\partial}{\partial v}. \tag{4.27}$$

The actions of the extensions of the generators (4.26) are summarized in Table 2. To apply this theorem, the generators (4.26) must satisfy the following invariance condition

$$X^{(1)}(\mathcal{L}) + \mathcal{L}\mathrm{D}_i\left(\xi^i\right) = 0, \tag{4.28}$$

where $X^{(1)}$ is the first order prolongation of X.

The invariance condition (4.28) can be replaced by the divergence condition

$$X^{(1)}(\mathcal{L}) + \mathcal{L}\mathrm{D}_i\left(\xi^i\right) = \mathrm{D}_i\left(B^i\right). \tag{4.29}$$

Calculations show that $X_1^{(1)}$ and $X_2^{(1)}$ satisfy the invariance condition (4.28) while $X_3^{(1)}$ and $X_4^{(1)}$ satisfy the divergence condition (4.29) as follows:

$$X_3^{(1)}(\mathcal{L}) + \mathcal{L}[0] = \frac{1}{2}U_t = \mathrm{D}_t\left(\frac{1}{2}U\right),$$

$$X_4^{(1)}(\mathcal{L}) + \mathcal{L}[0] = \frac{1}{2}xU_t + V_x = \mathrm{D}_t\left(\frac{1}{2}xU\right) + \mathrm{D}_x(V).$$

TABLE 2. Point symmetry X_i, first prolongation of X_i, local symmetry $\hat{X}_i$ and prolongation of $\hat{X}_i$

X_i	$X_i^{(1)}$	$\hat{X}_i$	$\hat{X}_i^{\infty}$
X_1	∂_t	$-u_t\partial_u - v_t\partial_v$	$\hat{X}_1 - u_{tt}\partial_{u_t} + \dots$
X_2	∂_x	$-u_x\partial_u - v_x\partial_v$	$\hat{X}_2 - u_{xx}\partial_{u_x} + \dots$
X_3	∂_v	∂_v	∂_v
X_4	$x\partial_v + \partial_{v_x}$	$x\partial_v$	$\hat{X}_4 + \partial_{v_x}$
X_5	$X_5 - 2u_t\partial_{u_t} + \left(\frac{1}{2} - 2v_t\right)\partial_{v_t} - \frac{3}{2}u_x\partial_{u_x} - \frac{3}{2}v_x\partial_{v_x}$	$\left(-u - \frac{1}{2}\right)\partial_u - \left(tu_t + \frac{1}{2}xu_x\right)\partial_u + \left(\frac{1}{2}t - v - tv_t - \frac{1}{2}xv_x\right)\partial_v$	$\hat{X}_5 - (2u_t + tu_{tt})\,\partial_{u_t} + \left(\frac{1}{2} - 2v_t - tv_{tt}\right)\partial_{v_t} + \cdots$

The prolongation $X_5^{(1)}$ did not satisfy any of the two conditions above, hence X_5 does not generate a variational symmetry group of $\mathcal{L}$. Using X_1, X_2, X_3 and X_4 respectively in the vector field (4.18), we obtain the conservation law

$$\mathrm{D}_t\left(C^1\right) + \mathrm{D}_x\left(C^2\right) = 0,$$

with components of the conserved vector $C = (C^1, C^2)$ given by

$$\begin{aligned}
&\left(C_1^1,\, C_1^2\right) = \left(\frac{1}{2}V_x^2 + \frac{1}{2}U_x^2 - \frac{1}{2}U^2 - \frac{1}{3}U^3,\, -U_xU_t - V_xV_t\right),\\
&\left(C_2^1,\, C_2^2\right) = \frac{1}{2}\left(UV_x - \frac{1}{2}VU_x,\, VU_t - UV_t - U^2 - \frac{2}{3}U^3 - V_x^2 - U_x^2\right),\\
&\left(C_3^1,\, C_3^2\right) = \left(-U,\, V_x\right),\\
&\left(C_4^1,\, C_4^2\right) = \left(-xU,\, xV_x - V\right).
\end{aligned}$$

4.3.5. *Construction of conservation laws for equation* (1.4) *via Boyer's formulation.* In particular, the actions of the extensions of the generators (4.26) on the Lagrangian (4.25) yield the divergence expressions:

$$\begin{aligned}
\hat{X}_1^{\infty}\mathcal{L} =&\mathrm{D}_t\left(2U^2 + 3U^3 + 2UU_{xx} + 2VV_{xx} - 2VU_t + \frac{1}{2}UV_t\right)\\
&+ \mathrm{D}_x\left(-\frac{1}{2}U_xU_t - \frac{1}{2}V_xV_t - \frac{1}{2}UU_{tx} - \frac{1}{2}VV_{tx}\right),
\end{aligned}$$

$$\hat{X}_2^{\infty}\mathcal{L} =\mathrm{D}_x\left(\frac{1}{2}UV_t + 2U^2 + 3U^3 - 2V_x^2 - 2U_x^2 - 2VU_t\right), \tag{4.30}$$

$$\hat{X}_3^{\infty}\mathcal{L} = \mathrm{D}_t\left(\frac{1}{2}U\right),$$
$$\hat{X}_4^{\infty}\mathcal{L} = \mathrm{D}_t\left(\frac{1}{2}xU\right) + \mathrm{D}_x\left(\frac{1}{2}V\right).$$

Hence through Boyer's formulation of Noether's theorem [Theorem (4.5)], the four symmetries (4.26) yield four conservation laws with eight multipliers given by

$$\begin{aligned}
&\Lambda_1^1[U,\,V] = \hat{\eta}_1^1[U,\,V] = -U_t, && \Lambda_1^2[U,\,V] = \hat{\eta}_1^2[U,\,V] = -V_t,\\
&\Lambda_2^1[U,\,V] = \hat{\eta}_2^1[U,\,V] = -U_x, && \Lambda_2^2[U,\,V] = \hat{\eta}_2^2[U,\,V] = -V_x,\\
&\Lambda_3^1[U,\,V] = \hat{\eta}_3^1[U,\,V] = 0, && \Lambda_3^2[U,\,V] = \hat{\eta}_3^2[U,\,V] = 1,\\
&\Lambda_4^1[U,\,V] = \hat{\eta}_4^1[U,\,V] = 0, && \Lambda_4^2[U,\,V] = \hat{\eta}_4^2[U,\,V] = x.
\end{aligned}$$

In this example, due to the simplicity of the form of multipliers and the given PDE, fluxes of the four conservation laws are readily found through integration by parts. However, in more complicated practical situations when Noether's theorem is used, one would normally compute fluxes using the formula (4.24) which involves no integration. Here we illustrate the use of this formula. Denoting $x^1 = t$, $x^2 = x$, we first compute the quantities $R^i\,[U, V, w^1, w^2]$ (4.22), $i = 1, 2$, using the Lagrangian (4.25):

$$R^1\left[U, V, w^1, w^2\right] = \frac{1}{2}\hat{\eta}^1 V - \frac{1}{2}\hat{\eta}^2 U, \quad R^2\left[U, V, w^1, w^2\right] = \hat{\eta}^1 U_x + \hat{\eta}^2 V_x.$$

Then for the four conservation laws, from (4.30), we identify

$$\begin{aligned}
\left(A_1^1, A_1^2\right) =& (2U^2 + 3U^3 + 2UU_{xx} + 2VV_{xx} - 2VU_t + \frac{1}{2}UV_t,\\
&- \frac{1}{2}U_xU_t - \frac{1}{2}V_xV_t - \frac{1}{2}UU_{tx} - \frac{1}{2}VV_{tx}),\\
\left(A_2^1, A_2^2\right) =& \left(0, \frac{1}{2}UV_t + 2U^2 + 3U^3 - 2V_x^2 - 2U_x^2 - 2VU_t\right),\\
\left(A_3^1, A_3^2\right) =& \left(\frac{1}{2}U, 0\right),\\
\left(A_4^1, A_4^2\right) =& \left(\frac{1}{2}xU, \frac{1}{2}V\right).
\end{aligned}$$

Therefore from (4.24), the four conservation laws of PDE (1.4) corresponding to variational symmetries (4.26) have the form

$$\begin{aligned}
&\mathrm{D}_t\,(-U) + \mathrm{D}_x\,(V_x) = 0,\\
&\mathrm{D}_t\,(0) + \mathrm{D}_x\left(xV_x - \frac{1}{2}V\right) = 0,\\
&\mathrm{D}_t\left(-\frac{1}{2}VU_x + \frac{1}{2}UV_x\right) + \mathrm{D}_x\left(-U_xU_t + V_x^2 - \frac{1}{2}UV_t - 2U^2 - 3U^3 + 2U_x^2 + VU_t\right) = 0,\\
&\mathrm{D}_t\left(-\frac{1}{2}VU_t + \frac{1}{2}UV_t - 2U^2 - 3U^3 - 2UU_{xx} - 2VV_{xx} + 2VU_t - \frac{1}{2}UV_t\right)
\end{aligned}$$

$$+\mathrm{D}_x\left(-\frac{1}{2}U_xU_t-\frac{1}{2}V_xV_t+\frac{1}{2}UU_{tx}+\frac{1}{2}VV_{tx}\right)=0.$$

One can check that the symmetry (4.27) does not yield a variational symmetry of the action functional $\mathcal{J}[U,V]$ with Lagrangian (4.25). This is considered from three points of view: Noether's formulation of Noether's theorem, Boyer's formulation of Noether's theorem and, finally, the direct method.

(1) Noether's formulation of Noether's theorem. First of all, in terms of using Noether's formulation, the additional infinitesimal generator (4.27) corresponds to the scaling symmetry $t^*=\lambda t$, $x^*=\lambda^{\frac{1}{2}}x$, $u^*=\lambda^{-1}\left(u+\frac{1}{2}\right)-\frac{1}{2}$ and $v^*=\lambda^{-1}\left(v-\frac{t}{4}\right)+\frac{1}{4}\lambda t$. Now one checks whether the scaling transformation $t^*=\lambda t$, $x^*=\lambda^{\frac{1}{2}}x$, $U^*=\lambda^{-1}\left(U+\frac{1}{2}\right)-\frac{1}{2}$ and $V^*=\lambda^{-1}\left(V-\frac{t}{4}\right)+\frac{1}{4}\lambda t$ is a symmetry of the action functional $\mathcal{J}[U,V]$. In particular,

$$\mathcal{J}[U^*,V^*]=\int\mathcal{L}[U^*,V^*]\,\mathrm{d}t^*\mathrm{d}x^*=\lambda^{\frac{3}{2}}\int\mathcal{L}[U^*,V^*]\,\mathrm{d}t\mathrm{d}x,$$

but $\mathcal{L}[U^*,V^*]=\lambda^{-3}\mathcal{L}[U,V]$. Hence $\mathcal{J}[U^*,V^*]=\lambda^{-\frac{3}{2}}\mathcal{J}[U,V]\neq\mathcal{J}[U,V]$. Thus, using Noether's formulation of Noether's theorem, the scaling symmetry (4.27) does not yield an additional conservation law of the Boussinesq system (1.4).

(2) Boyer's formulation of Noether's theorem. Secondly, in terms of the more general Boyer's formulation of Noether's theorem, using the extension of the infinitesimal generator (4.27) with $u(x)$ replaced by an arbitrary function $U(x)$, one obtains the expression

$$\begin{aligned}\hat{X}_5^\infty\mathcal{L}[U,V]=&\frac{1}{4}U+\frac{3}{2}U^2+U^3+\frac{1}{2}x\,UU_x+\frac{1}{2}xU^2\,U_x-\frac{3}{2}U_x^2-\frac{3}{2}V_x^2+\frac{1}{4}V_t\\&-\frac{1}{2}xU_xU_{xx}-\frac{1}{2}xV_xV_{xx}-\frac{1}{4}tU_t+tUU_t+tU^2U_t-\frac{3}{2}VU_t\\&-\frac{1}{4}xV_xU_t+\frac{3}{2}UV_t+\frac{1}{4}xU_xV_t-\frac{1}{2}tVU_{tt}+\frac{1}{2}tUV_{tt}.\end{aligned}\tag{4.31}$$

The right-hand side of (4.31) can not be expressed as a divergence expression. To show this, it is best to directly apply the Euler operator (4.3) with respect to U and V to this expression. In particular, one obtains

$$\begin{aligned}E_U\left(\hat{X}_5^\infty\mathcal{L}[U,V]\right)&=\frac{1}{4}\left(2+6U+6U^2+10U_{xx}+7V_t\right)\neq0,\\E_V\left(\hat{X}_5^\infty\mathcal{L}[U,V]\right)&=\frac{5}{2}V_{xx}-\frac{7}{4}U_t\neq0,\end{aligned}$$

which means that $\hat{X}_5^\infty\mathcal{L}[U,V]$ is not a divergence expression, and hence X_5 does not yield a variational symmetry of the action functional. Thus this scaling symmetry does not yield a conservation law multiplier in terms of using Boyer's formulation of Noether's theorem.

(3) Direct method Finally, it is easy to show that the scaling symmetry (4.27) does not yield a variational symmetry through using the direct method. In section (4.1), we have checked whether (ξ_5, φ_5) is a set of multipliers for a conservation law. In particular, one merely applies the Euler operator (4.3) with respect to U, V, i.e., E_U, E_V given by (4.3), to the expression

$$\begin{aligned}\xi^5\Delta^1 + \varphi^5\Delta^2 =& 7\left(-\frac{7}{5}tV_t - xV_x + \frac{2}{5}t - \frac{1}{5}V\right)(U_t - V_{xx})\\ &+ \left(\frac{7}{5}tU_t + xU_x + \frac{4}{5}U + \frac{2}{5}\right)\left(V_t - U_{xx} + U + U^2\right),\end{aligned}$$

to show that

$$E_V\left[\xi^5\Delta^1 + \varphi^5\Delta^2\right] = 0, \quad E_U\left[\xi^5\Delta^1 + \varphi^5\Delta^2\right] = \frac{-6}{5}U_{xx} \neq 0,$$

for arbitrary functions $U(t,x)$ and $V(t,x)$. The right-hand side of the above expression is not equivalent to 0, which means that X_5 does not yield a variational symmetry.

The conserved densities and fluxes corresponding to the symmetry (4.27) can be computed using the following approaches: Homotopy operator method and Ibragimov's theorem.

4.4. **Homotopy operator method.** In the case of complicated forms of multipliers and/or equations, for the inversion of divergence operators, one can use homotopy operators that arise in differential geometry and reduce the problem of finding fluxes to a problem of integration in single-variable calculus. We begin this part by a brief definition of a homotopy operator [15].

Definition 4.6. Let $f\left(t, x, \mathbf{u}^{(M)}(t,x)\right)$ be an exact differential function involving two independent variables (t,x). The second homotopy operator is a vector operator with two components

$$\left(\mathcal{H}^t_{\mathbf{u}(t,x)}(f), \mathcal{H}^x_{\mathbf{u}(t,x)}(f)\right), \tag{4.32}$$

where

$$\mathcal{H}^t_{\mathbf{u}(t,x)}(f) = \int_{\lambda_0}^1 \left(\sum_{j=1}^N \mathcal{I}^t_{u^j(t,x)}(f)\right)[\lambda\mathbf{u}]\frac{d\lambda}{\lambda}, \tag{4.33}$$

$$\mathcal{H}^x_{\mathbf{u}(t,x)}(f) = \int_{\lambda_0}^1 \left(\sum_{j=1}^N \mathcal{I}^x_{u^j(t,x)}(f)\right)[\lambda\mathbf{u}]\frac{d\lambda}{\lambda}. \tag{4.34}$$

The t-integrand, $\mathcal{I}^t_{u^j(t,x)}(f)$, is defined as

$$\mathcal{I}^t_{u^j(t,x)}(f) = \sum_{k_1=1}^{M_1^j}\sum_{k_2=0}^{M_2^j}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^t u^j_{i_1 t\, i_2 x}\left(-\mathrm{D}_t\right)^{k_1-i_1-1}\left(-\mathrm{D}_x\right)^{k_2-i_2}\right)\frac{\partial f}{\partial u^j_{k_1 t\, k_2 x}}, \tag{4.35}$$

with combinatorial coefficient $B^t = B(i_1, i_2, k_1, k_2)$ defined as

$$B(i_1, i_2, k_1, k_2) = \frac{\binom{i_1+i_2}{i_1}\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}}. \tag{4.36}$$

Similarly, the $x-$integrand, $\mathcal{I}^x_{u^j(t,x)}(f)$, is given by

$$\mathcal{I}^x_{u^j(t,x)}(f) = \sum_{k_1=0}^{M_1^j}\sum_{k_2=1}^{M_2^j}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^x u^j_{i_1 t\, i_2 x}(-\mathrm{D}_t)^{k_1-i_1}(-\mathrm{D}_x)^{k_2-i_2-1}\right)\frac{\partial f}{\partial u^j_{k_1 t\, k_2 x}}, \tag{4.37}$$

where $B^x = B(i_2, i_1, k_2, k_1)$.

The homotopy with $\lambda_0 = 0$ is used, except when singularities at $\lambda = 0$ occur. Using homotopy operator Div^{-1}, we have the following theorem [15, 17].

Theorem 4.6. *Let $f = f\left(t, x, \mathbf{u}^{(M)}(t,x)\right)$ be exact, i.e., $f = \mathrm{Div}F$ for some $F = F\left(t, x, \mathbf{u}^{(M-1)}(t,x)\right)$. Then $F = \mathrm{Div}^{-1}(f) = \left(\mathcal{H}^t_{\mathbf{u}(t,x)}(f), \mathcal{H}^x_{\mathbf{u}(t,x)}(f)\right)$.*

Theorem (4.6) helps us to find the flux and density of the equation (1.4) by using the multipliers (4.9) and (4.11). Once we have one component of the conservation law, e.g., the density or a component of the flux, we can compute the remaining components using the homotopy operator.

4.4.1. *Flux and density of the set of multipliers* (4.9). For example, one can show that PDE (1.4) has a local conservation law arising from the multiplier $(\xi_1, \varphi_1) = (-v_t, u_t)$, i.e.,

$$f = -v_t(u_t - v_{xx}) + u_t\left(v_t - u_{xx} + u + u^2\right),$$

is a divergence expression. We now reconstruct the corresponding density Ψ and flux Φ. Using (4.35), we obtain

$$\begin{aligned}
\mathcal{I}^t_{u(t,x)}(f) &= \sum_{k_1=1}^{1}\sum_{k_2=0}^{2}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^t u_{i_1 t\, i_2 x}(-\mathrm{D}_t)^{k_1-i_1-1}(-\mathrm{D}_x)^{k_2-i_2}\right)\frac{\partial f}{\partial u_{k_1 t\, k_2 x}},\\
&= u\frac{\partial f}{\partial u_t} = -uu_{xx} + u^2 + u^3,\\
\mathcal{I}^t_{v(t,x)}(f) &= \sum_{k_1=1}^{1}\sum_{k_2=0}^{2}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^t v_{i_1 t\, i_2 x}(-\mathrm{D}_t)^{k_1-i_1-1}(-\mathrm{D}_x)^{k_2-i_2}\right)\frac{\partial f}{\partial v_{k_1 t\, k_2 x}},\\
&= v\frac{\partial f}{\partial v_t} = vv_{xx}.
\end{aligned}$$

Likewise, using (4.37), we have

$$\begin{aligned}
\mathcal{I}^x_{u(t,x)}(f) &= \sum_{k_1=0}^{1}\sum_{k_2=1}^{2}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^x u_{i_1 t\, i_2 x}(-\mathrm{D}_t)^{k_1-i_1}(-\mathrm{D}_x)^{k_2-i_2-1}\right)\frac{\partial f}{\partial u_{k_1 t\, k_2 x}}\\
&= u(-\mathrm{D}_x)\frac{\partial f}{\partial u_{xx}} + u_x\frac{\partial f}{\partial u_{xx}} = uu_{tx} - u_t u_x,
\end{aligned}$$

$$\begin{aligned}\mathcal{I}^x_{v(t,x)}(f) &= \sum_{k_1=0}^{1}\sum_{k_2=1}^{2}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^x v_{i_1 t\, i_2 x}\,(-\mathrm{D}_t)^{k_1-i_1}\,(-\mathrm{D}_x)^{k_2-i_2-1}\right)\frac{\partial f}{\partial v_{k_1 t\, k_2 x}}\\ &= v\,(-\mathrm{D}_x)\,\frac{\partial f}{\partial v_{xx}} + v_x\frac{\partial f}{\partial v_{xx}} = v_x v_t - v v_{tx}.\end{aligned}$$

Then, using (4.33) and (4.34), yields

$$\begin{aligned}\mathcal{H}^t_{\mathbf{u}(t,x)}(f) &= \int_0^1 \left(\mathcal{I}^t_{u(t,x)}(f)[\lambda\mathbf{u}] + \mathcal{I}^t_{v(t,x)}(f)[\lambda\mathbf{u}]\right)\frac{d\lambda}{\lambda}\\ &= \frac{1}{2}\left(v v_{xx} - u u_{xx} + u^2\right) + \frac{1}{3}u^3,\\ \mathcal{H}^x_{\mathbf{u}(t,x)}(f) &= \int_0^1 \left(\mathcal{I}^x_{u(t,x)}(f)[\lambda\mathbf{u}] + \mathcal{I}^x_{v(t,x)}(f)[\lambda\mathbf{u}]\right)\frac{d\lambda}{\lambda}\\ &= \frac{1}{2}\left(u u_{tx} - u_t u_x - v v_{tx} + v_t v_x\right).\end{aligned}$$

Thus, the homotopy operator gives the vector

$$\mathbf{F} = \mathrm{D}iv^{-1} f = \frac{1}{2}\begin{pmatrix} v v_{xx} - u u_{xx} + u^2 + \frac{2}{3}u^3 \\ u u_{tx} - u_t u_x - v v_{tx} + v_t v_x \end{pmatrix}.$$

Similarly, we have computed the densities and fluxes corresponding to the other multipliers. The results are presented in Table 3.

4.4.2. *Flux and dentity of the set of multipliers* (4.11). For equation (1.4), we found out that (Λ_1, Λ_2) are the set of multipliers and are candidates for reconstructing the flux and density. For the homotopy operator (4.32) and for (4.11) with $\mathbf{u} = (u, v)$ the differential function f is defined as

$$\begin{aligned} f &= \Lambda^1\Delta^1 + \Lambda^2\Delta^2\\ &= \left(-v_{xx} + u_t\right)\left(-u_{xx}\,(3u+1) - 3/2\,u_x^2 - 1/2\,v_x^2 + 2/3\,u^3 + 1/2\,u^2 + u_{4x}\right)\\ &\quad + \left(-u_{xx} + v_t + u + u^2\right)\left(v_{xx}u + v_x u_x - v_{4x}\right).\end{aligned}$$

Thus, the $t-$integrand (4.37) is given by

$$\begin{aligned}\mathcal{I}^t_{u(t,x)}(f) &= \sum_{k_1=1}^{1}\sum_{k_2=0}^{4}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^t u_{i_1 t\, i_2 x}(-\mathrm{D}_t)^{k_1-i_1-1}(-\mathrm{D}_x)^{k_2-i_2}\right)\frac{\partial f}{\partial u_{k_1 t\, k_2 x}}\\ &= -u u_{xx} - \frac{3}{2}u u_x^2 - \frac{1}{2}u v_x^2 - 3u^2 u_{xx} + \frac{1}{2}u^3 + u u_{4x} + \frac{2}{3}u^4,\end{aligned}\tag{4.38}$$

$$\begin{aligned}\mathcal{I}^t_{v(t,x)}(f) &= \sum_{k_1=1}^{1}\sum_{k_2=0}^{4}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^t v_{i_1 t\, i_2 x}(-\mathrm{D}_t)^{k_1-i_1-1}(-\mathrm{D}_x)^{k_2-i_2}\right)\frac{\partial f}{\partial v_{k_1 t\, k_2 x}},\\ &= +u v v_{xx} + v v_x u_x - v v_{4x}.\end{aligned}\tag{4.39}$$

TABLE 3. Flux and density with multipliers (4.9) of the Boussinesq system (1.4).

Multiplier	Flux and Density
$(\xi_1, \varphi_1) = (-v_t, u_t)$	$\mathcal{H}^t_{\mathbf{u}(t,x)}(f) = \frac{1}{2}\left(vv_{xx} - uu_{xx} + u^2\right) + \frac{1}{3}u^3$, $\mathcal{H}^x_{\mathbf{u}(t,x)}(f) = \frac{1}{2}\left(uu_{tx} - u_t u_x - vv_{tx} + v_t v_x\right)$
$(\xi_2, \varphi_2) = (-v_x, u_x)$	$\mathcal{H}^t_{\mathbf{u}(t,x)}(f) = \frac{1}{2}\left(vu_x - uv_x\right)$, $\mathcal{H}^x_{\mathbf{u}(t,x)}(f) = \frac{1}{2}\left(uv_t - vu_t + u^2 - u_x^2 + v_x^2 + \frac{2}{3}u^3\right)$
$(\xi_3, \varphi_3) = (1, 0)$	$\mathcal{H}^t_{\mathbf{u}(t,x)}(f) = u$, $\mathcal{H}^x_{\mathbf{u}(t,x)}(f) = -v_x$
$(\xi_4, \varphi_4) = (x, 0)$	$\mathcal{H}^t_{\mathbf{u}(t,x)}(f) = xu$, $\mathcal{H}^x_{\mathbf{u}(t,x)}(f) = v - xv_x$
$(\xi_5, \varphi_5) = -\left(\frac{7}{5}tv_t - xv_x + \frac{2}{5}t - \frac{1}{5}v, \frac{7}{5}tu_t + xu_x + \frac{4}{5}u + \frac{2}{5}\right)$	$\mathcal{H}^t_{\mathbf{u}(t,x)}(f) = \frac{1}{2}vu_t - \frac{1}{2}uv_x + \frac{2}{5}tu + \frac{7}{10}tu^2 + \frac{7}{15}tu^3 + \frac{3}{10}uv + \frac{7}{5}v_t - \frac{7}{5}tv_t - \frac{7}{10}tuu_{xx} + \frac{2}{5}v - \frac{7}{10}tvu_t + \frac{1}{2}xvu_x + \frac{7}{10}tvv_{xx}$, $\mathcal{H}^x_{\mathbf{u}(t,x)}(f) = \frac{1}{2}xu^2 + \frac{1}{2}xu^3 + \frac{1}{2}xuv_t + \frac{1}{2}uu_x - \frac{2}{5}u_x - \frac{1}{2}xu_x^2 - \frac{7}{10}tu_tu_x + \frac{7}{10}tuu_{tx} - \frac{2}{5}tv_x + \frac{1}{2}xv_x^2 - \frac{1}{2}vv_x - \frac{1}{2}xvu_t - \frac{7}{10}tvv_{tx} + \frac{7}{10}tv_xv_t$

Substituiting (4.38) and (4.39) in the first component of the homotopy operator (4.32) yields

$$\begin{aligned}\mathcal{H}^t_{\mathbf{u}(t,x)}(f) &= \int_0^1 \left(\mathcal{I}^t_{u(t,x)}(f)[\lambda\mathbf{u}] + \mathcal{I}^t_{v(t,x)}(f)[\lambda\mathbf{u}]\right)\frac{d\lambda}{\lambda} \\ &= -\frac{1}{2}uu_{xx} - \frac{1}{2}uu_x^2 - \frac{1}{6}uv_x^2 - 3u^2u_{xx} + \frac{1}{6}u^3 \\ &\quad + \frac{1}{2}uu_{4x} + \frac{1}{6}u^4 + \frac{1}{3}uvv_{xx} + \frac{1}{3}vv_xu_x - \frac{1}{2}vv_{4x},\end{aligned} \tag{4.40}$$

which is a flux corresponding to (Λ_1, Λ_2). Similarly the $x-$integrand (4.35) is given by

$$\mathcal{I}^x_{u(t,x)}(f) = \sum_{k_1=0}^{1}\sum_{k_2=1}^{4}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^x u_{i_1t\,i_2x}(-\mathrm{D}_t)^{k_1-i_1}(-\mathrm{D}_x)^{k_2-i_2-1}\right)\frac{\partial f}{\partial u_{k_1t\,k_2x}} \tag{4.41}$$

$$
\begin{aligned}
=& - 3uu_xu_t + 5uu_xv_{xx} - uv_xu_{xx} + uv_tv_x + u^2v_x + u^3v_x + uu_{tx} \\
&- uu_{t3x} - uv_{3x} - 2u^2v3x - uu_xv_{xx} + uu_{xx}v_x + 3u^2u_{tx} - u_xu_t \\
&+ u_xv_{4x} + u_xv_{xx} - v_xu_x^2 - u_{3x}v_{xx} + u_{3x}u_t + u_{xx}v_{3x} - u_{xx}u_{tx} \\
&- u_{tx}v_{4x} + u_xu_{txx},
\end{aligned}
$$

$$
\mathcal{I}^x_{v(t,x)}(f) = \sum_{k_1=0}^{1}\sum_{k_2=1}^{4}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^x v_{i_1t\, i_2x}(-\mathrm{D}_t)^{k_1-i_1}(-\mathrm{D}_x)^{k_2-i_2-1}\right)\frac{\partial f}{\partial v_{k_1t\, k_2x}} \tag{4.42}
$$

$$
\begin{aligned}
=& -vv_xu_t + \frac{1}{2}v_x^3 + \frac{1}{3}u^3v_x + \frac{1}{2}u^2v_x - v_xu_{4x} + uv_tv_x - uvv_{tx} \\
&- 3vu_xu_{xx} + u_{xx}v_{3x} - v_tv_{3x} - uv3x - u^2v_{3x} + 2uu_xv_{xx} + u_xv_{xx} \\
&+ v_{xx}v_{tx} - u_{3x}v_{xx} + v_xu_{4x} - \frac{1}{2}v_xu_x^2 - v_xv_{txx} + vv_{t3x} + 3vu_xu_{xx}.
\end{aligned}
$$

Substituiting (4.46) and (4.47) in the second component of the homotopy operator (4.37),, yields a density for equation (1.4) corresponding to (Λ_1, Λ_2):

$$
\begin{aligned}
\mathcal{H}^x_{\mathbf{u}(t,x)}(f) =& \int_0^1 \left(\mathcal{I}^x_{u(t,x)}(f)[\lambda\mathbf{u}] + \mathcal{I}^x_{v(t,x)}(f)[\lambda\mathbf{u}]\right)\frac{d\lambda}{\lambda} \\
=& - uu_xu_t + \frac{2}{3}uu_xv_{xx} - \frac{1}{3}uv_xu_{xx} + \frac{2}{3}uv_tv_x + \frac{1}{2}u^2v_x + \frac{1}{3}u^3v_x \\
&+ \frac{1}{2}uu_{tx} - uv_{3x} - u^2v3x + \frac{1}{3}uu_{xx}v_x + u^2u_{tx} - \frac{1}{2}u_xu_t + \frac{1}{2}u_xv_{4x} \\
&+ u_xv_{xx} - \frac{1}{2}v_xu_x^2 + \frac{4}{3}uu_xv_{xx} - u_{3x}v_{xx} + \frac{1}{2}u_{3x}u_t + \frac{1}{2}u_{xx}v_{3x} \\
&- \frac{1}{2}u_{xx}u_{tx} - \frac{1}{2}uu_{t3x} - \frac{1}{2}u_{tx}v_{4x} + \frac{1}{2}u_xu_{txx} - \frac{1}{3}vv_xu_t + \frac{1}{6}v_x^3 \\
&- \frac{1}{3}uvv_{tx} - \frac{1}{2}v_tv_{3x} + \frac{1}{2}v_{xx}v_{tx} - \frac{1}{2}v_xv_{txx} + \frac{1}{2}vv_{t3x} + \frac{1}{2}v_{xx}u_{xx}.
\end{aligned}
$$

Setting $\Phi = \mathcal{H}^x_{\mathbf{u}(t,x)}(f)$ and $\Psi = \mathcal{H}^t_{\mathbf{u}(t,x)}(f)$, conservation laws corresponding to multipliers (4.11) for the system of equations (1.4) is given by

$$
\mathrm{D}_t\Psi + \mathrm{D}_x\Phi = 0.
$$

4.5. **Ibragimov's theorem.** In this section we obtain conservation laws for the $(1+1)$-dimensional Boussinesq system (1.4) using Ibragimov theorem [19], but first we give some definitions and notations which we will utilize later.

We begin with nonlinear self-adjointness introduced by Ibragimov [18], whose main idea is first to turn the system of PDEs into Lagrangian equations by artificially adding new variables, and then applying the theorem proved in [19] to construct local and nonlocal conservation laws.

Let $\mathcal{L}$ be the formal Lagrangian of system (4.1) written as

$$
\mathcal{L} = \vartheta^\beta\Delta^\beta\left(x, u, \partial u, \ldots, \partial^k u\right), \tag{4.43}
$$

where ϑ^β is the new introduced dependent variable, then the adjoint equations of system (4.1) are defined by

$$(\Delta^\sigma)^* \left(x, u, \vartheta, \partial u, \ldots, \partial^k u, \partial^k \vartheta\right) = \frac{\delta \mathcal{L}}{\delta u^\sigma} = 0, \tag{4.44}$$

where $\vartheta = (\vartheta^1, \ldots, \vartheta^m)$ and $\delta/\delta u^\sigma$ is the Euler operator defined as (4.3).

Then the definition of nonlinear self-adjointness of system (4.1) is given as follows.

Definition 4.7. The system (4.1) is said to be nonlinearly self-adjoint if the adjoint system (4.44) is satisfied for all solutions u of system (4.1) upon a substitution $\vartheta = \varphi(x, u)$ such that $\varphi(x, u) \neq 0$.

Here, $\varphi(x, u) = (\varphi^1(x, u), \ldots, \varphi^m(x, u))$ and $\vartheta = \varphi(x, u)$ means $\vartheta^i = \varphi^i(x, u)$, $\varphi(x, u) \neq 0$ means that not all elements of $\varphi(x, u)$ equal zero and is called a nontrivial substitution. Definition (4.7) is equivalent to the following identities holding for the undetermined functions $\lambda_\sigma^\beta = \lambda_\sigma^\beta \left(x, u, \partial u, \ldots, \partial^k u\right)$

$$(\Delta^\sigma)^* \left(x, u, \vartheta, \partial u, \ldots, \partial^k u, \partial^k \vartheta\right)_{|\vartheta=\varphi} = \lambda_\sigma^\beta \Delta^\beta,$$

which is applicable in the proofs and computations.

Nonlinear self-adjointness contains three subclasses. In particular, if the substitution $\vartheta = \varphi(x, u)$ becomes $\vartheta = u$, then system (4.1) is called strictly self-adjointness. If $\vartheta = \varphi(u)$, then it is termed quasi self-adjointness. If $\vartheta = \varphi(x, u)$ involving x and u, then it is called weakly self-adjointness. As an extension of the substitution, if $\vartheta = \varphi(x, u, \partial u, \ldots, \partial^s u)$, then it is called nonlinearly self-adjoint with differential substitution.

Let us now assume the system (4.1) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\sigma \frac{\partial}{\partial u^\sigma}. \tag{4.45}$$

Then the system of adjoint equations (4.44) admits the operator

$$\begin{aligned} Y &= \xi^i \frac{\partial}{\partial x^i} + \eta^\sigma \frac{\partial}{\partial u^\sigma} + \eta_*^\sigma \frac{\partial}{\partial \vartheta^\sigma}, \\ \eta_*^\sigma &= -\left[\lambda_\sigma^\beta \vartheta^\beta + \vartheta^\sigma \mathrm{D}_i\left(\xi^i\right)\right], \end{aligned} \tag{4.46}$$

where the operator (4.46) is an extension of (4.45) to the variable ϑ^σ and the λ_σ^β are obtainable from $X(\Delta^\sigma) = \lambda_\sigma^\beta \Delta^\beta$.

We now state the following theorem.

Theorem 4.7 (see [19]). *Every Lie point, Lie-Bäcklund, and nonlocal symmetry (4.45) admitted by the system of (4.1) leads to a conservation law $\mathrm{D}_i(T^i) = 0$, where the components T^i constructed by the formula*

$$T^i = \xi^i \mathcal{L} + W^\sigma \left[\frac{\partial \mathcal{L}}{\partial u_i^\sigma} - \mathrm{D}_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\sigma}\right) + \mathrm{D}_j \mathrm{D}_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\sigma}\right) - \cdots\right] + \mathrm{D}_j(W^\sigma) \tag{4.47}$$

$$\times\left[\frac{\partial\mathcal{L}}{\partial u^\sigma_{ij}}-\mathrm{D}_k\left(\frac{\partial\mathcal{L}}{\partial u^\sigma_{ijk}}\right)+\cdots\right]+\mathrm{D}_j\mathrm{D}_k\left(W^\sigma\right)\left[\frac{\partial\mathcal{L}}{\partial u^\sigma_{ijk}}-\cdots\right]+\cdots,$$

where $W^\sigma=\eta^\sigma-\xi^j u^\sigma_j$ *and* $\mathcal{L}$ *is the formal Lagrangian* (4.43) *which is written in the symmetric form about the mixed derivatives.*

4.5.1. *Construction of Conservation Laws for* (1.4) *via Ibragimov's theorem.* In Section 2, we derived five Lie point symmetries (2.3) of SNLB (1.4).

Corresponding to each of these five Lie point symmetries we shall construct five conserved vectors. The adjoint equations of (1.4), by invoking (4.44), are

$$\begin{aligned}&\frac{\delta}{\delta u}\left[P(u_t-v_{xx})+Q(v_t-u_{xx}+u+u^2)\right]=0,\\&\frac{\delta}{\delta v}\left[P(u_t-v_{xx})+Q(v_t-u_{xx}+u+u^2)\right]=0,\end{aligned}\tag{4.48}$$

where $P=P(t,x)$ and $Q=Q(t,x)$ are new dependent variables and (4.48) gives

$$\Delta^*=\begin{cases}P_t=-Q_{xx}+Q+2uQ,\\Q_t=-P_{xx}.\end{cases}\tag{4.49}$$

Clearly, (1.4) is not self-adjoint. By recalling (4.43), we get the following Lagrangian for the system of equations (1.4) and (4.49):

$$\mathcal{L}=P(u_t-v_{xx})+Q(v_t-u_{xx}+u+u^2).\tag{4.50}$$

(i) We first consider the Lie point symmetry generator $X_1=\frac{\partial}{\partial t}$. It can be verified from (4.46) that the operator Y_1 is the same as X_1 and the Lie characteristic functions are $W_1=-u_t$, $W_2=-v_t$. Thus, by using (4.47), the components T^i, $i=1,2$, of the conserved vector $T=(T^1,T^2)$ are given by

$$\begin{aligned}T^1&=Q(u+u^2-u_{xx})-Pv_{xx},\\T^2&=Qu_{tx}+Pv_{tx}-Q_xu_t-P_xv_t.\end{aligned}$$

Remark 4.1. The conserved vector T contains the arbitrary solutions P,Q of the adjoint equation (4.49) and hence gives an infinite number of conservation laws.

The same remark applies to all the following four cases.

(ii) Now for the second symmetry generator $X_2=\frac{\partial}{\partial x}$, we have $W_1=-u_x$, $W_2=-v_x$. Hence, by invoking (4.47), the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$\begin{aligned}T^1&=-Pu_x-Qv_x,\\T^2&=Pu_t+Q(u+u^2+v_t)-Q_xu_x-P_xv_x.\end{aligned}$$

(iii) For the third symmetry $X_3=\frac{\partial}{\partial v}$, we have $W_1=0$, $W_2=1$ and the corresponding components of the conserved vector are:

$$T^1=Q,\quad T^2=P_x.$$

(iv) The fourth symmetry $X_4 = x\frac{\partial}{\partial v}$, gives $W_1 = 0$, $W_2 = x$ and the corresponding components of the conserved vector are:

$$T^1 = xQ, \quad T^2 = xP_x - P.$$

(v) Finally, for the symmetry

$$X_5 = t\frac{\partial}{\partial t} + \left(\frac{x}{2}\right)\frac{\partial}{\partial x} - \left(u + \frac{1}{2}\right)\frac{\partial}{\partial u} + \left(\frac{t}{2} - v\right)\frac{\partial}{\partial v},$$

the value of Y_5 is not the same as X_5 and in fact is given by

$$Y_5 = t\frac{\partial}{\partial t} + \left(\frac{x}{2}\right)\frac{\partial}{\partial x} - \left(u + \frac{1}{2}\right)\frac{\partial}{\partial u} + \left(\frac{t}{2} - v\right)\frac{\partial}{\partial v} + \frac{1}{2}P\frac{\partial}{\partial P} + \frac{1}{2}Q\frac{\partial}{\partial Q}.$$

In this case the Lie characteristic functions are $W_1 = -u - \frac{1}{2} - tu_t - \frac{1}{2}xu_x$, $W_2 = -v + \frac{t}{2} - tv_t - \frac{1}{2}xv_x$. So using (4.47), one can obtain the conserved vector T whose components are given by

$$\begin{aligned}
T^1 =& tQ(u + u^2 - u_{xx}) - tPv_{xx} - P\left(u + \frac{1}{2} + \frac{1}{2}xu_x\right) + Q\left(-v + \frac{t}{2} - \frac{1}{2}xv_x\right),\\
T^2 =& \frac{x}{2}Q(u + u^2 - v_t) + \frac{x}{2}Pu_t - P\left(\frac{3}{2}v_x + tv_{tx}\right) + Q\left(\frac{3}{2}u_x + tu_{tx}\right)\\
& - Q_x\left(u + tu_t + \frac{1}{2} + \frac{1}{2}xu_x\right) + P_x\left(-v + \frac{t}{2} - tv_t - \frac{1}{2}xv_x\right).
\end{aligned}$$

Conclusion

In this paper a comprehensive analysis of the symmetries and conservation laws of the Boussinesq system of PDEs (SNLB), is presented. The classical symmetries of the SNLB are computed by applying the criterion of invariance of the equations under the infinitesimal prolonged infinitesimal generators. The one parameter groups and symmetry transformations associated to infinitesimal symmetries are determined. Meanwhile, the Lie invariants and the similarity reduced equations associated to infinitesimal symmetry generators are computed. Finally, the conservation laws of the SNLB are determined via five different methods.

(a) Noether's theorem. We have found an exact Lagrangian of equation (1.4), then have obtained local variational symmetries. There is a one-to-one correspondence between local symmetries $\hat{X}_i$ in Table 2 and local conservation law multipliers which were presented in Table 4.
(b) Boyer's generalization of Noether's theorem. We outline a generalization of Noether's theorem which includes higher-order symmetries due to Boyer's formulation. It is straightforward to apply Noether's theorem to obtain a conservation law for any one-parameter higher-order transformation leaving invariant the action functional to within a divergence term.
(c) Direct method. We present the direct (multiplier) method of local conservation law construction, which is applicable to both variational and non-variational

TABLE 4. Comparison of Lie point symmetries (2.3) in the evolutionary form Table 2 and local conservation law multipliers (4.9) of the Boussinesq system (1.4).

Symmetry generator	Conservation law multipliers	
$\hat{X}_1 = -u_t\partial_u - v_t\partial_v$	$\Lambda_1 = v_t,$	$\Lambda_2 = -u_t$
$\hat{X}_2 = -u_x\partial_u - v_x\partial_v$	$\Lambda_1 = v_x,$	$\Lambda_2 = -u_x$
$\hat{X}_3 = \partial_v$	$\Lambda_1 = 1,$	$\Lambda_2 = 0$
$\hat{X}_4 = x\partial_v$	$\Lambda_1 = x,$	$\Lambda_2 = 0$
$\hat{X}_5 = \left(-u - \frac{1}{2} - tu_t - \frac{1}{2}xu_x\right)\partial_u$ $+\left(\frac{1}{2}t - v - tv_t - \frac{1}{2}xv_x\right)\partial_v$	$\Lambda_1 = 5tv_t + xv_x - t - v,$ $\Lambda_2 = -5tu_t - xu_x - 1 - 2u$	

PDE systems. This method yields the multipliers for conservation laws as well as an integral formula for corresponding conserved densities.

(d) Homotopy operator method. After the multipliers are computed, fluxes and densities of the corresponding divergence expression can be reconstructed, here we are applying the homotopy operator.

(e) Ibragimov's theorem. The conservation laws for the Boussinesq system were also derived by using the new conservation theorem due to Ibragimov in order to find formal Lagrangians of the Boussinesq system without variational structure.

All five of these methods have some limitations in their use.

On comparing (a): we note that there are several limitations to Noether's theorem. It is restricted to variational systems. Consequently, to be applicable to a given system as written, the given system must be of even order, have the same number of dependent variables as the number of equations in the system and have no dissipation. There is also the difficulty of finding symmetries admitted by the action functional. Moreover, the use of Noether's theorem to find conservation laws is coordinate-dependent.

On comparing (b): this method is generalises Noether's definition of invariance of an action integral. Note that the four conservation laws which we have obtained by Boyer's formulation, have been also resulted from the celebrated Noether's theorem. Also, it is restricted to variational systems and one must find an explicit Lagrangian.

In comparing the direct method (c) with Noether's theorem, it is important to reiterate that conservation laws arise from multipliers for both approaches. But unlike Noether's theorem (also, Ibragimov's theorem), the direct method is not limited to PDE systems arising from some variational principle (i.e., self-adjoint PDE systems). None of these complications arise when one computes conservation law multipliers through the direct method. Indeed, the multiplier determining equations are solved off the solution space of the given PDEs.

On comparing (d): as was observed the major weakness of the direct method is in calculating the fluxes and the densities of the conservation law. Whereas when one employs the Homotopy operator method either shall be one is able to find in a very direct fashion two polynomials in terms of the dependent variables and their derivatives. These may then be expressed as a divergence expression via using special formulas which usually include the integration of multiple sentences. While in complicated cases where the system of PDEs includes higher-order derivatives, integration is seldom possible. This problem can be resolved by combining Noether's theorem and the direct method but as previously mentioned, Noether's theorem has several limitation. In the case of complicated forms of multipliers and/or equations, for the inversion of divergence operators, one can use homotopy operators.

On comparing (e) with Noether's theorem: we find that this method is similar to Noether's theorem which requires a Lagrangian to exist, however, Lagrangians exists only for very special types of DEs within the construct of Noether's theorem. Ibragimov has attempted to overcome this difficulty by defining an adjoint equation for non-linear DEs and constructing a Lagrangian for an arbitrary (linear and non-linear) equation considered together with its adjoint equation. Comparing the conservation laws obtained in Subsection 4.3.4 by the exact Lagrangian to thoese in Subsection 4.5.1 obtained from Ibragimov's theorem, we can see that they are equivalent.

Finally, we find that there are only two means by which one may obtain all fluxes and densities: the Homotopy operator method and Ibragimov's theorem.

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