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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR $(g,h,\alpha-m)$-CONVEX FUNCTIONS AND CONSEQUENCE RESULTS

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ABSTRACT. This paper aims to define a new class of functions, which will be called $(g,h;\alpha-m)$-convex functions. We prove Hermite-Hadamard type inequalities for this new class. Many already known classes of functions can be reproduced from new notion of $(g,h;\alpha-m)$-convexity. Some published results are special cases of inequalities established in this article.

1. INTRODUCTION

The following definition is taken from [1].

Definition 1.1. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative function. Let $I\subset(0,+\infty)$ be a real interval and $p\in\mathbb{R}\setminus\{0\}$. A function $f:I\to\mathbb{R}$ is said to be exponentially $(\alpha$, h-$m)$-p-convex, if

$$f\left((ta^p+m(1-t)b^p)^{\frac{1}{p}}\right)\leq\frac{h(t^\alpha)f(a)}{e^{\zeta a}}+\frac{mh(1-t^\alpha)f(b)}{e^{\zeta b}} \tag{1.1}$$

holds provided $a,b,\left((ta^p+m(1-t)b^p)^{\frac{1}{p}}\right)\in I$ for $t\in(0,1)$, $\zeta\in\mathbb{R}$ and $(\alpha,m)\in[0,1]^2$.

A chain of definitions of convexities can be deduced from (1.1), and is mentioned in [1, Remark 1]. For a detailed study on convex functions and their extensions can be found in [2–5]. We are motivated to extend the results of [1] by giving a simple inequality as compare to (1.1), see Section 2, Inequality (2.1). The following inequalities hold for exponentially $(\alpha$, h-$m)$-p-convex functions.

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(i) If $p>0$, then

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)\tag{1.2}$$
$$\leq\frac{\Gamma(\tau+1)}{(mb^p-a^p)^\tau}\left(\mathfrak{G}_1(\zeta)h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{a^p+}f\circ\phi\right)(mb^p)\right.$$
$$\left.+\mathfrak{G}_2(\zeta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^\tau_{b^p-}f\circ\phi\right)\left(\frac{a^p}{m}\right)\right)$$
$$\leq\tau\left\{\left(\mathfrak{G}_1(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\zeta a}}+\mathfrak{G}_2(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}\right)\int_0^1 t^{\tau-1}h\left(t^\alpha\right)dt\right.$$
$$+m\left(\mathfrak{G}_1(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}+\mathfrak{G}_2(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\frac{1}{e^{(\zeta a/m^2)}}\right)$$
$$\left.\times\int_0^1 t^{\tau-1}h\left(1-t^\alpha\right)dt\right\},$$

where $\phi(z)=z^{\frac{1}{p}}$, $z\in[a^p,mb^p]$, $\mathfrak{G}_1(\zeta)=e^{-\zeta bm^{\frac{1}{p}}}$ for $\zeta<0$, $\mathfrak{G}_1(\zeta)=e^{-\zeta a}$ for $\zeta\geq 0$, $\mathfrak{G}_2(\zeta)=e^{-\zeta(a/m^{\frac{1}{p}})}$ for $\zeta>0$, $\mathfrak{G}_2(\zeta)=e^{-\zeta b}$ for $\zeta\leq 0$.

(ii) If $p<0$, then

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)$$
$$\leq\frac{\Gamma(\tau+1)}{(a^p-mb^p)^\tau}\left(\mathfrak{G}_3(\zeta)h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{a^p-}f\circ\phi\right)(mb^p)\right.$$
$$\left.+\mathfrak{G}_4(\zeta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^\tau_{b^p+}f\circ\phi\right)\left(\frac{a^p}{m}\right)\right)$$
$$\leq\tau\left\{\left(\mathfrak{G}_3(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\zeta a}}+\mathfrak{G}_4(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}\right)\int_0^1 t^{\tau-1}h\left(t^\alpha\right)dt\right.$$
$$+m\left(\mathfrak{G}_3(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}+\mathfrak{G}_4(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\frac{1}{e^{(\zeta a/m^2)}}\right)$$
$$\left.\times\int_0^1 t^{\tau-1}h\left(1-t^\alpha\right)dt\right\},$$

where $\phi(z)=z^{\frac{1}{p}}$, $z\in[mb^p,a^p]$, $\mathfrak{G}_3(\zeta)=e^{-\zeta bm^{\frac{1}{p}}}$ for $\zeta<0$, $\mathfrak{G}_3(\zeta)=e^{-\zeta a}$ for $\zeta\geq 0$, $\mathfrak{G}_4(\zeta)=e^{-\zeta(a/m^{\frac{1}{p}})}$ for $\zeta<0$, $\mathfrak{G}_4(\zeta)=e^{-\zeta b}$ for $\zeta\geq 0$.

(iii) If $p>0$, then

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)$$
$$\leq\Gamma(\tau+1)\left(\frac{2}{mb^p-a^p}\right)^\tau$$

$$\times\left(\mathfrak{G}_1(\zeta)h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{\left(\frac{a^p+mb^p}{2}\right)^+}f\circ\phi\right)(mb^p)\right.$$
$$\left.+\mathfrak{G}_2(\zeta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^\tau_{\left(\frac{a^p+mb^p}{2m}\right)^-}f\circ\phi\right)\left(\frac{a^p}{m}\right)\right)$$
$$\leq\tau\left\{\left(\mathfrak{G}_1(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\zeta a}}+\mathfrak{G}_2(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.$$
$$+m\left(\mathfrak{G}_1(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}+\mathfrak{G}_2(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\frac{1}{e^{(\zeta a/m^2)}}\right)$$
$$\left.\times\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},$$

where $\phi(z)=z^{\frac{1}{p}}$, $z\in[a^p,mb^p]$, $\mathfrak{G}_1(\zeta),\mathfrak{G}_2(\zeta)$ are given in (i).

(iv) If $p<0$, then

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{\frac{1}{p}}\right)\tag{1.3}$$
$$\leq\Gamma(\tau+1)\left(\frac{2}{a^p-mb^p}\right)^\tau$$
$$\times\left(\mathfrak{G}_3(\zeta)h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{\left(\frac{a^p+mb^p}{2}\right)^-}f\circ\phi\right)(mb^p)\right.$$
$$\left.+\mathfrak{G}_4(\zeta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^\tau_{\left(\frac{a^p+mb^p}{2m}\right)^+}f\circ\phi\right)\left(\frac{a^p}{m}\right)\right)$$
$$\leq\tau\left\{\left(\mathfrak{G}_3(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\zeta a}}+\mathfrak{G}_4(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.$$
$$+m\left(\mathfrak{G}_3(\zeta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\zeta b}}+\mathfrak{G}_4(\zeta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\frac{1}{e^{(\zeta a/m^2)}}\right)$$
$$\left.\times\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},$$

where $\phi(z)=z^{\frac{1}{p}}$, $z\in[mb^p,a^p]$, $\mathfrak{G}_3(\zeta),\mathfrak{G}_4(\zeta)$ are given in (ii).

In the following, definitions of Riemann-Liouville fractional integrals of order $\tau>0$, denoted by $I^\tau_{a^+}f$ and $I^\tau_{b^-}f$ used in above inequalities are given [6]:

$$I^\tau_{a^+}f(x)=\frac{1}{\Gamma(\tau)}\int_a^x(x-t)^{\tau-1}f(t)dt,\quad x>a,\tag{1.4}$$

$$I^\tau_{b^-}f(x)=\frac{1}{\Gamma(\tau)}\int_x^b(t-x)^{\tau-1}f(t)dt,\quad x<b,\tag{1.5}$$

where $\Gamma(\cdot)$ is the gamma function and $f\in L_1[a,b]$.

In [1], the generalizations of above inequalities were also proved. The motivation for establishing the above inequalities is the following well-known Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.6}$$

where f is convex function on $[a,b]$.

A convex function on $[a,b]$ satisfies the inequality $f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$, $t \in [0,1]$, $x, y \in [a,b]$. The inequalities stated in above are actually generalizations/extensions of the Hermite-Hadamard inequality given in (1.6). The trend of exploring the Hermite-Hadamard inequality is very common, which is why many articles and books have been published on it, see [1, 7–10].

In the upcoming section, we will define a new class of functions. By applying this definition, along with definitions of Riemann-Liouville fractional integrals, two variants of Hermite-Hadamard inequality are presented (see, Theorem 2.1 and Theorem 2.5). Some new inequalities are deduced as applications of Theorem 2.1 and Theorem 2.5 (see, Theorem 2.2 to Theorem 2.4, Theorem 2.6 to Theorem 2.8). In Section 3, two variants of Fejér-Hermite-Hadamard inequality are presented (see, Theorem 3.1 and Theorem 3.4). New Fejér-Hermite-Hadamard type inequalities are deduced as applications of Theorem 3.1 and Theorem 3.4 (see Theorem 3.2, Theorem 3.3, Theorem 3.5, Theorem 3.6).

2. Hermite-Hadamard Type Inequalities for Fractional Riemann-Liouville Integrals

First, we are interested in giving the definition of $(g,h;\alpha-m)$-convex function and its consequences. In the whole paper we use the notation $P(\cdot) := (f.g)(\cdot)$.

Definition 2.1. Let h be a non-negative function on $J \subset \mathbb{R}$, $(0,1) \subset J$, $h \neq 0$ and let g be a positive function on $I \subset \mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be $(g,h;\alpha-m)$-convex if it is non-negative and satisfies the following inequality

$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda^\alpha)f(x)g(x) + mh(1-\lambda^\alpha)f(y)g(y), \tag{2.1}$$

where $\alpha, m \in (0,1]$, $\lambda \in [0,1]$, $x, y \in I$.

If we set $g(x) = \exp(-\eta x)$, $\eta \in \mathbb{R}$, in (2.1), we obtain the following inequality:

$$f(\lambda x + m(1-\lambda)y) \leq \frac{h(\lambda^\alpha)f(x)}{e^{\eta x}} + \frac{mh(1-\lambda^\alpha)f(y)}{e^{\eta y}}. \tag{2.2}$$

A function satisfying the inequality (2.2), is called exponentially $(\alpha, h-m)$-convex function, see [1, Definition 4], for $\alpha = 1$ definition of $(g,h;m)$-convexity is obtained, see [11]. Also, quasi F-$(g,h;\alpha-m)$-convex function is deduced in (2.11).

First, we state and prove the following Hermite-Hadamard type inequality for $(g,h;\alpha-m)$-convex functions.

Theorem 2.1. *Let $f : I \subset (0, +\infty) \to \mathbb{R}$ be $(g, h; \alpha - m)$-convex function as defined in Definition 2.1. The following inequality is valid:*

$$(2.3)\quad f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma(\tau+1)}{(mb-a)^\tau}\left(h\left(\frac{1}{2^\alpha}\right) I^\tau_{a+}P(mb) + m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right) I^\tau_{b-}P\left(\frac{a}{m}\right)\right)$$
$$\leq M\left\{\left(h\left(\frac{1}{2^\alpha}\right) f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f(b)\right)\int_0^1 t^{\tau-1}h\left(t^\alpha\right)dt\right.$$
$$\left.+m\left(h\left(\frac{1}{2^\alpha}\right) f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(1-t^\alpha\right)dt\right\},$$

where $\tau > 0$, $a, b \in I$, $a < b$ and $M = \max_{x\in[a,b]}(g(x))$.

Proof. Using (2.1), one can have the following inequality:

$$(2.4)\qquad f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right) f(x)g(x) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f(y)g(y),$$

where we have used $\lambda = \frac{1}{2}$, in (2.1). For $x = ta + m(1-t)b$ and $y = tb + (1-t)\frac{a}{m}$ in (2.4), we get

(2.5)
$$f\left(\frac{a+mb}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right) P\left(ta + m(1-t)b\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) P\left(tb + (1-t)\frac{a}{m}\right).$$

Multiplying the above inequality with $t^{\tau-1}$ on both sides and integrating over $[0, 1]$, we have

$$(2.6)\qquad f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\tau-1}dt \leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(ta+m(1-t)b\right)dt$$
$$+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(tb+(1-t)\frac{a}{m}\right)dt.$$

Let $ta+m(1-t)b = x$ and $tb+(1-t)\frac{a}{m} = y$ on right hand side of the above inequality. Then after some calculations one can obtain the first inequality of (2.3).

On the other hand by (2.1) on the right hand side of (2.5), one can obtain the inequality:

$$(2.7)\qquad h\left(\frac{1}{2^\alpha}\right) P\left(ta + m(1-t)b\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) P\left(tb + (1-t)\frac{a}{m}\right)$$
$$\leq h\left(\frac{1}{2^\alpha}\right) g\left(ta+m(1-t)b\right)\left(h(t^\alpha)P(a) + mh(1-t^\alpha)P(b)\right)$$
$$+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right) g\left(tb+(1-t)\frac{a}{m}\right)\left(h(t^\alpha)P(b) + mh(1-t^\alpha)P\left(\frac{a}{m^2}\right)\right).$$

Multiplying with $t^{\tau-1}$, and then integrating over $[0, 1]$, one can get

$$(2.8)\quad h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(ta+m(1-t)b\right)dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(tb+(1-t)\frac{a}{m}\right)dt$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\left(P(a)\int_0^1 t^{\tau-1}g\left(ta+m(1-t)b\right)h(t^\alpha)dt + mP(b)\int_0^1 t^{\tau-1}g\left(ta+m(1-t)b\right)\right.$$

$$\times h(1-t^\alpha)dt) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(P(b)\int_0^1 t^{\tau-1}g\left(tb+(1-t)\frac{a}{m}\right)h(t^\alpha)dt\right.$$
$$\left.+mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1}g\left(tb+(1-t)\frac{a}{m}\right)h(1-t^\alpha)dt\right).$$

Setting $ta+m(1-t)b=x$, that is $t=\frac{mb-x}{mb-a}$ and $tb+(1-t)\frac{a}{m}=y$, that is $t=\frac{y-\frac{a}{m}}{b-\frac{a}{m}}$ in integrals on the left hand side of the above inequality (2.8) we get

$$\frac{\Gamma(\tau+1)}{(mb-a)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{a+}P\left(mb\right)+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{b-}P\left(\frac{a}{m}\right)\right) \tag{2.9}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\left(P(a)\int_0^1 t^{\tau-1}g\left(ta+m(1-t)b\right)h(t^\alpha)dt + mP(b)\int_0^1 t^{\tau-1}g\left(ta+m(1-t)b\right)\right.$$
$$\times h(1-t^\alpha)dt) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(P(b)\int_0^1 t^{\tau-1}g\left(tb+(1-t)\frac{a}{m}\right)h(t^\alpha)dt\right.$$
$$\left.+mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1}g\left(tb+(1-t)\frac{a}{m}\right)h(1-t^\alpha)dt\right).$$

Replacing g with M in the above inequality yields the second required inequality. □

Next, we use the inequality (2.1) to define a new class of functions. By making a substitution, for a strictly monotone continuous function F, $x=F(X)$ and $y=F(Y)$ in (2.1), we obtain the following inequality:

$$\begin{aligned} & f\left(\lambda F(X)+m(1-\lambda)F(Y)\right) \\ \leq & h(\lambda^\alpha)f(F(X))g(F(X))+mh(1-\lambda^\alpha)f(F(Y))g(F(Y)). \end{aligned} \tag{2.10}$$

Replacing f with $f(F^{-1})$ and g with $g(F^{-1})$, the following inequality is obtained:

$$f\left(F^{-1}\left(\lambda F(X)+m(1-\lambda)F(Y)\right)\right)\leq h(\lambda^\alpha)f(X)g(X)+mh(1-\lambda^\alpha)f(Y)g(Y). \tag{2.11}$$

This leads to the following definition.

Definition 2.2. Let h be a non-negative function on $J\subset\mathbb{R}$, $(0,1)\subset J$, $h\neq 0$ and let g be a positive function on $I\subset\mathbb{R}$. Furthermore, let α, $m\in(0,1]$. A function $f:I\to\mathbb{R}$ is said to be quasi $F-(h,g;\alpha-m)$-convex function if it is non-negative and if

$$f(F^{-1}(\lambda F(a)+(1-\lambda)F(mb))\leq h(\lambda^\alpha)f(a)g(a)+mh(1-\lambda^\alpha)f(b)g(b) \tag{2.12}$$

provided $F:I\to\mathbb{R}$ is strictly monotone, where $\lambda\in[0,1]$, $a,b\in I$.

In the following we give the version of Theorem 2.1 for quasi F-$(g,h;\alpha-m)$-convex function.

Theorem 2.2. *Let f be quasi F-$(g,h;\alpha-m)$-convex function as given in Definition 2.2. Then, the following inequality holds:*

$$f\left(F^{-1}\left(\frac{F(u)+mF(v)}{2}\right)\right) \tag{2.13}$$

$$\leq \frac{\Gamma(\tau+1)}{(mF(v)-F(u))^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{F(u)^{+}} P\left(F^{-1}\left(mF(v)\right)\right)\right.$$
$$\left. +m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{F(v)^{-}} P\left(F^{-1}\left(\frac{F(u)}{m}\right)\right)\right)$$
$$\leq M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right) \int_0^1 t^{\tau-1} h\left(t^{\alpha}\right) dt\right.$$
$$\left. +m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{u}{m^2}\right)\right) \int_0^1 t^{\tau-1} h\left(1-t^{\alpha}\right) dt\right\},$$

where $u, u \in I$, $M = \max_{x\in[u,v]}(g(x))$.

Proof. The inequality (2.13) can be obtained, first by setting $a = F(u)$, $b = F(v)$, where F is strictly monotone function, in (2.3) and then replacing f with $f(F^{-1})$ and g with $g(F^{-1})$ in the resulting inequality. □

Theorem 2.3. *Let* $f : I \subset (0,+\infty) \to \mathbb{R}$ *be quasi* $F-(g,h;\alpha-m)$*-convex function as given in Definition* 2.2.

(*i*) *If* $p > 0$, *then we have*

(2.14)
$$f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\tau+1)}{(mv^p-u^p)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{u^{p}+}(P\circ k)\left(mv^p\right)\right.$$
$$\left. +m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{v^{p}-}(P\circ k)\left(\frac{u^p}{m}\right)\right)$$
$$\leq M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right) \int_0^1 t^{\tau-1} h\left(t^{\alpha}\right) dt\right.$$
$$\left. +m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{u}{m^2}\right)\right) \int_0^1 t^{\tau-1} h\left(1-t^{\alpha}\right) dt\right\},$$

where $k(t) = t^{\frac{1}{p}}$, $t \in [u^p, mv^p]$, $u, v \in I$.

(*ii*) *If* $p < 0$, *then we have*

(2.15)
$$f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\tau+1)}{(u^p-mv^p)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{u^{p}-}(P\circ k)\left(mv^p\right)\right.$$
$$\left. +m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{v^{p}+}(P\circ k)\left(\frac{u^p}{m}\right)\right)$$
$$\leq M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right) \int_0^1 t^{\tau-1} h\left(t^{\alpha}\right) dt\right.$$
$$\left. +m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{u}{m^2}\right)\right) \int_0^1 t^{\tau-1} h\left(1-t^{\alpha}\right) dt\right\},$$

where $k(t) = t^{\frac{1}{p}}$, $t \in [mv^p, u^p]$.

Proof. Let $F(t) = t^p$, $p \neq 0$. Then, for $p > 0$, F is strictly increasing and if $p < 0$, then F is strictly decreasing on I. Hence, by setting this power function for $p > 0$ the inequality (2.14) required in (i) can be achieved, and for $p<0$ the inequality (2.15) required in (ii) can be achieved. □

Remark 2.1. By choosing $g(x)=\exp(-\eta x)$ in Theorem 2.3, one can obtain [1, Theorem 1].

Theorem 2.4. *Let $f: I\subset(0,+\infty)\to\mathbb{R}$ be quasi $F-(g,h;\alpha-m)$-convex function as given in Definition 2.2. Then, the following inequalities hold*

$$\begin{aligned}(2.16)\quad f\left(\sqrt{uv^m}\right)\leq&\frac{\Gamma(\tau+1)}{(\log(\frac{v^m}{u}))^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\log u^+}P\left(v^m\right)\right.\\&\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\log v^-}P\left(u^{\frac{1}{m}}\right)\right)\\\leq&M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\int_0^1 t^{\tau-1}h\left(t^\alpha\right)dt\right.\\&\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(1-t^\alpha\right)dt\right\},\end{aligned}$$

where $M=\max_{x\in[u,v]}(g(x))$, $u,v\in I$.

Proof. Since $\log(\cdot)$ is a strictly increasing function on $(0,+\infty)$, then by setting $F(t)=\log t$ in (2.13), and with some computations the required inequality can be obtained. □

In the next theorem we state and prove another Hermite-Hadamard type inequality for Riemann-Liouville fractional integrals.

Theorem 2.5. *Let $f: I\subset(0,+\infty)\to\mathbb{R}$ be $(g,h;\alpha-m)$-convex function as defined in Definition 2.1. Then, we have*

(2.17)
$$\begin{aligned}f\left(\frac{a+mb}{2}\right)\leq&\frac{2^\tau\Gamma(\tau+1)}{(mb-a)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\{\frac{a+mb}{2}\}^+}P\left(mb\right)\right.\\&\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\{\frac{a+mb}{2m}\}^-}P\left(\frac{a}{m}\right)\right)\\\leq&M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\&\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(b)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},\end{aligned}$$

where $\tau>0$, $a,b\in I$, $a<b$ and $M=\max_{x\in[a,b]}(g(x))$.

Proof. By setting $x=\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b$ and $y=\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}$ in (2.4), we get

$$\begin{aligned}(2.18)\qquad f\left(\frac{a+mb}{2}\right)\leq&h\left(\frac{1}{2^\alpha}\right)P\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)\\&+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)P\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right).\end{aligned}$$

Multiplying the above inequality by $t^{\tau-1}$ on both sides and integrating over $[0,1]$, we have

$$\begin{aligned}(2.19)\quad f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\tau-1}dt \leq & h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt\\ &+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)dt.\end{aligned}$$

The first inequality in (2.3) can be achieved by using substitutions $x=\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b$ and $y=\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}$ in the integrals appearing on right hand side of the above inequality. On the other hand, using (2.1) on the right hand side of (2.5), one can obtain the inequality:

$$\begin{aligned}(2.20)\quad & h\left(\frac{1}{2^\alpha}\right)P\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)P\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)\\ \leq & h\left(\frac{1}{2^\alpha}\right)g\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)\left(h\left(\left(\frac{t}{2}\right)^\alpha\right)P(a)+mh\left(1-\left(\frac{t}{2}\right)^\alpha\right)P(b)\right)\\ &+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)\\ &\times\left(h\left(\left(\frac{t}{2}\right)^\alpha\right)P(b)+mh\left(1-\left(\frac{t}{2}\right)^\alpha\right)P\left(\frac{a}{m^2}\right)\right).\end{aligned}$$

Multiplying with $t^{\tau-1}$, and then integrating over $[0,1]$, one can get

$$\begin{aligned}(2.21)\quad & h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt\\ &+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}P\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)dt\\ \leq & h\left(\frac{1}{2^\alpha}\right)\left(P(a)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\ &\left.+mP(b)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right)\\ &+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(P(b)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\ &\left.+mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right).\end{aligned}$$

Setting $\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b=x$, and $\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}=y$, in integrals on the left hand side of the above inequality (2.8), we get

$$\begin{aligned}(2.22)\quad & \frac{2^\tau\Gamma(\tau+1)}{(mb-a)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\left\{\frac{a+mb}{2}\right\}^+}P(mb)+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\left\{\frac{a+mb}{2m}\right\}^-}P\left(\frac{a}{m}\right)\right)\\ \leq & h\left(\frac{1}{2^\alpha}\right)\left(P(a)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\ &\left.+mP(b)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right)\end{aligned}$$

$$+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(P(b)\int_0^1 t^{\tau-1}\ g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.$$
$$\left.+mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)\ h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right).$$

Replacing g with M in the above inequality yields the second required inequality. □

In the following we give the version of Theorem 2.5 for quasi F-$(g,h;\alpha-m)$-convex function.

Theorem 2.6. *The following inequality holds for quasi F-$(g,h;\alpha-m)$-convex function defined in Definition 2.2:*

$$\begin{aligned}
(2.23)\qquad & f\left(F^{-1}\left(\frac{F(u)+mF(v)}{2}\right)\right)\\
\le&\frac{\Gamma(\tau+1)}{(mF(v)-F(u))^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\left\{\frac{F(u)+mF(v)}{2}\right\}^+}P\left(F^{-1}\left(mF(v)\right)\right)\right.\\
&\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\left\{\frac{F(u)+mF(v)}{2m}\right\}^-}P\left(F^{-1}\left(\frac{F(u)}{m}\right)\right)\right)\\
\le&M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\
&\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},
\end{aligned}$$

where $M=\max_{x\in[u,v]}\left(g(x)\right)$, $u,v\in I$.

Proof. The inequality (2.13) can be obtained, first by setting $a=F(u)$, $b=F(v)$, where F is strictly monotone function, in (2.17) and then replacing f with $f(F^{-1})$ and g with $g(F^{-1})$ in the resulting inequality. □

Theorem 2.7. *Let f be quasi F-$(g,h;\alpha-m)$-convex function as given in Definition 2.2. Then, the following inequalities hold.*

(*i*) *If $p>0$, then we have*

$$\begin{aligned}
& f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right)\\
\le&\frac{\Gamma(\tau+1)}{(mv^p-u^p)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\left\{\frac{u^p+mv^p}{2}\right\}^+}(P\circ k)\left(mv^p\right)\right.\\
&\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\left\{\frac{u^p+mv^p}{2m}\right\}^-}(P\circ k)\left(\frac{u^p}{m}\right)\right)\\
\le&M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\
&\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},
\end{aligned}$$

where $k(t)=t^{\frac{1}{p}}$, $t\in[u^p,mv^p]$, $u,v\in I$.

(ii) If $p<0$, *then we have*

$$\begin{aligned}
&f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right)\\
\leq&\frac{\Gamma(\tau+1)}{(u^p-mv^p)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\left\{\frac{u^p+mv^p}{2}\right\}^-}(P\circ k)(mv^p)\right.\\
&\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\left\{\frac{u^p+mv^p}{2m}\right\}^+}(P\circ k)\left(\frac{u^p}{m}\right)\right)\\
\leq&M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\
&\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},
\end{aligned}$$

where $k(t)=t^{\frac{1}{p}}$, $t\in[mv^p,u^p]$, $u,v\in I$.

Proof. The proof is similar to that of Theorem 2.3. □

Remark 2.2. By choosing $g(x)=\exp(-\eta x)$ in Theorem 2.5, one can obtain [1, Theorem 2].

Theorem 2.8. *Let* f *be quasi* F*-*$(g,h;\alpha-m)$*-convex function as given in Definition* 2.2. *Then, the following inequalities hold*

$$\begin{aligned}
f\left(\sqrt{uv^m}\right)\leq&\frac{\Gamma(\tau+1)}{\left(\log\left(\frac{v^m}{u}\right)\right)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{\log\sqrt{uv^m}+}P\left(v^m\right)\right.\\
&\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{\log\sqrt{u^{\frac{1}{m}}v}-}P\left(u^{\frac{1}{m}}\right)\right)\\
\leq&M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\right.\\
&+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)\\
&\left.\times\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},
\end{aligned}\tag{2.24}$$

where $M=\max_{x\in[u,v]}(g(x))$, $u,v\in I$.

Proof. Since $\log(\cdot)$ is a strictly increasing function on $(0,+\infty)$, then by setting $F(t)=\log t$ in (2.17), and with some computations the required inequality can be obtained. □

3. Fejér-Hadamard Type Inequalities for Fractional Riemann-Liouville Integrals

In the following we present Fejér-Hadamard type inequalities.

Theorem 3.1. *Under the assumptions of Theorem* 2.1, *moreover, we have* $f\left(\frac{a+mb-x}{m}\right) = f(x)$, $g\left(\frac{a+mb-x}{m}\right) = g(x)$. *Then, the following inequalities hold:*

(3.1)
$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) I_{a+}^{\tau} p\,(mb)\\
\leq&\frac{\Gamma(\tau+1)}{(mb-a)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I_{a+}^{\tau}(f\cdot p\cdot g)\,(mb) + m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I_{b-}^{\tau}(f\cdot p\cdot g)\left(\frac{a}{m}\right)\right)\\
\leq& M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(a) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(b)\right)\int_0^1 t^{\tau-1} p\,(ta+m(1-t)b)\, h\,(t^{\alpha})\, dt\right.\\
&\left. + m\left(h\left(\frac{1}{2^{\alpha}}\right) f(b) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1} p\,(ta+m(1-t)b)\, h\,(1-t^{\alpha})\, dt\right\},
\end{aligned}$$

where $M = \max_{x\in[a,b]}\left(g(x)\right)$, $a, b \in I$.

Proof. Multiplying the inequality (2.5) with $t^{\tau-1}p\,(ta+m(1-t)b)$ on both sides and integrating over $[0,1]$, we get

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\tau-1} p\,(ta+m(1-t)b)\, dt\\
\leq& h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1}(f.g.p)\,(ta+m(1-t)b)\, dt\\
&+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1} P\left(tb+(1-t)\frac{a}{m}\right) p\,(ta+m(1-t)b)\, dt.
\end{aligned} \tag{3.2}$$

Let $ta+m(1-t)b = z$. Then, $t = \frac{mb-z}{mb-a}$ and $tb+(1-t)\frac{a}{m} = \frac{a+mb-z}{m}$, also $P\left(\frac{a+mb-z}{m}\right) = P(z)$, using these facts one can obtain the first inequality of (3.1).

On the other hand multiplying (2.7) with $t^{\tau-1}p\,(ta+m(1-t)b)$, and then integrating over $[0,1]$, one can get

$$\begin{aligned}
&h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1}(f\cdot p\cdot g)\,(ta+m(1-t)b)\, dt\\
&+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1} P\left(tb+(1-t)\frac{a}{m}\right) p\,(ta+m(1-t)b)\, dt\\
\leq& h\left(\frac{1}{2^{\alpha}}\right)\left(P(a)\int_0^1 t^{\tau-1}(p\cdot g)\,(ta+m(1-t)b)\, h(t^{\alpha}) dt\right.\\
&\left.+ mP(b)\int_0^1 t^{\tau-1}(p\cdot g)\,(ta+m(1-t)b)\, h(1-t^{\alpha}) dt\right)\\
&+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(P(b)\int_0^1 t^{\tau-1} g\left(tb+(1-t)\frac{a}{m}\right) p\,(ta+m(1-t)b)\, h(t^{\alpha}) dt\right.\\
&\left.+ mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1} g\left(tb+(1-t)\frac{a}{m}\right) p\,(ta+m(1-t)b)\, h(1-t^{\alpha}) dt\right).
\end{aligned} \tag{3.3}$$

By setting $ta + m(1-t)b = z$, and using $P\left(\frac{a+mb-z}{m}\right) = P(z)$, $g\left(\frac{a+mb-z}{m}\right) = g(z)$ in integrals on the left hand side of the above inequality (3.10), we get

$$\begin{aligned}
(3.4)\qquad & \frac{\Gamma(\tau+1)}{(mb-a)^\tau}\left(h\left(\frac{1}{2^\alpha}\right) I^\tau_{a^+}(f\cdot p\cdot g)(mb) + m^{\tau+1} h\left(\frac{2^\alpha-1}{2^\alpha}\right) I^\tau_{b^-}(f\cdot p\cdot g)\left(\frac{a}{m}\right)\right)\\
\leq & h\left(\frac{1}{2^\alpha}\right)\left(P(a)\int_0^1 t^{\tau-1}(p\cdot g)(ta+m(1-t)b)\,h(t^\alpha)dt\right.\\
& \left.+ mP(b)\int_0^1 t^{\tau-1}(p\cdot g)(ta+m(1-t)b)\,h(1-t^\alpha)dt\right)\\
& + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(P(b)\int_0^1 t^{\tau-1} g\left(tb+(1-t)\frac{a}{m}\right) p\,(ta+m(1-t)b)\,h(t^\alpha)dt\right.\\
& \left.+ mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1} g\left(tb+(1-t)\frac{a}{m}\right) p\,(ta+m(1-t)b)\,h(1-t^\alpha)dt\right).
\end{aligned}$$

Replacing g with M in the above inequality yields the second required inequality. ☐

Theorem 3.2. *Under the assumptions of Theorem* 3.1*, the following Fejér-Hadamard type inequality holds for quasi F-$(g,h;\alpha-m)$-convex functions:*

(3.5)

$$\begin{aligned}
& f\left(F^{-1}\left(\frac{F(u)+mF(v)}{2}\right)\right) I^\tau_{F(u)^+} p\left(F^{-1}(mF(v))\right)\\
\leq & \frac{\Gamma(\tau+1)}{(mF(v)-F(u))^\tau}\left(h\left(\frac{1}{2^\alpha}\right) I^\tau_{F(u)^+}(f\cdot p\cdot g)\left(F^{-1}(mF(v))\right)\right.\\
& \left.+ m^{\tau+1} h\left(\frac{2^\alpha-1}{2^\alpha}\right) I^\tau_{F(v)^-}(f\cdot p\cdot g)\left(F^{-1}\left(\frac{F(u)}{m}\right)\right)\right)\\
\leq & M\left\{\left(h\left(\frac{1}{2^\alpha}\right) f(u) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f(v)\right)\right.\\
& \times \int_0^1 t^{\tau-1} p\left(F^{-1}(tF(u)+m(1-t)F(v))\right) h\left(t^\alpha\right) dt + m\left(h\left(\frac{1}{2^\alpha}\right) f(v)\right.\\
& \left.\left.+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\frac{u}{m^2}\right)\right)\int_0^1 t^{\tau-1} p\left(F^{-1}(tF(u)+m(1-t)F(v))\right) h\left(1-t^\alpha\right) dt\right\},
\end{aligned}$$

where $M = \max_{x\in[u,v]}(g(x))$, $u, v \in I$.

Proof. The inequality (3.5) can be obtained, first by setting $a = F(u)$, $b = F(v)$, where F is strictly monotone function, in (3.1) and then replacing f with $f(F^{-1})$, p with $p(F^{-1})$ and g with $g(F^{-1})$ in the resulting inequality. ☐

Theorem 3.3. *Let the assumptions of Theorem* 3.2 *hold.*

(*i*) *If* $p > 0$*, then we have*

$$(3.6)\qquad f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right) I^\tau_{u^p+} p\circ k\,(mv^p)$$

$$\leq \frac{\Gamma(\tau+1)}{(mv^p-u^p)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{u^p+}((f\cdot p\cdot g)\circ k)\,(mv^p)\right.$$
$$\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{v^p-}((f\cdot p\cdot g)\circ k)\left(\frac{u^p}{m}\right)\right)$$
$$\leq M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\right.$$
$$\times\int_0^1 t^{\tau-1}p\circ k\,(tu^p+m(1-t)v^p)\,h\,(t^\alpha)\,dt$$
$$+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)$$
$$\left.\times\int_0^1 t^{\tau-1}p\circ k\,(tu^p+m(1-t)v^p)\,h\,(1-t^\alpha)\,dt\right\},$$

where $k(t)=t^{\frac{1}{p}}$, $t\in[u^p,mv^p]$, $u,v\in I$.

(ii) If $p<0$, *then we have*

$$f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right)I^\tau_{u^p-}p\circ k\,(mv^p) \tag{3.7}$$
$$\leq \frac{\Gamma(\tau+1)}{(u^p-mv^p)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)I^\tau_{u^p-}((f\cdot p\cdot g)\circ k)\,(mv^p)\right.$$
$$\left.+m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I^\tau_{v^p+}((f\cdot p\cdot g)\circ k)\left(\frac{u^p}{m}\right)\right)$$
$$\leq M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\right.$$
$$\times\int_0^1 t^{\tau-1}p\circ k\,(tu^p+m(1-t)v^p)\,h\,(t^\alpha)\,dt$$
$$+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u}{m^2}\right)\right)$$
$$\left.\times\int_0^1 t^{\tau-1}p\circ k\,(tu^p+m(1-t)v^p)\,h\,(1-t^\alpha)\,dt\right\},$$

where $k(t)=t^{\frac{1}{p}}$, $t\in[mv^p,u^p]$, $u,v\in I$.

Proof. By setting $F(t)=t^p$ in (3.5), inequalities (3.6) and (3.7) can be obtained for $p<0$ and $p>0$, respectively. □

Remark 3.1. By choosing $g(x)=\exp(-\eta x)$ in Theorem 3.3, one can obtain [1, Theorem 1].

Theorem 3.4. *Let the assumptions of Theorem* 3.1 *hold, also let* $f\left(\frac{a+mb-x}{m}\right)=f(x)$, $g\left(\frac{a+mb-x}{m}\right)=g(x)$. *Then, the following inequalities hold:*

(3.8)

$$f\left(\frac{a+mb}{2}\right)I^\tau_{\left\{\frac{a+mb}{2}\right\}^+}p\,(mb)$$

$$\begin{aligned}
\leq & \frac{\Gamma(\tau+1)}{(mb-a)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right)I^{\tau}_{\left\{\frac{a+mb}{2}\right\}^{+}}(f\cdot p\cdot g)(mb)\right.\\
&\left.+m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)I^{\tau}_{\left\{\frac{a+mb}{2m}\right\}^{-}}(f\cdot p\cdot g)\left(\frac{a}{m}\right)\right)\\
\leq & M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right)f(a)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)f(b)\right)\int_0^1 t^{\tau-1}p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right.\\
&+m\left(h\left(\frac{1}{2^{\alpha}}\right)f(b)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)f\left(\frac{a}{m^2}\right)\right)\\
&\left.\times\int_0^1 t^{\tau-1}p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right\},
\end{aligned}$$

where $\tau>0$, $a,b\in I$, $a<b$, and $M=\max_{x\in[a,b]}(g(x))$.

Proof. Multiplying the inequality (2.18) with $t^{\tau-1}p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)$ on both sides and integrating over $[0,1]$, we get

$$\begin{aligned}
(3.9)\qquad & f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\tau-1}p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt\\
\leq & h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1}(f.g.p)\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt\\
&+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1}P\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt.
\end{aligned}$$

Let $\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b=z$. Then, $\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}=\frac{a+mb-z}{m}$, also $P\left(\frac{a+mb-z}{m}\right)=P(z)$, using these facts one can obtain the first inequality of (3.8).
On the other hand multiplying (2.20) with $t^{\tau-1}p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)$, and then integrating over $[0,1]$, one can get

(3.10)

$$\begin{aligned}
& h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1}(f\cdot p\cdot g)\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt\\
&+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 t^{\tau-1}P\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)dt\\
\leq & h\left(\frac{1}{2^{\alpha}}\right)\left(P(a)\int_0^1 t^{\tau-1}(p\cdot g)\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right.\\
&\left.+mP(b)\int_0^1 t^{\tau-1}(p\cdot g)\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\\
&\times\left(P(b)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right.\\
&\left.+mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right).
\end{aligned}$$

By setting $\frac{t}{2}a + m\left(1-\frac{t}{2}\right)b = z$, and using $f\left(\frac{a+mb-z}{m}\right) = P(z)$, $g\left(\frac{a+mb-z}{m}\right) = g(z)$ in integrals on the left hand side of the above inequality (3.10), we get

(3.11)

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) I^{\tau}_{\left\{\frac{a+mb}{2}\right\}^+} p\,(mb) \\
\leq&\frac{\Gamma(\tau+1)}{(mb-a)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{a+mb}{2}\right\}^+}(f\cdot p\cdot g)\,(mb)\right. \\
&\left.+m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{a+mb}{2m}\right\}^-}(f\cdot p\cdot g)\left(\frac{a}{m}\right)\right) \\
\leq&h\left(\frac{1}{2^{\alpha}}\right)\left(P(a)\int_0^1 t^{\tau-1}(p\cdot g)\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right. \\
&\left.+mP(b)\int_0^1 t^{\tau-1}(p\cdot g)\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right) \\
&+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(P(b)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right. \\
&\left.+mP\left(\frac{a}{m^2}\right)\int_0^1 t^{\tau-1}g\left(\frac{t}{2}b+\left(1-\frac{t}{2}\right)\frac{a}{m}\right)p\left(\frac{t}{2}a+m\left(1-\frac{t}{2}\right)b\right)h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right).
\end{aligned}$$

Replacing g with M in the above inequality yields the second required inequality. □

Theorem 3.5. *Let the assumptions of Theorem* 3.4 *hold. Then, the following Fejér-Hadamard type inequality holds for quasi F-$(g,h;\alpha-m)$-convex functions:*

$$\begin{aligned}
(3.12)\qquad &f\left(F^{-1}\left(\frac{F(u)+mF(v)}{2}\right)\right) I^{\tau}_{\left\{\frac{F(u)+mF(v)}{2}\right\}^+} p\left(F^{-1}\left(mF(v)\right)\right) \\
\leq&\frac{\Gamma(\tau+1)}{(mF(v)-F(u))^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{F(u)+mF(v)}{2}\right\}^+}(f\cdot p\cdot g)\left(F^{-1}\left(mF(v)\right)\right)\right. \\
&\left.+m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{F(u)+mF(v)}{2m}\right\}^-}(f\cdot p\cdot g)\left(F^{-1}\left(\frac{F(u)}{m}\right)\right)\right) \\
\leq&M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right)f(u)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)f(v)\right)\right. \\
&\times\int_0^1 t^{\tau-1}p\left(F^{-1}\left(\frac{t}{2}F(u)+m\left(1-\frac{t}{2}\right)F(v)\right)\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt \\
&+ m\left(h\left(\frac{1}{2^{\alpha}}\right)f(v)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)f\left(\frac{u}{m^2}\right)\right) \\
&\left.\times\int_0^1 t^{\tau-1}p\left(F^{-1}\left(\frac{t}{2}F(u)+m\left(1-\frac{t}{2}\right)F(v)\right)\right)h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right\},
\end{aligned}$$

where $M = \max_{x\in[u,v]}(g(x))$, $u,v\in I$.

Proof. The inequality (3.12) can be obtained, first by setting $a = F(u)$, $b = F(v)$, where F is strictly monotone function, in (3.8) and then replacing f with $f(F^{-1})$, p with $p(F^{-1})$ and g with $g(F^{-1})$ in the resulting inequality. □

Theorem 3.6. *The following Fejér-Hadamard type inequality holds for quasi F-$(g, h; \alpha - m)$-convex function defined in Definition 2.2.*

(i) If $p > 0$, then we have:

$$\begin{aligned}
(3.13)\qquad & f\left(\left(\frac{u^p + mv^p}{2}\right)^{\frac{1}{p}}\right) I^{\tau}_{\left\{\frac{u^p+mv^p}{2}\right\}^+} p \circ k\left(mv^p\right) \\
\leq & \frac{\Gamma(\tau+1)}{(mv^p - u^p)^{\tau}} \left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{u^p+mv^p}{2}\right\}^+} \left(\left(f \cdot p \cdot g\right) \circ k\right)\left(mv^p\right)\right. \\
& \left. + m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{u^p+mv^p}{2m}\right\}^-} \left(\left(f \cdot p \cdot g\right) \circ k\right)\left(\frac{u^p}{m}\right)\right) \\
\leq & M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right)\right. \\
& \times \int_0^1 t^{\tau-1} p \circ k\left(\frac{t}{2}u^p + m\left(1 - \frac{t}{2}\right)v^p\right) h\left(\frac{t^{\alpha}}{2^{\alpha}}\right) dt. \\
& + m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{u}{m^2}\right)\right) \\
& \left. \times \int_0^1 t^{\tau-1} p \circ k\left(\frac{t}{2}u^p + m\left(1 - \frac{t}{2}\right)v^p\right) h\left(1 - \left(\frac{t}{2}\right)^{\alpha}\right) dt\right\},
\end{aligned}$$

where $k(t) = t^{\frac{1}{p}}$, $t \in [u^p, mv^p]$.

(ii) If $p < 0$, then we have

$$\begin{aligned}
(3.14)\qquad & f\left(\left(\frac{u^p + mv^p}{2}\right)^{\frac{1}{p}}\right) I^{\tau}_{u^p-} p \circ k\left(mv^p\right) \\
\leq & \frac{\Gamma(\tau+1)}{(u^p - mv^p)^{\tau}} \left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{u^p-} \left(\left(f \cdot p \cdot g\right) \circ k\right)\left(mv^p\right)\right. \\
& \left. + m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{v^p+} \left(\left(f \cdot p \cdot g\right) \circ k\right)\left(\frac{u^p}{m}\right)\right) \\
\leq & M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right)\right. \\
& \times \int_0^1 t^{\tau-1} p \circ k\left(\frac{t}{2}u^p + m\left(1 - \frac{t}{2}\right)v^p\right) h\left(\left(\frac{t}{2}\right)^{\alpha}\right) dt \\
& + m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{u}{m^2}\right)\right) \\
& \left. \times \int_0^1 t^{\tau-1} p \circ k\left(\frac{t}{2}u^p + m\left(1 - \frac{t}{2}\right)v^p\right) h\left(1 - \left(\frac{t}{2}\right)^{\alpha}\right) dt\right\},
\end{aligned}$$

where $k(t) = t^{\frac{1}{p}}$, $t \in [mv^p, u^p]$.

Proof. See the proof of Theorem 2.3. □

Remark 3.2. By choosing $g(x) = \exp(-\eta x)$ in Theorem 3.6, one can obtain [1, Theorem 4].

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