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BROWDER-WEYL PERTURBATION THEORY AND SOME RESULTS OF ESSENTIAL SPECTRA IN BANACH ALGEBRAS

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ABSTRACT. The aim of this manuscript is to extend established results in Fredholm and perturbation theory within Banach spaces to the framework of Fredholm perturbation theory in a Banach algebra with respect to a bounded homomorphism. The findings facilitate the characterization of the stability of Browder, Weyl, left-right Browder, and left-right Weyl spectra. Moreover, we present results concerning the sums of two Browder (Weyl, left-right Browder, left-right Weyl) elements in a Banach algebra with respect to a bounded homomorphism.

1. INTRODUCTION AND PRELIMINARIES

The use of perturbation theory is a pivotal tool in spectral theory, especially in exploring the stability and characterization of the essential spectra of linear operators in a Banach space. Many mathematicians are engaged in such studies, and a substantial body of literature on this subject is readily available (see, for example, [1, 3–5, 7–11, 13–15, 17, 19–21]). In recent years, a new line of research on Fredholm and perturbation theory within Banach algebras has expanded the scope of results in the spectral theory of linear operators. The study of Fredholm theory in relation to Banach algebra homomorphisms has been a focal point for several decades, with numerous spectral properties being thoroughly investigated. Some foundational contributions in this area are credited to F. Abdmouleh, H. Khlif, and I. Walha in [2], H. Baklouti in [6], and T. Mouton and H. Raubenheimer in [18], offering a comprehensive body of work containing many intriguing results.

Key words and phrases. Perturbation theory, Banach algebra homomorphisms, left-right Browder and Weyl spectra.

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To the best of our knowledge, except for the work of F. Abdmouleh et al in [2], no work has been published on the analysis of the stability problem of perturbed left-right Browder and left-right Weyl elements in the Banach algebra relative to a bounded homomorphism. Hence, our interest in this paper is to develop some stability results of left-right Browder and left-right Weyl spectrum of elements in a Banach algebra relative to a bounded homomorphism. The primary tool employed in our investigations involves an elegant utilization of the definition of left-right Browder, left-right Weyl, and left-right Browder elements introduced in Definition 1.1, by employing the method linked to a Banach algebra homomorphism, we aim to establish the spectrum stability of the sum of Weyl, Browder, Weyl left-right and Browder left-right elements when subjected to slight disturbances.

The purpose of this section is to recall some fundamental concepts and results from Fredholm theory in Banach algebra. To this end, let us consider $T : \mathcal{U} \to \mathcal{U}$ as a homomorphism of a Banach algebra, where $\mathcal{U}$ is a complex Banach algebra with identity $1 \neq 0$. We define the set of invertible elements in $\mathcal{U}$ by

$$Inv(\mathcal{U}) = \{t \in \mathcal{U} : t^{-1} \in \mathcal{U}\}.$$

Moreover, we define the set of left (resp. right) invertible elements in $\mathcal{U}$ as

$$Inv^l(\mathcal{U}) = \{t \in \mathcal{U} \ : \ \text{there exists } t' \in \mathcal{U} \text{ such that } t't = 1\},$$

resp.,

$$Inv^r(\mathcal{U}) = \{t \in \mathcal{U} \ : \ \text{there exists } t' \in \mathcal{U} \text{ such that } tt' = 1\}.$$

For $t \in \mathcal{U}$, we define the resolvent (resp. spectrum) set of elements in $\mathcal{U}$ denoted by $\rho(t)$ (resp. $\sigma(t)$) by

$$\rho(t) = \{\lambda \in \mathbb{C} \ : \ \lambda - t \in Inv(\mathcal{U})\} \quad (\text{resp. } \sigma(t) = \mathbb{C}\backslash\rho(t)).$$

The left resolvent (resp. spectrum) set of elements in $\mathcal{U}$ are defined as

$$\rho^l(t) = \{\lambda \in \mathbb{C} \ : \ \lambda - t \in Inv^l(\mathcal{U})\} \quad (\text{resp. } \sigma^l(t) = \mathbb{C}\backslash\rho^l(t)).$$

Furthermore, we define the right resolvent (resp. spectrum) set of elements in $\mathcal{U}$ by

$$\rho^r(t) = \{\lambda \in \mathbb{C} \ : \ \lambda - t \in Inv^r(\mathcal{U})\} \quad (\text{resp. } \sigma^r(t) = \mathbb{C}\backslash\rho^r(t)).$$

Below, we present the definitions and propositions introduced by F. Abdmouleh, H. Khlif and I. Walha in [2], as well as those by R. Harte in [12].

Definition 1.1. Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism.

(1) $t \in \mathcal{U}$ is a Weyl element if and only if $t \in Inv(\mathcal{U}) \uplus T^{-1}(0)$, which means that t is the sum of an invertible element and a one whose image is zero; W_T denotes the set of Weyl elements of $\mathcal{U}$ relative to T [12, p. 431 (i)].

(2) $t \in \mathcal{U}$ is a Browder element if and only if $t \in Inv(\mathcal{U}) \uplus T^{-1}(0) = \{b + c : b \in Inv(\mathcal{U}),\ c \in T^{-1}(0) \ \text{ and } bc = cb\}$ the commuting sum of an invertible and an image-zero; B_T denotes the set of Browder elements of $\mathcal{U}$ relative [12, p. 431 (ii)].

(3) $t \in \mathcal{U}$ is a left Weyl element if and only if $t \in Inv^l(\mathcal{U}) \uplus T^{-1}(0)$, which means that t is the sum of a left invertible element and a one whose image is zero; W_T^l denotes the set of left Weyl elements of $\mathcal{U}$ relative to T [2, Definition 2.4 (iii)].

(4) $t \in \mathcal{U}$ is a right Weyl element if and only if $t \in Inv^r(\mathcal{U}) \uplus T^{-1}(0)$, which means that t is the sum of a left invertible element and a one whose image is zero; W_T^r denotes the set of right Weyl elements of $\mathcal{U}$ relative to T [2, Definition 2.4 (iv)].

(5) $t \in \mathcal{U}$ is a left Browder element if and only if $t \in Inv^l(\mathcal{U}) \uplus T^{-1}(0) = \{b + c \ : b \in Inv^l(\mathcal{U}), c \in T^{-1}(0) \ \text{ and } bc = cb\}$, which means that t is the sum of a left invertible element and a one whose image is zero; B_T^l denotes the set of left Browder elements of $\mathcal{U}$ relative to T [2, Definition 2.4 (v)].

(6) $t \in \mathcal{U}$ is a right Browder element if and only if $t \in Inv^r(\mathcal{U}) \uplus T^{-1}(0) = \{b + c \ : b \in Inv^r(\mathcal{U}), c \in T^{-1}(0) \ \text{ and } bc = cb\}$, which means that t is the sum of a right invertible element and a one whose image is zero; B_T^r denotes the set of right Browder elements of $\mathcal{U}$ relative to T [2, Definition 2.4 (vi)].

Proposition 1.1. *Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism. Then, we have*

(1) [12, (1.4)]

$$Inv(\mathcal{U}) \Rightarrow B_T \Rightarrow W_T \Rightarrow F_T,$$

(2) [2, Remark 2.1]

$$Inv^l(\mathcal{U}) \Rightarrow B_T^l \Rightarrow W_T^l \Rightarrow F_T^l,$$
$$Inv^r(\mathcal{U}) \Rightarrow B_T^r \Rightarrow W_T^r \Rightarrow F_T^r.$$

In order to study the essential spectra of an element $t \in \mathcal{U}$, we define the Weyl spectrum, the left and right Weyl spectra, as well as the Browder spectrum and the left and right Browder spectra of an element $t \in \mathcal{U}$, resp., by

$$\begin{aligned}
\sigma_{W_T}(t) &= \{\lambda \in \mathbb{C} \text{ such that } \lambda - t \notin W_T\} = \mathbb{C} \backslash W_T,\\
\sigma_{W_T}^l(t) &= \{\lambda \in \mathbb{C} \text{ such that } \lambda - t \notin W_T^l\} = \mathbb{C} \backslash W_T^l,\\
\sigma_{W_T}^r(t) &= \{\lambda \in \mathbb{C} \text{ such that } \lambda - t \notin W_T^r\} = \mathbb{C} \backslash W_T^r,\\
\sigma_{B_T}(t) &= \{\lambda \in \mathbb{C} \text{ such that } \lambda - t \notin B_T\} = \mathbb{C} \backslash B_T,\\
\sigma_{B_T}^l(t) &= \{\lambda \in \mathbb{C} \text{ such that } \lambda - t \notin B_T^l\} = \mathbb{C} \backslash B_T^l,\\
\sigma_{B_T}^r(t) &= \{\lambda \in \mathbb{C} \text{ such that } \lambda - t \notin B_T^r\} = \mathbb{C} \backslash B_T^r.
\end{aligned}$$

In the following, we introduce the sets of Browder and Weyl perturbations, as well as left and right Browder and Weyl perturbations in a Banach algebra relative to a bounded homomorphism T through the following definition.

Definition 1.2. Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism and let $t \in \mathcal{U}$.

(1) An element p is called Weyl perturbation if $t + p \in \mathcal{W}_T$ whenever $t \in \mathcal{W}_T$. The set of Weyl perturbations is denoted by $Pr(\mathcal{W}_T)$.

(2) An element p is called left Weyl perturbation if $t+p\in\mathcal{W}_T^l$ whenever $t\in\mathcal{W}_T^l$. The set of left Weyl perturbations is denoted by $Pr(\mathcal{W}_T^l)$.
(3) An element p is called right Weyl perturbation if $t+p\in\mathcal{W}_T^r$ whenever $t\in\mathcal{W}_T^r$. The set of right Weyl perturbations is denoted by $Pr(\mathcal{W}_T^r)$.
(4) An element p is called Browder perturbation if $t+p\in\mathcal{B}_T$ whenever $t\in\mathcal{B}_T$. The set of Browder perturbations is denoted by ${}^{com}Pr(\mathcal{B}_T)$.
(5) An element p is called left Browder perturbation if $t+p\in\mathcal{B}_T^l$ whenever $t\in\mathcal{B}_T^l$. The set of left Browder perturbations is denoted by ${}^{com}Pr(\mathcal{B}_T^l)$.
(6) An element p is called right Browder perturbation if $t+p\in\mathcal{B}_T^r$ whenever $t\in\mathcal{B}_T^r$. The set of right Browder perturbations is denoted by ${}^{com}Pr(\mathcal{B}_T^r)$.

In [16], A. Lebow and M. Schechter proved that, under certain conditions, the perturbation class of a subset of the algebra $\mathcal{U}$ is a closed ideal. The motivation for the work in Section 2 stems from the observation that the perturbation class of an open semigroup in a Banach algebra is a closed two-sided ideal. In many applications, we deal with two elements t and t' of a Banach algebra such that $t'=t+s$, where s is not necessarily included in the sets of perturbation. However, we possess information about $(\mu-t)^{-1}-(\mu-t')^{-1}$ for $\mu\in\rho(t)\cap\rho(t')$. In Section 3, we prove that, under certain conditions, $\sigma_{W_T}(t)=\sigma_{W_T}(t')$, and the same holds for the other essential spectra. Finally, in Section 4, we investigate the left-right Browder and left-right Weyl spectra of the sum of two elements in a Banach algebra relative to a bounded homomorphism T, utilizing the left-right Browder and left-right Weyl spectra of each of the two elements, where their products are elements in the previously mentioned perturbation sets.

2. Weyl and Browder Perturbation Theory in a Banach Algebra Relative to a Bounded Homomorphism

The purpose of this section is to establish certain properties of the sets of Browder, Weyl, left-right Browder, and left-right Weyl perturbations. We investigate how these perturbation classes relate to the essential spectra of elements in a Banach algebra under the framework of a bounded homomorphism. By analyzing these properties, we aim to contribute to a deeper understanding of the stability and spectral characteristics of elements in this algebraic setting.

Theorem 2.1. *Let $T:\mathcal{U}\to\mathcal{U}$ be a Banach algebra homomorphism. Then, the following hold.*

(1) *Let $t\in\mathcal{B}_T\cap\mathcal{U}$ and commuting with all elements in $\mathcal{B}_T^l$ (resp. $\mathcal{B}_T^r$) $p\in{}^{com}Pr(\mathcal{B}_T^l)$ (resp. ${}^{com}Pr(\mathcal{B}_T^r)$). Then,*
   (i) *$tp\in{}^{com}Pr(\mathcal{B}_T^l)$ (resp. ${}^{com}Pr(\mathcal{B}_T^r)$),*
   (ii) *$pt\in{}^{com}Pr(\mathcal{B}_T^l)$ (resp. ${}^{com}Pr(\mathcal{B}_T^r)$).*
(2) *Let $t\in\mathcal{B}_T\cap\mathcal{U}$ and commuting with all elements in $\mathcal{B}_T$, $p\in{}^{com}Pr(\mathcal{B}_T)$. Then,*
   (i) *$tp\in{}^{com}Pr(\mathcal{B}_T)$,*
   (ii) *$pt\in{}^{com}Pr(\mathcal{B}_T)$.*

(3) *Let* $t \in \mathcal{W}_T \cap \mathcal{U}$ *and* $p \in Pr(\mathcal{W}_T^l)$ *(resp.* $Pr(\mathcal{W}_T^r)$*). Then,*
  (i) $tp \in Pr(\mathcal{W}_T^l)$ *(resp.* $Pr(\mathcal{W}_T^r)$*),*
  (ii) $pt \in Pr(\mathcal{W}_T^l)$ *(resp.* $Pr(\mathcal{W}_T^r)$*).*
(4) *Let* $t \in \mathcal{W}_T \cap \mathcal{U}$ *and* $p \in Pr(\mathcal{W}_T)$*. Then,*
  (i) $tp \in Pr(\mathcal{W}_T)$*,*
  (ii) $pt \in Pr(\mathcal{W}_T)$*.*

Proof. (1) **(i)** To initiate the proof. Let t be an element in $\mathcal{B}_T \cap \mathcal{U}$. This implies the existence of $b \in Inv(\mathcal{U})$ and $c \in T^{-1}(0)$ such that $t = b + c$ and $bc = cb$. Now, let $t' \in \mathcal{B}_T^l$, and express $t' = b' + c'$, where $b' \in Inv^l(\mathcal{U})$, $c' \in T^{-1}(0)$, and $b'c' = c'b'$.

It is evident that $b^{-1}t' \in \mathcal{B}_T^l$, as shown by $b^{-1}t' = b^{-1}b' + b^{-1}c$, where $b^{-1}b' \in Inv^l(\mathcal{U})$ and $b^{-1}c \in T^{-1}(0)$. Moreover,

$$(b^{-1}b')(b^{-1}c) = b^{-1}b'cb^{-1} = b^{-1}cb'b^{-1} = (b^{-1}c)(b^{-1}b'),$$

demonstrating that $b^{-1}t' + p \in \mathcal{B}_T^l$. According to [2, Theorem 3.2 (i)], we have $t(b^{-1}t' + p) \in \mathcal{B}_T^l$, i.e., $(b + c)b^{-1}t' + tp = t' + cb^{-1}t' + tp \in \mathcal{B}_T^l$. As $cb^{-1}t' \in T^{-1}(0)$ and t, t', and p commute, we can observe that

$$(t' + pt)(t'b^{-1}c) = t't'b^{-1}c + ptt'b^{-1}c = t'b^{-1}ct' + t'b^{-1}cpt = (t'b^{-1}c)(t' + pt).$$

Thus, $t' + tp \in \mathcal{B}_T^l$, implying that $tp \in {}^{com}Pr(\mathcal{B}_T^l)$.

The prove of **(ii)** is the same as that of **(i)**.

(2) The validity of (2) follows directly from (1).

(3) **(i)** Consider $t \in \mathcal{W}_T \cap \mathcal{U}$. This implies the existence of $b \in Inv(\mathcal{U})$ and $c \in T^{-1}(0)$ such that $t = b + c$. Now, let $t' \in \mathcal{W}_T^l$ be expressed as $t' = b' + c'$, where $b' \in Inv^l(\mathcal{U})$ and $c \in T^{-1}(0)$.

It's clear that $b^{-1}t' \in \mathcal{W}_T^l$, $b^{-1}t' = b^{-1}b' + b^{-1}c$, implying $b^{-1}t' + p \in \mathcal{W}_T^l$. From [2, Theorem 3.2 (i)], we have $t(b^{-1}t' + p) \in \mathcal{W}_T^l$, which translates to $(b + c)b^{-1}t' + tp = t' + cb^{-1}t' + tp \in \mathcal{W}_T^l$. As $cb^{-1}t' \in T^{-1}(0)$, it follows that $t' + tp \in \mathcal{W}_T^l$. Consequently, $tp \in Pr(\mathcal{W}_T^l)$.

(ii) Next, we prove that $pt \in Pr(\mathcal{W}_T^l)$.

Clearly, $t'b^{-1} \in \mathcal{W}_T^l$, and therefore, $t'b^{-1} + p \in \mathcal{W}_T^l$. Applying [2, Theorem 3.2 (i)], we get $(t'b^{-1} + p)t \in \mathcal{W}_T^l$, i.e., $(t'b^{-1} + p)(b + c) = t' + t'b^{-1}c + pt \in \mathcal{W}_T^l$. Since $t'b^{-1}c \in T^{-1}(0)$, it follows that $t' + pt \in \mathcal{W}_T^l$. Consequently, $pt \in Pr(\mathcal{W}_T^l)$.

(4) The proof of (4) is a direct consequence of (3). □

Theorem 2.2. $Pr(\mathcal{W}_T^l)$*,* $Pr(\mathcal{W}_T^r)$*,* $Pr(\mathcal{W}_T)$*,* ${}^{com}Pr(\mathcal{B}_T^l)$*,* ${}^{com}Pr(\mathcal{B}_T^r)$ *and* ${}^{com}Pr(\mathcal{B}_T)$ *are closed two-sided ideals of* $\mathcal{U}$*.*

Proof. We prove first that $Pr(\mathcal{W}_T^l)$ is a two-sided ideal of $\mathcal{U}$ and the rest will be done the same way. Let $p \in Pr(\mathcal{W}_T^l)$ and $t \in \mathcal{U}$.

Let $t' \in \mathcal{W}_T \cap \mathcal{U}$, by [2, Theorem 3.1 (i), Theorem 3.2 (i)], t can be represented as the sum of two elements in $\mathcal{W}_T \cap \mathcal{U}$: $t = (t - \lambda t') + \lambda t'$. Using the previous theorem we have $tp(t - \lambda t')p + \lambda t'p \in Pr(\mathcal{W}_T^l)$. Also, in the same way $pt \in Pr(\mathcal{W}_T^l)$. Thus, $Pr(\mathcal{W}_T^l)$ is a two-sided ideal of $\mathcal{U}$.

To prove now that $Pr(\mathcal{W}_T^l)$ is a closed subset of $\mathcal{U}$, we consider a sequence $(p_n)_{n\in\mathbb{N}} \subseteq Pr(\mathcal{W}_T^l)$, such that $p_n \to p$ for $\|\cdot\|_{\mathcal{U}}$. Let $t \in \mathcal{W}_T^l$. By [2, Theorem 3.1 (i)] there exists an $\eta > 0$ such that for all $t \in \mathcal{U}$, with $\|t'\| < \eta$ we have $t + t' \in \mathcal{W}_T^l$. Since $p_n \to p$ then there exists N such that $\|p - p_N\|_{\mathcal{U}} < \eta$, therefore $t + p - p_N \in \mathcal{W}_T^l$. Since $p_N \in Pr(\mathcal{W}_T^l)$ then $t + p - p_N + p_N = t + p \in \mathcal{W}_T^l$. Thus, $p \in Pr(\mathcal{W}_T^l)$ and $Pr(\mathcal{W}_T^l)$ is a closed subset of $\mathcal{U}$.

We use the same reasoning as above to prove that $Pr(\mathcal{W}_T^r)$, $Pr(\mathcal{W}_T)$, ${}^{com}Pr(\mathcal{B}_T^l)$, ${}^{com}Pr(\mathcal{B}_T^r)$ and ${}^{com}Pr(\mathcal{B}_T)$ are closed two-sided ideals too. ☐

Remark 2.1. Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism. Then, the following statements hold

(1) $Pr(\mathcal{W}_T) \subset Pr(\mathcal{W}_T^l)$,
(2) $Pr(\mathcal{W}_T) \subset Pr(\mathcal{W}_T^r)$.

Indeed, let $p \in Pr(\mathcal{W}_T)$ and $t \in \mathcal{W}_T^l$, then there exist $b \in Inv^l(\mathcal{U})$, $c \in T^{-1}(0)$ such that $t = b + c$. Since $b \in Inv^l(\mathcal{U})$, so there exist $d \in \mathcal{U}$ such that $db = 1$. Hence, $dt = db + dc = 1 + dc \in \mathcal{W}_T^l$. Since $Pr(\mathcal{W}_T)$ is an ideal of $\mathcal{U}$, then $dp \in Pr(\mathcal{W}_T)$. Therefore, $dp + dt \in \mathcal{W}_T$. From [2, Theorem 3.2 (i)], we have that $t(dt + dp) \in \mathcal{W}_T^l$.

On the other hand,

$$t(dt + dp) =(b + c)(dt + dp) = (1 + cd)(t + p) = t + p + cd(t + p) \in \mathcal{W}_T^l.$$

Since $cd(t + p) \in T^{-1}(0)$, then $t + p \in \mathcal{W}_T^l$. Thus, $p \in Pr(\mathcal{W}_T^l)$, consequently, $Pr(\mathcal{W}_T) \subset Pr(\mathcal{W}_T^l)$.

The proof is the same for the other inclusions.

3. Disturbances Arising from a Weyl-Browder Spectrum

The aim of this section is to study the essential spectra of an element $t \in \mathcal{U}$ and to explore their stability. We will analyze how perturbations affect the Weyl and Browder spectra, focusing on the conditions under which these spectra remain invariant. This investigation is crucial for understanding the behavior of spectral properties in the context of perturbations within a Banach algebra.

Theorem 3.1. *Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism. Assume that $\gamma \in \rho(t)$. Then, the following statements hold.*

(1) *For $\lambda \neq \gamma$,*

$$(\lambda - t) \in \mathcal{W}_T \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \mathcal{W}_T.$$

(2) *For $\lambda \neq \gamma$,*

$$(\lambda - t) \in \mathcal{W}_T^l \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \mathcal{W}_T^l.$$

(3) *For $\lambda \neq \gamma$,*

$$(\lambda - t) \in \mathcal{W}_T^r \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \mathcal{W}_T^r.$$

(4) *For* $\lambda \neq \gamma$,

$$(\lambda - t) \in \mathcal{B}_T \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \mathcal{B}_T.$$

(5) *For* $\lambda \neq \gamma$,

$$(\lambda - t) \in \mathcal{B}_T^l \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \mathcal{B}_T^l.$$

(6) *For* $\lambda \neq \gamma$,

$$(\lambda - t) \in \mathcal{B}_T^r \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \mathcal{B}_T^r.$$

Proof. Since $\gamma \in \rho(t)$, we have $(\gamma - t)^{-1} \in \mathcal{U}$. To establish (1), let's initially emphasize an equality that will be utilized repeatedly:

$$(\lambda - t)(\gamma - t)^{-1} = (\gamma - \lambda)\Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big]. \tag{3.1}$$

Now, suppose $\lambda - t \in \mathcal{W}_T$. This implies the existence of $b \in Inv(\mathcal{U})$, $c \in T^{-1}(0)$ such that $\lambda - t = b + c$. Applying (3.1), we obtain

$$\begin{aligned}(\gamma - \lambda)^{-1} - (\gamma - t)^{-1} =& (\gamma - \lambda)^{-1}(\gamma - t)^{-1}(\lambda - t)\\ =& \underbrace{(\gamma - \lambda)^{-1}(\gamma - t)^{-1}b}_{Inv(\mathcal{U})} + \underbrace{(\gamma - \lambda)^{-1}(\gamma - t)^{-1}c}_{T^{-1}(0)}.\end{aligned}$$

Thus, $\Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big] \in \mathcal{W}_T$. Consequently,

$$(\lambda - t) \in \mathcal{W}_T \Rightarrow \Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big] \in \mathcal{W}_T.$$

To establish the converse, assume $\Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big] \in \mathcal{W}_T$. This implies $\Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big] = b + c$, where $b \in Inv(\mathcal{U})$ and $c \in T^{-1}(0)$. Utilizing (3.1), we derive

$$\begin{aligned}\lambda - t =& (\gamma - t)(\gamma - \lambda)\Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big]\\ =& \underbrace{(\gamma - t)(\gamma - \lambda)b}_{Inv(\mathcal{U})} + \underbrace{(\gamma - t)(\gamma - \lambda)c}_{T^{-1}(0)}.\end{aligned}$$

Thus, $\lambda - t \in \mathcal{W}_T$. Consequently,

$$\Big[(\gamma - \lambda)^{-1} - (\gamma - t)^{-1}\Big] \in \mathcal{W}_T \Rightarrow (\lambda - t) \in \mathcal{W}_T.$$

The proof of (2), and (3) is the same as (1).

(4) To demonstrate this, it suffices to establish in (1) that $(\gamma - t)^{-1}b$ and $(\gamma - t)^{-1}c$ commute. Indeed, the following hold. As b and c commute, the same holds for $(\gamma - t)$, b, and c. Thus,

$$\begin{aligned}b(\gamma - t)^{-1} - (\gamma - t)^{-1}b =& (\gamma - t)^{-1}(\gamma - t)\Big[b(\gamma - t)^{-1} - (\gamma - t)^{-1}b\Big]\\ =& (\gamma - t)^{-1}\Big[(\gamma - t)b(\gamma - t)^{-1} - b\Big]\\ =& (\gamma - t)^{-1}\Big(b - b\Big) = 0.\end{aligned}$$

Consequently, $b(\gamma-t)^{-1}$ and $(\gamma-t)^{-1}b$ commute. Analogously, we can show that $c(\gamma-t)^{-1}$ and $(\gamma-t)^{-1}c$ commute. This implies that $(\gamma-t)^{-1}b$ and $(\gamma-t)^{-1}c$ commute, leading to

$$(\lambda-t)\in\mathcal{B}_T\Leftrightarrow(\gamma-\lambda)^{-1}-(\gamma-t)^{-1}\in\mathcal{B}_T.$$

The proof of (5) and (6) is the same as (4). ☐

Corollary 3.1. *Let $T:\mathcal{U}\to\mathcal{U}$ be a Banach algebra homomorphism. Assume that $0\in\rho(t)$. Then, the following statements hold.*

(1) $(\lambda-t)\in\mathcal{W}_T\Leftrightarrow(\lambda^{-1}-t^{-1})\in\mathcal{W}_T$.
(2) $(\lambda-t)\in\mathcal{W}^l_T\Leftrightarrow(\lambda^{-1}-t^{-1})\in\mathcal{W}^l_T$.
(3) $(\lambda-t)\in\mathcal{W}^r_T\Leftrightarrow(\lambda^{-1}-t^{-1})\in\mathcal{W}^r_T$.
(4) $(\lambda-t)\in\mathcal{B}_T\Leftrightarrow(\lambda^{-1}-t^{-1})\in\mathcal{B}_T$.
(5) $(\lambda-t)\in\mathcal{B}^l_T\Leftrightarrow(\lambda^{-1}-t^{-1})\in\mathcal{B}^l_T$.
(6) $(\lambda-t)\in\mathcal{B}^r_T\Leftrightarrow(\lambda^{-1}-t^{-1})\in\mathcal{B}^r_T$.

Proof. The proof of this corollary follows immediately from the proof of the Theorem 3.1 (just take $\gamma=0$). ☐

Corollary 3.2. *Let $T:\mathcal{U}\to\mathcal{U}$ be a Banach algebra homomorphism. Assume that $0\in\rho(t)$. Then, the following statements hold.*

(1) $\lambda\in\sigma_{W_T}(t)\Leftrightarrow\lambda^{-1}\in\sigma_{W_T}(t^{-1})$.
(2) $\lambda\in\sigma^l_{W_T}(t)\Leftrightarrow\lambda^{-1}\in\sigma^l_{W_T}(t^{-1})$.
(3) $\lambda\in\sigma^r_{W_T}(t)\Leftrightarrow\lambda^{-1}\in\sigma^r_{W_T}(t^{-1})$.
(4) $\lambda\in\sigma_{B_T}(t)\Leftrightarrow\lambda^{-1}\in\sigma_{B_T}(t^{-1})$.
(5) $\lambda\in\sigma^l_{B_T}(t)\Leftrightarrow\lambda^{-1}\in\sigma^l_{B_T}(t^{-1})$.
(6) $\lambda\in\sigma^r_{B_T}(t)\Leftrightarrow\lambda^{-1}\in\sigma^r_{B_T}(t^{-1})$.

Theorem 3.2. *Let $T:\mathcal{U}\to\mathcal{U}$ be a Banach algebra homomorphism. If $0\in\rho(t)$, then the following statements hold.*

(1) $\sigma_{W_T}(t+p)=\sigma_{W_T}(t)$, *for all* $p\in Pr(\mathcal{W}_T)$.
(2) $\sigma^l_{W_T}(t+p)=\sigma^l_{W_T}(t)$, *for all* $p\in Pr(\mathcal{W}^l_T)$.
(3) $\sigma^r_{W_T}(t+p)=\sigma^r_{W_T}(t)$, *for all* $p\in Pr(\mathcal{W}^r_T)$.
(4) $\sigma_{B_T}(t+p)=\sigma_{B_T}(t)$, *for all* $p\in {}^{com}Pr(\mathcal{B}_T)$.
(5) $\sigma^l_{B_T}(t+p)=\sigma^l_{B_T}(t)$, *for all* $p\in {}^{com}Pr(\mathcal{B}^l_T)$.
(6) $\sigma^r_{B_T}(t+p)=\sigma^r_{B_T}(t)$, *for all* $p\in {}^{com}Pr(\mathcal{B}^r_T)$.

Proof. Let $\lambda\neq0$. Since $0\in\rho(t)$, then we can write

$$t^{-1}(\lambda-t-p)=-\lambda(\lambda^{-1}-t^{-1})-t^{-1}p. \tag{3.2}$$

Let $\lambda\notin\sigma_{W_T}(t)$. Then, $\lambda-t\in\mathcal{W}_T$. Corollary 3.1 implies that $\lambda^{-1}-t^{-1}\in\mathcal{W}_T$. Since $p\in Pr(\mathcal{W}_T)$ and $Pr(\mathcal{W}^l_T)$ is a two-sided ideal of $\mathcal{U}$, then $t^{-1}p\in Pr(\mathcal{W}_T)$. Hence, $-\lambda(\lambda^{-1}-t^{-1})-t^{-1}p\in\mathcal{W}_T$, i.e., $t^{-1}(\lambda-t-p)\in\mathcal{W}_T$. Since $0\in\rho(t)$, then $(\lambda-t-p)\in\mathcal{W}_T$. Therefore, $\lambda\notin\sigma_{W_T}(t+p)$. Consequently, $\sigma^l_{W_T}(t+p)\subseteq\sigma^l_{W_T}(t)$.

To prove the other way around. Let $\lambda \notin \sigma_{W_T}(t+p)$, then $\lambda - t - p \in \mathcal{W}_T$. By (3.2), we have that $-\lambda(\lambda^{-1} - t^{-1}) - t^{-1}p \in \mathcal{W}_T$. Since $p \in Pr(\mathcal{W}_T)$ and $Pr(\mathcal{W}^l_T)$ is a two-sided ideal of $\mathcal{U}$, then $t^{-1}p \in Pr(\mathcal{W}_T))$. Hence, $\lambda^{-1} - t^{-1} \in \mathcal{W}_T$. Corollary 3.1 implies that $\lambda - t \in \mathcal{W}_T$. Therefore, $\lambda \notin \sigma_{W_T}(t)$. Consequently, $\sigma^l_{W_T}(t) \subseteq \sigma^l_{W_T}(t+p)$.

The prove of (2) and (3) is the same as that of (1).

(4) To show this, it is enough to show in (1) that $t^{-1}(\lambda - t - p)$ and $(\lambda - t - p)t^{-1}$ are commutating, in fact the following hold. Since $p(\lambda^{-1} - t^{-1})$ and $(\lambda^{-1} - t^{-1})p$ are commutating, so the same for $t^{-1}p$ and pt^{-1}. Hence, the result follows.

The prove of (5) and (6) is the same as that of (4). ☐

Lemma 3.1. *Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism. Let $t \in \mathcal{U}$ and $\lambda \in \mathbb{C}$ such that $(\lambda - t)^{-1} \in \mathcal{U}$. Then,*

$$s = tp + q,\ p, q \in \mathcal{X} \Leftrightarrow (\lambda - t)^{-1}s \in \mathcal{X}$$

and

$$s = pt + q,\ p, q \in \mathcal{X} \Leftrightarrow s(\lambda - t)^{-1} \in \mathcal{X},$$

for all $\mathcal{X} \in \left\{Pr(\mathcal{W}_T), Pr(\mathcal{W}^l_T), Pr(\mathcal{W}^r_T), {}^{com}Pr(\mathcal{B}_T), {}^{com}Pr(\mathcal{B}^l_T), {}^{com}Pr(\mathcal{B}^r_T)\right\}$.

Proof. Assume $s = tp + q$, where $p, q \in \mathcal{X}$. Given that $\mathcal{X}$ is an ideal of $\mathcal{U}$, we have

$$\begin{aligned}(\lambda - t)^{-1}s =&(\lambda - t)^{-1}(tp + q)\\ =&(\lambda - t)^{-1}[-(\lambda - t)p + \lambda p + q]\\ =&-p + \lambda(\lambda - t)^{-1}p + (\lambda - t)^{-1}q \in \mathcal{X}.\end{aligned}$$

This implies that $(\lambda - t)^{-1}s \in \mathcal{X}$.

Now, let's prove the reverse implication. Suppose $(\lambda - t)^{-1}s = p \in \mathcal{X}$, then

$$s = (\lambda - t)p = -tp + \lambda p = tp' + q', \quad \text{where } p', q' \in \mathcal{X}.$$

☐

Theorem 3.3. *Let $T : \mathcal{U} \to \mathcal{U}$ be a Banach algebra homomorphism. If $t, t' \in \mathcal{U}$, then the following hold.*

(1) *If, for some $\mu \in \rho(t) \cap \rho(t')$, $(\mu - t)^{-1} - (\mu - t')^{-1} \in Pr(\mathcal{W}_T)$, then*

$$\sigma_{\mathcal{W}_T}(t) = \sigma_{\mathcal{W}_T}(t').$$

(2) *If, for some $\mu \in \rho(t) \cap \rho(t')$, $(\mu - t)^{-1} - (\mu - t')^{-1} \in Pr(\mathcal{W}^l_T)$, then*

$$\sigma^l_{\mathcal{W}_T}(t) = \sigma^l_{\mathcal{W}_T}(t').$$

(3) *If, for some $\mu \in \rho(t) \cap \rho(t')$, $(\mu - t)^{-1} - (\mu - t')^{-1} \in Pr(\mathcal{W}^r_T)$, then*

$$\sigma^r_{\mathcal{W}_T}(t) = \sigma^r_{\mathcal{W}_T}(t').$$

(4) *If, for some $\mu \in \rho(t) \cap \rho(t')$, $(\mu - t)^{-1} - (\mu - t')^{-1} \in {}^{com}Pr(\mathcal{B}_T)$, then*

$$\sigma_{\mathcal{B}_T}(t) = \sigma_{\mathcal{B}_T}(t').$$

(5) *If, for some* $\mu \in \rho(t) \cap \rho(t')$, $(\mu - t)^{-1} - (\mu - t')^{-1} \in {}^{com}Pr(\mathcal{B}^l_T)$, *then*

$$\sigma^l_{\mathcal{B}_T}(t) = \sigma^l_{\mathcal{B}_T}(t').$$

(6) *If, for some* $\mu \in \rho(t) \cap \rho(t')$, $(\mu - t)^{-1} - (\mu - t')^{-1} \in {}^{com}Pr(\mathcal{B}^r_T)$, *then*

$$\sigma^r_{\mathcal{B}_T}(t) = \sigma^r_{\mathcal{B}_T}(t').$$

Proof. (1) Suppose $\lambda \notin \sigma_{\mathcal{W}_T}(t)$; then, by Theorem 3.1, we deduce that $\lambda - t \in \mathcal{W}_T$. Additionally, for $\lambda \neq \mu$, we can apply Theorem 3.1 to obtain $(\mu-\lambda)^{-1} - (\mu-t)^{-1} \in \mathcal{W}_T$. Since $[(\mu - t)^{-1} - (\mu - t')^{-1}] \in Pr(\mathcal{W}_T)$, it follows that $(\mu - \lambda)^{-1} - (\mu - t')^{-1} \in \mathcal{W}_T$. Reapplying Theorem 3.1, we conclude that $(\lambda - t') \in \mathcal{W}_T$. Consequently, $\sigma_{\mathcal{W}_T}(t) = \sigma_{\mathcal{W}_T}(t')$.

The proof of (2), (3), (4), (5) and (6) is the same as (1). ☐

Theorem 3.4. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be a Banach algebra homomorphism. Let* $s = tp + q$, *where* $p, q \in Pr(\mathcal{W}_T)$ *(resp.* $Pr(\mathcal{W}^l_T)$, $Pr(\mathcal{W}^r_T)$, ${}^{com}Pr(\mathcal{B}_T)$, ${}^{com}Pr(\mathcal{B}^l_T)$, ${}^{com}Pr(\mathcal{B}^r_T)$*) such that* $r_{\sigma_{\mathcal{U}}}(\lambda - t)^{-1}s) < 1$ *for some* $\lambda \in \rho_{\mathcal{U}}(t)$, *then*

$$\sigma_{\mathcal{W}_T}(t + s) = \sigma_{\mathcal{W}_T}(t)$$

(resp. $\sigma^l_{\mathcal{W}_T}(t + s) = \sigma^l_{\mathcal{W}_T}(t)$, $\sigma^r_{\mathcal{W}_T}(t + s) = \sigma^r_{\mathcal{W}_T}(t)$, $\sigma_{\mathcal{B}_T}(t + s) = \sigma_{\mathcal{B}_T}(t)$, $\sigma^l_{\mathcal{B}_T}(t + s) = \sigma^l_{\mathcal{B}_T}(t)$, $\sigma^r_{\mathcal{B}_T}(t + s) = \sigma^r_{\mathcal{B}_T}(t)$*).*

Proof. Consider $s = tp + q$, where $p, q \in Pr(\mathcal{W}_T)$. Observe that $\lambda - t - s = (t - \lambda)[1 - (\lambda - t)^{-1}s]$, implying $(1 - (\lambda - t)^{-1}s) = (\lambda - t)^{-1}(\lambda - t - s)$. Given that $r\sigma_{\mathcal{U}}((\lambda - t)^{-1}s) < 1$, we can express $(\lambda - t - s)^{-1}$ as

$$(\lambda - t - s)^{-1} = \sum_{k=0}^{n}((\lambda - t)^{-1}s)^n(\lambda - t)^{-1} = (\lambda - t)^{-1} + \sum_{k=1}^{n}((\lambda - t)^{-1}s)^n(\lambda - t)^{-1}.$$

This yields

$$(\lambda - t - s)^{-1} - (\lambda - t)^{-1} = \sum_{k=1}^{n}((\lambda - t)^{-1}s)^n(\lambda - t)^{-1}.$$

By Lemma 3.1, since $s = tp + q$, with $p, q \in Pr(\mathcal{W}_T)$, and since $Pr(\mathcal{W}_T)$ is an ideal of $\mathcal{U}$, we conclude that $(\lambda - t - s)^{-1} - (\lambda - t)^{-1} \in Pr(\mathcal{W}_T)$. Utilizing the previous theorem, we then have $\sigma_{\mathcal{W}_T}(t + s) = \sigma_{\mathcal{W}_T}(t)$.

This equivalence also holds for $\sigma^l_{\mathcal{W}_T}(\cdot)$, $\sigma^r_{\mathcal{W}_T}(\cdot)$, $\sigma_{\mathcal{B}_T}(\cdot)$, $\sigma^l_{\mathcal{B}_T}(\cdot)$ and $\sigma^r_{\mathcal{B}_T}(\cdot)$. ☐

Theorem 3.5. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be a Banach algebra homomorphism. Let* $t \in \mathcal{U}$. *If* $t \in \mathcal{X}$, *then* $t^n \in \mathcal{X}$ *for every* $n \in \mathbb{N}$, *for all* $\mathcal{X} \in \left\{\mathcal{W}_T, \mathcal{W}^l_T, \mathcal{W}^r_T, \mathcal{B}_T, \mathcal{B}^l_T, \mathcal{B}^r_T\right\}$.

Proof. Let $t \in \mathcal{W}_T$. Then, there exist $b \in Inv(\mathcal{U})$ and $c \in T^{-1}(0)$ such that $t = b + c$. We will now prove that $t^n = b^n + x_n$, where $x_n \in T^{-1}(0)$, for every $n \in \mathbb{N}$. This is evident for $n = 1$. Suppose the statement holds for $k \in \mathbb{N}$. Then,

$$t^{k+1} = t^k t = (b^k + x_k)(b + c) = b^{k+1} + (b^k c + x_k b + x_k c).$$

Clearly, $b^k c + x_k b + x_k c \in T^{-1}(0)$. As a result, since $b^n \in Inv(\mathcal{U})$, $t^n \in \mathcal{W}_T$, for all $n \in \mathbb{N}$. The same argument holds for the other sets. ☐

Corollary 3.3. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be a Banach algebra homomorphism. Let* $t, s \in \mathcal{U}$. *If* $t \in \mathcal{X}$ *such that* $ts \in T^{-1}(0)$, *then* $(t+s)^n \in \mathcal{X}$ *for every* $n \in \mathbb{N}$, *for all* $\mathcal{X} \in \left\{\mathcal{W}_T, \mathcal{W}_T^l, \mathcal{W}_T^r, \mathcal{B}_T, \mathcal{B}_T^l, \mathcal{B}_T^r\right\}$.

Theorem 3.6. *Let* $\mathcal{U}$ *be a unital Banach algebra and let* $T : \mathcal{U} \to \mathcal{U}$ *be a homomorphism. Then, the following statements hold.*

(1) $\sigma_{W_T}(t)$ *and* $\sigma_{B_T}(t)$ *are regularities.*

(2) *If* $f : U \to \mathbb{C}$ *is analytic on a neighborhood* U *of* $\sigma(t)$ *and is non-constant on each connected component of* U, *then*

$$\sigma_{W_T}(f(t)) = f(\sigma_{W_T}(t)) \quad \textit{and} \quad \sigma_{B_T}(f(t)) = f(\sigma_{B_T}(t)).$$

(3) $\sigma_{W_T}(t)$ *and* $\sigma_{B_T}(t)$ *are closed subsets of* $\mathbb{C}$.

(4) *If* $t, s \in \mathcal{U}$, *then*

$$\sigma_{W_T}(ts) = \sigma_{W_T}(st) \quad \textit{and} \quad \sigma_{B_T}(ts) = \sigma_{B_T}(st).$$

(5) $\sigma_{W_T}(t)$ *is countable if and only if* $\sigma_{B_T}(t)$ *is countable.*

Proof. To begin with, recall that $W_T = Inv(\mathcal{U}) + T^{-1}(0)$ and

$$B_T = \{b + c : b \in Inv(\mathcal{U}),\ c \in T^{-1}(0),\ bc = cb\}.$$

(1) We first prove that W_T is a regularity.

(i) Let $t \in W_T$. Then $t = b + c$ with $b \in Inv(\mathcal{U})$ and $c \in T^{-1}(0)$. Since invertibility is stable under powers and $T^{-1}(0)$ is an ideal, it follows that t^n admits the same type of decomposition. Hence, $t^n \in W_T$ for every $n \in \mathbb{N}$.

Conversely, if $t^n \in W_T$ for some n, the spectral mapping property for invertibility implies that t must also admit such a decomposition. Hence, $t \in W_T$.

(ii) Let $a, b \in \mathcal{U}$ such that $ab \in W_T$. Write $ab = u + k$ with u invertible and $k \in T^{-1}(0)$. Using the classical identity

$$(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$$

whenever defined, one deduces that ba admits the same decomposition. Hence, $ba \in W_T$. Therefore, W_T is a regularity. The same argument, keeping the commutativity condition, shows that B_T is also a regularity.

(2) Since W_T and B_T are regularities, the general spectral theory of regularities yields that for any analytic function f defined on a neighbourhood of $\sigma(t)$ and non-constant on each connected component, we have

$$\sigma_{W_T}(f(t)) = f(\sigma_{W_T}(t))$$

and

$$\sigma_{B_T}(f(t)) = f(\sigma_{B_T}(t)).$$

(3) The spectrum associated with any regularity is closed. Hence, $\sigma_{W_T}(t)$ and $\sigma_{B_T}(t)$ are closed subsets of $\mathbb{C}$.

(4) Let $t, s \in \mathcal{U}$. Since regularity spectra satisfy the standard relation

$$\sigma_{\mathcal{R}}(ts) = \sigma_{\mathcal{R}}(st),$$

the conclusion follows for both W_T and B_T.

(5) Finally, for regularity spectra, countability is preserved under the functional calculus, hence the equivalence follows.

This completes the proof. □

4. Essential Cpectra of the Elements in a Banach Algebra Relative to a Bounded Homomorphism

The following theorem establishes a connection between the essential spectra of the sum of two elements in a Banach algebra and the essential spectra of each of these elements individually. This relationship is explored under the condition that their products belong to the sets of perturbations. By examining this connection, we aim to provide insights into the stability of essential spectra in the context of bounded homomorphisms.

Lemma 4.1. *Let $T:\mathcal{U}\to\mathcal{U}$ be a Banach algebra homomorphism and let $t,t'\in\mathcal{U}$. Then, the following assertions hold.*

1. *If $t,t'\in\mathcal{W}_T$, then $tt'\in\mathcal{W}_T$.*
2. *If $t,t'\in\mathcal{W}^l_T$, then $tt'\in\mathcal{W}^l_T$.*
3. *If $t,t'\in\mathcal{W}^r_T$, then $tt'\in\mathcal{W}^r_T$.*
4. *If $tt'\in\mathcal{W}_T$ such that $0\in\rho_{\mathcal{U}}(t)$ (resp. $0\in\rho_{\mathcal{U}}(t')$), then $t'\in\mathcal{W}_T$ (resp. $t\in\mathcal{W}_T$).*
5. *If $tt'\in\mathcal{W}^l_T$ such that $0\in\rho_{\mathcal{U}}(t)$, then $t'\in\mathcal{W}^l_T$.*
6. *If $tt'\in\mathcal{W}^r_T$ such that $0\in\rho_{\mathcal{U}}(t')$, then $t\in\mathcal{W}^r_T$.*

Proof. (1) Let $t,t'\in\mathcal{W}_T$. Then, there exist $b,b'\in Inv(\mathcal{U})$ and $c,c'\in T^{-1}(0)$ such that $t=b+c$ and $t'=b'+c'$. We have

$$tt'=(b+c)(b'+c')=\underbrace{bb'}_{Inv(\mathcal{U})}+\underbrace{bc'+cb'+cc'}_{T^{-1}(0)}=m+n,$$

where $m=bb'$ and $n=bc'+cb'+cc'$. From Definition 1.1 (1), we deduce that $tt'\in\mathcal{W}_T$. The assertions (2) and (3) can be established by using the same argument as in (1).

(4) Assume that $tt'\in\mathcal{W}_T$. Then, there exist $b\in Inv(\mathcal{U})$ and $c\in T^{-1}(0)$ such that $tt'=b+c$. Since $0\in\rho_{\mathcal{U}}(t)$ (resp. $0\in\rho_{\mathcal{U}}(t')$), we obtain $t'=t^{-1}b+t^{-1}c$ (resp. $t=bt'^{-1}+ct'^{-1}$). Hence, $t'\in\mathcal{W}_T$ (resp. $t\in\mathcal{W}_T$).

The proofs of (5) and (6) follow the same argument as in (4). □

Theorem 4.1. *Let $T:\mathcal{U}\to\mathcal{U}$ be a Banach algebra homomorphism. If $t,t'\in\mathcal{U}$, then the following assertions hold.*

(1) *If $tt'\in Pr(\mathcal{W}_T)$, then*

$$\sigma_{\mathcal{W}_T}(t+t')\subset[\sigma_{\mathcal{W}_T}(t)\cup\sigma_{\mathcal{W}_T}(t')].$$

Moreover, if $\lambda\in\rho_{\mathcal{U}}(t)$, then

$$\sigma_{\mathcal{W}_T}(t+t')=[\sigma_{\mathcal{W}_T}(t)\cup\sigma_{\mathcal{W}_T}(t')].$$

(2) *If* $tt' \in Pr(\mathcal{W}_T^l)$, *then*

$$\sigma^l_{\mathcal{W}_T}(t+t') \subset [\sigma^l_{\mathcal{W}_T}(t) \cup \sigma^l_{\mathcal{W}_T}(t')].$$

Moreover, if $\lambda \in \rho^l_{\mathcal{U}}(t)$, *then*

$$\sigma^l_{\mathcal{W}_T}(t+t') = [\sigma^l_{\mathcal{W}_T}(t) \cup \sigma^l_{\mathcal{W}_T}(t')].$$

(3) *If* $tt' \in Pr(\mathcal{W}_T^r)$, *then*

$$\sigma^r_{\mathcal{W}_T}(t+t') \subset [\sigma^r_{\mathcal{W}_T}(t) \cup \sigma^r_{\mathcal{W}_T}(t')].$$

Moreover, if $\lambda \in \rho^r_{\mathcal{U}}(t)$, *then*

$$\sigma^r_{\mathcal{W}_T}(t+t') = [\sigma^r_{\mathcal{W}_T}(t) \cup \sigma^r_{\mathcal{W}_T}(t')].$$

Proof. For $\lambda \in \mathbb{C}$. We have

$$(t-\lambda)(t'-\lambda) = tt' - \lambda(t+t'-\lambda). \tag{4.1}$$

Suppose $\lambda \notin [\sigma_{\mathcal{W}_T}(t) \cup \sigma_{\mathcal{W}_T}(t')]$. Then, $t-\lambda \in \mathcal{W}_T$ and $t'-\lambda \in \mathcal{W}_T$. From Eq. (4.1), it follows that $tt' - \lambda(t+t'-\lambda) \in \mathcal{W}_T$. Since $tt' \in Pr(\mathcal{W}_T)$, we conclude that $t+t'-\lambda \in \mathcal{W}_T$. Therefore, $\lambda \notin \sigma_{\mathcal{W}_T}(t+t')$. This proves the inclusion

$$\sigma_{\mathcal{W}_T}(t+t') \subset [\sigma_{\mathcal{W}_T}(t) \cup \sigma_{\mathcal{W}_T}(t')]. \tag{4.2}$$

To prove the reverse inclusion in (4.2), assume that $\lambda \notin \sigma_{\mathcal{W}_T}(t+t')$, which implies $t+t'-\lambda \in \mathcal{W}_T$. Applying (4.1), since $tt' \in Pr(\mathcal{W}_T)$, we have $(t-\lambda)(t'-\lambda) \in \mathcal{W}_T$. Since $\lambda \in \rho_{\mathcal{U}}(t)$, as per Lemma 4.1, we conclude that $t-\lambda \in \mathcal{W}_T$ and $t'-\lambda \in \mathcal{W}_T$. Thus, $\lambda \notin [\sigma_{\mathcal{W}_T}(t) \cup \sigma_{\mathcal{W}_T}(t')]$. Consequently,

$$[\sigma_{\mathcal{W}_T}(t) \cup \sigma_{\mathcal{W}_T}(t')] \subset \sigma_{\mathcal{W}_T}(t+t').$$

Therefore,

$$\sigma_{\mathcal{W}_T}(t+t') = [\sigma_{\mathcal{W}_T}(t) \cup \sigma_{\mathcal{W}_T}(t')].$$

(2) The proof is analogous to (1), replacing $Pr(\mathcal{W}_T)$ and $\sigma_{\mathcal{W}_T}(\cdot)$ with $Pr(\mathcal{W}_T^l)$ and $\sigma^l_{\mathcal{W}_T}(\cdot)$, resp., and applying part (*ii*) of Lemma 4.1.

(3) Similarly, the proof is analogous to (1), substituting $Pr(\mathcal{W}_T)$ and $\sigma_{\mathcal{W}_T}(\cdot)$ with $Pr(\mathcal{W}_T^r)$ and $\sigma^r_{\mathcal{W}_T}(\cdot)$, resp., and applying part (*iii*) of Lemma 4.1. ☐

Lemma 4.2. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be a Banach algebra homomorphism. Let* $t, t' \in \mathcal{U}$ *such* $tt' = t't$. *Then, the following assertions hold.*

(i) *If* $tt' \in \mathcal{B}_T$ *such that* $0 \in \rho_{\mathcal{U}}(t)$ *(resp.* $0 \in \rho_{\mathcal{U}}(t')$*), then* $t' \in \mathcal{B}_T$ *(resp.* $t \in \mathcal{B}_T$*).*

(ii) *If* $tt' \in \mathcal{B}_T^l$ *such that* $0 \in \rho_{\mathcal{U}}(t)$, *then* $t' \in \mathcal{B}_T^l$.

(iii) *If* $tt' \in \mathcal{B}_T^r$ *such that* $0 \in \rho_{\mathcal{U}}(t')$, *then* $t' \in \mathcal{B}_T^r$.

Proof. To demonstrate this, it suffices to establish in the proof of Lemma 4.2 that $t^{-1}b$ and $t^{-1}c$ commute (refer to (4) in the proof of Theorem 3.1). ☐

Theorem 4.2. *Let* $T : \mathcal{U} \to \mathcal{U}$ *be a Banach algebra homomorphism. If* $t, t' \in \mathcal{U}$ *such that* $tt' = t't$, *then the following hold.*

(1) *If* $tt' \in {}^{com}Pr(\mathcal{B}_T)$, *then*

$$\sigma_{\mathcal{B}_T}(t+t') \subset [\sigma_{\mathcal{B}_T}(t) \cup \sigma_{\mathcal{B}_T}(t')].$$

Moreover, if $\lambda \in \rho_{\mathfrak{U}}(t)$, *then*

$$\sigma_{\mathcal{B}_T}(t+t') = [\sigma_{\mathcal{B}_T}(t) \cup \sigma_{\mathcal{B}_T}(t')].$$

(2) *If* $tt' \in {}^{com}Pr(\mathcal{B}_T^l)$, *then*

$$\sigma^l_{\mathcal{B}_T}(t+t') \subset [\sigma^l_{\mathcal{B}_T}(t) \cup \sigma^l_{\mathcal{B}_T}(t')].$$

Moreover, if $\lambda \in \rho^l_{\mathfrak{U}}(t)$, *then*

$$\sigma^l_{\mathcal{B}_T}(t+t') = [\sigma^l_{\mathcal{B}_T}(t) \cup \sigma^l_{\mathcal{B}_T}(t')].$$

(3) *If* $tt' \in {}^{com}Pr(\mathcal{B}_T^r)$, *then*

$$\sigma^r_{\mathcal{B}_T}(t+t') \subset [\sigma^r_{\mathcal{B}_T}(t) \cup \sigma^r_{\mathcal{B}_T}(t')].$$

Moreover, if $\lambda \in \rho^r_{\mathfrak{U}}(t)$, *then*

$$\sigma^r_{\mathcal{B}_T}(t+t') = [\sigma^r_{\mathcal{B}_T}(t) \cup \sigma^r_{\mathcal{B}_T}(t')].$$

Proof. (1) To establish this, it suffices to demonstrate in the proof of Theorem 4.1 (1) that $(\lambda - t)^{-1}b$ and $(\lambda - t)^{-1}c$ commute (refer to (4) in the proof of Theorem 3.1).

(2) The proof follows a similar structure to (1) with the replacement of ${}^{com}Pr(\mathcal{B}_T)$ and $\sigma_{\mathcal{B}_T}(\cdot)$ by ${}^{com}Pr(\mathcal{B}_T^l)$ and $\sigma^l_{\mathcal{B}_T}(\cdot)$, resp., and the application of part (*ii*) of Lemma 4.2.

(3) Analogously, the proof mirrors (1) with the substitution of $Pr(\mathcal{B}_T)$ and $\sigma_{\mathcal{B}_T}(\cdot)$ by ${}^{com}Pr(\mathcal{B}_T^r)$ and $\sigma^r_{\mathcal{B}_T}(\cdot)$, resp., and the utilization of part (*iii*) of Lemma 4.2. ☐

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