Kragujevac Journal of Mathematics
Volume 51(1) (2027), Pages 23–33.

SOME CLASSES OF IMPLICATIVE IDEALS IN BI-ALGEBRAS

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Abstract. In this paper, we introduce the concepts of implicative, positive implicative, weak positive implicative and sub-implicative ideals in (right-distributive) BI-algebras and study their properties. Moreover, we analyze the relations between them and also between ideals and these newly introduced ideals in this class of logical algebras.

1. Introduction

A. Borumand Saeid et al. introduced in a new algebra, called a BI-algebra, which a generalization of both a (dual) implication algebra (in terms of the articles [1, 5]) and an implicative BCK-algebra (in the sense of the paper [8]), and they discussed the basic properties of BI-algebras and studied some ideals and congruence relations ([4]). S. S. Ahn et al. introduced the concepts of (normal) sub-algebras and (normal) ideals in BI-algebras. In addition to the previous paper, in this paper the authors considered both the design of congruences on BI-algebras and the construction of quotient BI-algebras as well as the properties of homomorphisms between BI-algebras. In the paper [11], the author registers an additional property of ideals in right-distributive BI-algebras.

This paper is a continuation of the papers [3, 4, 11] in the literal sense. A good practice in researching a logical algebra is to consider not only its determination and its relation to other logical algebras, but also to understand the architecture of its internal structure and its substructures, which is a widely held and accepted view of research.

Key words and phrases. BI-algebra, right distributive BI-algebra, ideal, implicative ideal, positive implicative ideal, weak positive implicative ideal, sub-implicative ideal.

2020 *Mathematics Subject Classification.* Primary: 06F35. Secondary: 03G25.

DOI

Received: October 13, 2024.

Accepted: May 20, 2025.

In this article, relying primarily on the algebraic aspects of the internal architecture of a logical algebra as well as its sub-structural components, we look at the possible determination of the concepts of implicative, positive implicative, weak positive implicative and sub-implicative ideals and analyze their basic properties in the class of (right-distributive) BI-algebras. Section 3, the central part of this article, is devoted to this topic. Moreover, we analyze the relations between them and also between ideals and these newly introduced ideals in this class of logical algebras. It has been shown, among other things, that the generally accepted determination of implicative ideals does not lead to a new substructure in BI-algebras. Implicative ideals in BI-algebras coincide with ideals in these algebras. It has also been shown that the standard determination of the concept of weak positive implicative ideals does not lead to a new substructure in BI-algebras, but coincides with the structure of positive implicative ideals. It is thus shown that the concepts of ideals, implicative ideals, positive implicative ideals and weak positive implicative ideals coincide in the case when the BI-algebra is right-distributive. However, the relationship between sub-implicative ideals and positive implicative ideals remains an open problem.

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is common in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Rightarrow$, and so on are labels for the logical functions of conjunction, disjunction, implication and so on. Parentheses in formulas are also used in the usual way. All formulas that appear in this paper are closed by a quantifier. If one of the formulas is open, then the variables in it should be considered as free variables. In addition to the above, the sign $=:$, when using $A =: B$, should be understood in the sense that the mark A is the abbreviation for the formula B.

In this text, to mark recognizable formulas, we will use as much as possible their standard abbreviations, which appear in a very well-known paper [6].

Definition 2.1 ([4], Definition 3.1)**.** An algebra $\mathfrak{A} =: (A, \cdot, 0)$ of type $(2, 0)$ is called a BI-algebra if the following holds:

(Re) $(\forall x \in A)(x \cdot x = 0)$,

(Im) $(\forall x, y \in A)(x \cdot (y \cdot x) = x)$.

A BI-algebra $\mathfrak{A}$ is said to be right distributive if the following

(DR) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z))$

is valid.

According to [4, Proposition 3.9], left-distributivity in this class of logical algebras is only possible in the trivial case if $\mathfrak{A} = (\{0\}, \cdot, 0)$.

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. We introduce a relation $\preccurlyeq$ on the set A by

$$(\forall x, y \in A)(x \preccurlyeq y \Leftrightarrow x \cdot y = 0).$$

We note that $\preccurlyeq$ is not a partially order set, but only reflexive. It is shown [4, Proposition 3.14] that if $\mathfrak{A}$ is a right-distributive BI-algebra, the induced relation $\preccurlyeq$ is a transitive relation ([4, Proposition 3.14]). Thus, if $\mathfrak{A}$ is a right-distributive BI algebra, then $\preccurlyeq$ is a quasi-order on A that is right-compatible with the operation in $\mathfrak{A}$ ([4, Proposition 3.12 (iv)]).

Some of the important properties of this class of logical algebras are given by the following two propositions.

Proposition 2.1 ([4], Proposition 3.7)**.** *Let* $\mathfrak{A} =: (A, \cdot, 0)$ *be a BI-algebra. Then,*
(M) $(\forall x \in A)(x \cdot 0 = x)$,
(L) $(\forall x \in A)(0 \cdot x = 0)$,
(iii) $(\forall x, y \in A)(x \cdot y = (x \cdot y) \cdot y)$,
(vi) $(\forall x, y, z \in A)(x \cdot y = z \Rightarrow (z \cdot y = z \wedge y \cdot z = y))$.

It is obvious that, according to (L), it holds
(1) $(\forall x \in A)(0 \preccurlyeq x)$.

The properties of this relation $\preccurlyeq$ in the right distributive BI-algebra $\mathfrak{A} = (A, \cdot, 0)$ are summarized in the following proposition.

Proposition 2.2 ([4], Proposition 3.12)**.** *Let* $\mathfrak{A} =: (A, \cdot, 0)$ *be a right distributive BI-algebra. Then, the following holds:*
(2) $(\forall x, y \in A)(y \cdot x \preccurlyeq y)$,
(3) $(\forall x, y \in A)((y \cdot x) \cdot x \preccurlyeq y)$,
(4) $(\forall x, y, z \in A)((x \cdot z) \cdot (y \cdot z) \preccurlyeq x \cdot y)$,
(5) $(\forall x, y \in A)(x \preccurlyeq y \Rightarrow x \cdot z \preccurlyeq y \cdot z)$,
(6) $(\forall x, y, z \in A)((x \cdot y) \cdot z \preccurlyeq x \cdot (y \cdot z))$,
(7) $(\forall x, y, z \in A)(x \cdot y = z \cdot y \Rightarrow (x \cdot z) \cdot y = 0)$.

It should be noted here that, in the general case, this relation $\preccurlyeq$ is not left compatible with the internal operation in any right distributive BI-algebra.

3. On Ideals and Implicative Ideals in BI-Algebras

This section is the central part of this paper. Here, within the BI-algebra environment, we consider some of the well-known classes of ideals in logical algebras, such as implicative, positive implicative, weak positive implicative and sub-implicative ideals. The section consists of five subsections. In the first, three new statements about ideals in (right-distributive) BI-algebras are proved. In Subsection 3.2 it was shown that every ideal in BI-algebras is an implicative ideal. Subsection 3.3 is devoted to the concept of positive implicative ideals and their properties. In this class of logical algebras, positive implicative ideals and weakly positive implicative ideals coincide (Subsection 3.4). In Subsection 3.5 the concept of sub-implicative ideals in BI-algebras is discussed and some of its basic properties are analyzed.

The notion of ideal in BI-algebras is determined by the following definition.

Definition 3.1 ([4], Definition 4.1)**.** A subset J of a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called an ideal of $\mathfrak{A}$ if the following holds:
(J0) $0 \in J$,
(J1) $(\forall x, y \in A)((x \cdot y \in J \wedge y \in J) \Rightarrow x \in J)$.

For an ideal J in a BI-algebra $\mathfrak{A}$ holds [4, Proposition 4.5]
(J2) $(\forall x, y \in A)((x \preccurlyeq y \wedge y \in J) \Rightarrow x \in J)$.

Example 3.1. Let $A = \{0, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then, $\mathfrak{A} =: (A, \cdot, 0)$ is a BI-algebra [4, Example 3.3]. Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$, $J_3 = \{0, c\}$, $J_5 = \{0, a, c\}$ are ideals in $\mathfrak{A}$. Subset $S_4 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a \in S_4$ and $c \cdot a = b \in S_4$ but $c \notin S_4$. Also, subset $S_6 = \{0, b, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c \in S_6$ and $a \cdot c = b \in J_6$, but $a \notin J_6$.

Example 3.2. Let $A = \{1, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then, $\mathfrak{A} =: (A, \cdot, 0)$ is a right distributive BI-algebra [4, Example 3.10 (i)]. The relation $\preccurlyeq$ is given by

$$\preccurlyeq = \{(0,0), (0,a), (0,b), (0,c), (a,a), (a,c), (b,b), (b,c), (c,c)\}.$$

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$ are ideals in $\mathfrak{A}$. Subset $S_3 = \{0, c\}$ is not an ideal in $\mathfrak{A}$, because, for example, we have $a \cdot c = 0 \in S_3$ and $c \in S_3$ but $a \notin S_3$. Subset $S_4 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a \in S_4$ and $c \cdot a = b \in S_4$ but $c \notin S_4$. Subset $S_5 = \{0, a, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $b \cdot c = 0 \in S_5$ and $c \in S_5$ but $b \notin S_5$. Also, subset $S_6 = \{0, b, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c \in S_6$ and $a \cdot c = b \in S_6$, but $a \notin S_6$.

Ideals in right distributive BI-algebras have one additional property as shown in the following proposition.

Proposition 3.1 ([11], Theorem 3.1)**.** *Let J be an ideal in a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then,*
(J3) $(\forall x, y \in A)(x \in J \Rightarrow x \cdot y \in J)$.

3.1. **Something more about ideals.** The following theorem gives a criterion for recognizing ideals in BI-algebras.

Theorem 3.1. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. A subset J of A is an ideal in $\mathfrak{A}$ if and only if it holds*
(J4) $(\forall x, y, z \in A)((y \in J \wedge z \in J \wedge (x \cdot y) \cdot z = 0) \Rightarrow x \in J)$.

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $y, z \in J$ and $(x \cdot y) \cdot z = 0$. Then, $x \cdot y \preccurlyeq z \in J$. Thus, $x \cdot y \in J$ by (J2). Hence, $x \in J$ by (J1).

Conversely, let (J4) be valid and let $x, y \in A$ be such that $x \cdot y \in J$ and $y \in J$. Then, $(x \cdot y) \cdot (x \cdot y) = 0$, $y \in J$ and $x \cdot y \in J$ according to (Re). Thus, $x \in J$ by (J4). Let us show that (J0) holds. Putting $x = 0$ in (J4), we get $0 \in J$ with respect (L). Therefore, J is an ideal in $\mathfrak{A}$. □

In addition to the previous one, if $\mathfrak{A}$ is a right distributive BI-algebra, then we have additional possibilities.

Theorem 3.2. *A subset J of a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$, which satisfies the condition* (J0)*, is an ideal in $\mathfrak{A}$ if and only if it satisfies the condition*
(J5) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \Rightarrow x \cdot z \in J)$.

Proof. (i) If we put $z = 0$ in (J5), then, with respect to (M), we get (J1).

(ii) Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in J$ and $y \in J$. Since, according to (6), we have $(x \cdot y) \cdot z \preccurlyeq x \cdot (x \cdot z)$, then $x \cdot (y \cdot z) \in J$ implies $(x \cdot y) \cdot z \in J$ according to (J2). Then, $(x \cdot z) \cdot (y \cdot z) = (x \cdot y) \cdot z \in J$ by (DR). On the other hand, $y \in J$, according to (J3), implicit $y \cdot z \in J$. Now, from $(x \cdot z) \cdot (y \cdot z) \in J$ and $y \cdot z \in J$, according to (J1), we get $x \cdot z \in J$. This proves the validity of formula (J5). □

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra, X be a non-empty subset of A and $a \in A$ be an arbitrary element. Let us define $X_a =: \{x \in A : x \cdot a \in X\}$.

Theorem 3.3. *If J is an ideal of a right distributive BI-algebra $\mathfrak{A}$, then for any $a \in A$, the subset J_a is the smallest ideal in $\mathfrak{A}$ containing J and a.*

Proof. It is clear that $0 \in J_a$ holds because $0 \cdot a = 0 \in J$ by (L) and (J0).

Let $x, y \in A$ be such that $x \cdot y \in J_a$ and $y \in J_a$. This means $(x \cdot y) \cdot a \in J$ and $y \cdot a \in J$. Then, $(x \cdot a) \cdot (y \cdot a) \in J$ according to (DR). Thus, $x \cdot a \in J$ by (J1). Hence, $x \in J_a$. Therefore, J_a is an ideal in $\mathfrak{A}$.

Since $\mathfrak{A}$ is a right distributive BI-algebra, for all $x \in J$ we have $(x \cdot a) \cdot a \preccurlyeq x \cdot (a \cdot a) = x \cdot 0 = x \in J$ by (6), (Re) and (M). Then, $(x \cdot a) \cdot a \in J$ by (J2). Thus, $x \cdot a \in J$ by Proposition 2.1 (iii). This means $x \in J_a$. So, $J \subseteq J_a$. Also, it is clear that $a \in J_a$ because $a \cdot a = 0 \in J$.

Suppose that I is any ideal in $\mathfrak{A}$ containing J and a. Is $x \in J_a$, then $x \cdot a \in J \subseteq I$. Thus, $x \cdot a \in I$. Hence, $x \in I$ by (J1) since $a \in J \subseteq I$. So, $J \subseteq I$. This means that J_a is the least ideal containing J and a. □

3.2. **Implicative ideals.** If we transform formula (J1) using (Im) and putting $y = z$, we get

(IJ) $(\forall x, y, z \in A)(((x \cdot (y \cdot x)) \cdot z \in J \land z \in J) \Rightarrow x \in J)$.

The standard determibation within ideal theory in many logical algebras is that the condition (IJ) describes the concept of an implicative ideal in those algebras (see, for example, [8]).

Definition 3.2. Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. A subset J of A is an implicative ideal in $\mathfrak{A}$ if it satisfies the conditions (J0) and (IJ).

However, we have the following conclusion.

Theorem 3.4. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra and J be a subset of A such that $0 \in J$. Then, J is an implicative ideal in $\mathfrak{A}$ if and only if J is an ideal in $\mathfrak{A}$.*

3.3. **Positive implicative ideals.** The concept of positive implicative ideals in this class of logical algebras is correlated with the determination of the term positive implicative ideals in BCK-algebras (in terms of [8]) and BCI-algebras (in sense of [2]).

Definition 3.3. A subset J of a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called a positive implicative ideal of $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$,

(PIJ) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \land y \cdot z \in J) \implies x \cdot z \in J)$.

First, we have the following.

Proposition 3.2. *Any positive implicative ideal in a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is an ideal in $\mathfrak{A}$.*

Proof. If we put $z = 0$ in (PIJ), with respect to (M), we get (J1). $\square$

The following example shows that the converse of the Proposition 3.1 need not be valid.

Example 3.3. Let $\mathfrak{A} = (A, \cdot, 0)$ be BI-algebra as in Example 3.1.

The ideal $J_1 = \{0, a\}$ is not a positive implicative ideal in $\mathfrak{A}$ because, for example, we have $(c \cdot a) \cdot b = b \cdot b = 0 \in J_1$ and $a \cdot b = a \in J_1$, but $c \cdot b = c \notin J_1$.

The ideal $J_3 = \{0, c\}$ is not a positive implicative ideal in $\mathfrak{A}$ either because, for example, we have $(a \cdot c) \cdot b = b \cdot b = 0 \in J_3$ and $c \cdot b = c \in J_3$, but $a \cdot b = a \notin J_3$.

Remark 3.1. By putting $z = y$ into the formula (PIJ), we get with respect to (Re), the validity of the following formula

$$(\forall x, y \in A)((x \cdot y) \cdot y \in J \Rightarrow x \cdot y \in J).$$

Moreover, the above conclusion holds for every ideal J in a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ according to Proposition 2.1 (iii). But the validity of this formula is not sufficient for the subset J to be a positive implicative ideal in $\mathfrak{A}$, as the previous example shows. The ideals $J_1 = \{0, a\}$ and $J_3 = \{0, c\}$, although they satisfy the above formula, are still not positive implicative ideals in the BI algebra $\mathfrak{A}$.

In BI-algebras, as the previous example showed, not all ideals are positive implicative ideals. In the following two examples, we show that the ideals $J_0 = \{0\}$ and $J_2 = \{0, b\}$ are positive implicative ideals in a BI-algebra $\mathfrak{A} = (A, \cdot, 0)$ described in Example 3.1. For illustration, complete verifications of these assertions are shown in these examples.

Example 3.4. Let $\mathfrak{A} = (A, \cdot, 0)$ be BI-algebra as in Example 3.1. Let us show that $J_0 = \{0\}$ is a positive implicative ideal in $\mathfrak{A}$. Let $x, y, z \in A$ be such that $(x \cdot y) \cdot z = 0$ and $y \cdot z = 0$. From $y \cdot z = 0$, it follows $y = 0$ or $y = z$ and from $(x \cdot y) \cdot z = 0$ it follows $x \cdot y = 0$ or $x \cdot y = z$. We are considering the following possibilities.

(i) $x \cdot y = 0$. Then, $x = 0$ or $x = y$. Thus, $x = 0$ implies $x \cdot z = 0 \cdot z = 0$ and $x = y$ implies $x \cdot z = y \cdot z = 0$.

(ii) Let us assume that $x \cdot y = z$. If $y = 0$, then $z = x \cdot 0 = x$. So, $x \cdot z = z \cdot z = 0$. If $y = z$, then $x \cdot y = y$. Thus, by Proposition 2.1 (vi), we have $0 = y \cdot y = y$. This gives $z = x \cdot 0 = x$. Finally, we have $x \cdot z = x \cdot x = 0$.

Example 3.5. Let $\mathfrak{A} = (A, \cdot, 0)$ be BI-algebra as in Example 3.1. To prove that $J_2 = \{0, b\}$ is a positive implicative ideal in $\mathfrak{A}$, let us assume that $x, y, z \in A$ are such that $(x \cdot y) \cdot z \in J_2$ and $y \cdot z \in J_2$. Then, $y \cdot z = 0$ or $y \cdot z = b$.

If $y \cdot z = 0$, then $y = 0$ or $y = z$. In the first case, we have $J_2 \ni (x \cdot 0) \cdot z = x \cdot z$ by (M). In the second case, we have $J_2 \ni (x \cdot z) \cdot z = x \cdot z$ by Proposition 2.1 (iii).

Let, now, $y \cdot z = b$. This is possible in the following five cases.

1. $y = a$, $z = c$ and $(x \cdot a) \cdot c \in J_2$. If $(x \cdot a) \cdot c = 0$, then $x \cdot a = 0$ or $x \cdot a = c$. As the option $x \cdot a = c$ is not possible, we have the options $x = 0$ or $x = a$. If $x = 0$, this gives $0 \cdot c = 0 \in J_2$. If $x = a$, this gives $(a \cdot a) \cdot c = 0 \cdot c = 0 \in J_2$ and $a \cdot c = b \in J_2$. Assume that $(x \cdot a) \cdot c = b$. Since $x \cdot a = a$ is not possible, the possibility remains $x \cdot a = b$, which is possible only for $x = b$ or $x = c$. Further on, we have:

$$(b \cdot a) \cdot c = b \cdot c = b \in J_2 \quad \text{and} \quad a \cdot c = b \in J_2,$$
$$(c \cdot a) \cdot c = b \cdot c = b \in J_2 \quad \text{and} \quad a \cdot c = b \in J_2.$$

2. $y = b$, $z = 0$ and $(x \cdot b) \cdot 0 = x \cdot b \in J_2$. Since the option $x \cdot b = b$ is not possible, $x \cdot b = 0$ remains, which gives the options $x = 0$ or $x = b$. If $x = 0$, we have $0 \cdot 0 = 0 \in J_2$. If $x = b$, we have $b \cdot 0 = b \in J_2$.

3. $y = b$, $z = a$ and $(x \cdot b) \cdot a \in J_2$. The option $(x \cdot b) \cdot a = 0$ is possible if $x \cdot b = 0$ or $x \cdot b = a$. The second option is only possible if $x = a$. In this case $a \cdot a = 0 \in J_2$. The first option is achievable with the possibilities $x = 0$ or $x = b$. Then we have:

$$(0 \cdot b) \cdot a = 0 \cdot a = 0 \in J_2 \quad \text{and} \quad 0 \cdot a = 0 \in J_2,$$
$$(b \cdot b) \cdot a = 0 \cdot a = 0 \in J_2 \quad \text{and} \quad b \cdot a = b \in J_2.$$

Let us now assume that $(x \cdot b) \cdot a = b$, which is achievable only if $x \cdot b = b$ or $x \cdot b = c$. Since the option $x \cdot b = b$ is not possible, and the option $x \cdot b = c$ is achievable only if $x = c$, we have $c \cdot a = b \in J_2$.

4. $y = b$, $z = c$ and $(x \cdot b) \cdot c \in J_2$. Option $(x \cdot b) \cdot c = 0$ is possible if $x \cdot b = 0$ or $x \cdot b = c$. The second possibility is realizable only if $x = c$. In that case, we have

$c \cdot c = 0 \in J_2$. If it were $x \cdot b = 0$, we would have $x = 0$ or $x = b$. Then we would have:

$$(0 \cdot b) \cdot c = 0 \cdot c = 0 \in J_2 \quad \text{and} \quad 0 \cdot c = 0 \in J_2,$$
$$(b \cdot b) \cdot c = 0 \cdot c = 0 \in J_2 \quad \text{and} \quad b \cdot c = b \in J_2.$$

Suppose, now, that $(x \cdot b) \cdot c = b$. Possible options are $x \cdot b = a$ or $x \cdot b = b$. The second option is impossible. The first option is feasible if $x = a$. In this case we have $(a \cdot b) \cdot c = a \cdot c = b \in J_2$ and $a \cdot c = b \in J_2$.

5. $y = c$, $z = a$ and $(x \cdot c) \cdot a \in J_2$. If it were $(x \cdot c) \cdot a = b$, we would have $x \cdot c = b$ or $x \cdot c = c$. The second option is not possible. For the first option to be realized, it must be $x = a$ or $x = b$. In these cases, we have $a \cdot a = 0 \in J_2$, that is, $b \cdot a = b \in J_2$.

Let $(x \cdot c) \cdot a = 0$. This is feasible if $x \cdot c = 0$ or $x \cdot c = a$. Since the option $x \cdot c = a$ is not possible, we get the options $x = 0$ or $x = c$. In these cases, had $0 \cdot a = 0 \in J_2$, and $c \cdot a = b \in J_2$.

However, if $\mathfrak{A}$ is a right distributive BI-algebra, then the situation is completely different.

Theorem 3.5. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a right distributive BI-algebra. Then, every ideal in $\mathfrak{A}$ is a positive implicative ideal in $\mathfrak{A}$.*

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(x \cdot y) \cdot z \in J$ and $y \cdot z \in J$. Then, $(x \cdot z) \cdot (y \cdot z) \in J$ by (DR). Thus, $x \cdot z \in J$ according to (J1). □

We end this subsection with the following theorem.

Theorem 3.6. *Let J be an ideal in a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. If, for all $a \in A$, J_a is an ideal in $\mathfrak{A}$, then J is a positive implicative ideal in $\mathfrak{A}$.*

Proof. Let $x, y, z \in a$ be such that $(x \cdot y) \cdot z \in J$ and $y \cdot z \in J$. Then, $x \cdot y \in J_z$ and $y \in J_z$. Since J_z is an ideal in $\mathfrak{A}$, then $x \in J_z$ by (J1). So, $x \cdot z \in J$. Therefore, J is a positive implicative ideal in $\mathfrak{A}$. □

The following example, showing several applications of Theorem 3.6, enables a relatively simple check whether an ideal in the BI-algebra is a positive implicative ideal.

Example 3.6. Let $\mathfrak{A} = (A, \cdot, 0)$ be BI-algebra as in Example 3.1.

(i) Ideal $J =: \{0, a, c\}$ is a positive implicative ideal in $\mathfrak{A}$. Indeed, since all subsets $J_0 = \{x \in A : x \cdot 0 \in J\} = J$, $J_a = \{x \in A : x \cdot a \in J\} = \{0, a\}$, $J_b = \{x \in A : x \cdot b \in J\} = A$ and $J_c = \{x \in A : x \cdot x \in J\} = \{0, c\}$ are ideals in $\mathfrak{A}$, then J is a positive implicative ideal in $\mathfrak{A}$ according to the previous theorem.

(ii) Relying on Theorem 3.6, we again show that the subsets $\{0\}$ and $\{0, b\}$ are positive implicative ideals in $\mathfrak{A}$. As for the ideal $J =: \{0\}$, we have $J_0 = \{0\}$, $J_a = \{0, a\}$, $J_b = \{0, b\}$ and $J_c = \{0, c\}$, we conclude that $\{0\}$ is a positive implicative deal in $\mathfrak{A}$.

Since for the ideal $J =: \{0, b\}$ we have $J_0 = \{0, b\}$, $J_a = A$, $J_b = \{0, b\}$ and $J_c = A$, we conclude, according to Theorem 3.6, that J is a positive implicative ideal in $\mathfrak{A}$.

(iii) Theorem 3.6 does not ensure that the ideal $K =: \{0, a\}$ is a positive implicative ideal in $\mathfrak{A}$, because, for example, the subset $K_b = \{x \in A : x \cdot b \in K\} = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$.

Remark 3.2. The previous Examples 3.3–3.6 show that the property 'positive implicative ideal' is inherited neither in the ascending nor in the descending sequence of ideals in this class of logical algebras. Namely, although the ideal $\{0\}$ is a positive implicative ideal, the ideal $\{0, a\}$ which contains it is not a positive implicative ideal. Similarly, although $\{0, a, c\}$ is a positive implicative ideal, the ideal $\{0, c\}$, contained in $\{0, a, c\}$ is not a positive implicative ideal.

3.4. **Weak positive implicative ideals.** If we transform formula (PIJ), using equality (iii) in Proposition 2.1, we get

(wPIJ) $(\forall x, y, z \in A)((x \cdot y) \cdot z \in J \wedge y \cdot z \in J) \Rightarrow (x \cdot z) \cdot z \in J)$.

The last formula, as usual, determines the concept of weakly positive implicative ideals in many logical algebras (see, for example, [9, 10]). Therefore, the following holds.

Definition 3.4. Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. A subset J of A is an weak positive implicative ideal in $\mathfrak{A}$ if it satisfies the conditions (J0) and (wPIJ).

Theorem 3.7. *Any positive implicative ideal in a BI-algebra* $\mathfrak{A} =: (A, \cdot, 0)$ *is a weak positive implicative ideal in* $\mathfrak{A}$ *and vice versa.*

3.5. **Sub-implicative ideals.** The term 'sub-implicability' as a concept of sub-implicative ideas in BCI algebras was first determined in 2000 in [7] by L. Y. Lin and J. Meng. In this subsection, we introduce the concept of sub-implicative ideals in BI-algebras.

Definition 3.5. A subset J of a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called a sub-implicative ideal of $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$,

(SIJ) $(\forall x, y, z \in A)((((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot z \in J \wedge z \in J) \Rightarrow y \cdot (y \cdot x)) \in J)$.

The following example shows that there are such ideals in BI-algebra.

Example 3.7. Let $\mathfrak{A} = (A, \cdot, 0)$ be BI-algebra as in Example 3.1. The ideal $J_2 = \{0, b\}$ is a sub-implicative ideal in $\mathfrak{A}$. The ideal $J_5 = \{0, a, c\}$ is a sub-implicative ideal in $\mathfrak{A}$, too.

Proposition 3.3. *Any sub-implicative ideal in a BI-algebra* $\mathfrak{A} =: (A, \cdot, 0)$ *is an ideal in* $\mathfrak{A}$.

Proof. If we put $y = x$ in the valid formula (SIJ), we get

$$((x \cdot 0) \cdot 0) \cdot z = x \cdot z \in J \wedge z \in J \Rightarrow x \cdot 0 = x \in J,$$

with respect to (Re) and (M). ☐

Theorem 3.8. *Let J be an ideal in a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then, J is a sub-implicative ideal in $\mathfrak{A}$ if and only if the following holds*
(SIJa) $(\forall x, y \in A)(((x \cdot (x \cdot y)) \cdot (y \cdot x) \in J \Rightarrow y \cdot (y \cdot x) \in J)$.

Proof. Assume that J is a sub-implicative ideal in $\mathfrak{A}$ and let $x, y \in A$ be such that $(x \cdot (x \cdot y)) \cdot (y \cdot x) \in J$. Then, $((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot 0 \in J$ in accordance with (M). From here and from $0 \in J$, in accordance with (SIJ), it follows that $y \cdot (y \cdot x) \in J$, which proves the validity of the formula (SIJa).

Suppose, conversely, that an ideal J in $\mathfrak{A}$ satisfies the condition (SIJa). Let us prove (SIJ). Let $x, y, z \in A$ be arbitrary elements such that $((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot z \in J$ and $z \in J$. From here, we get $(x \cdot (x \cdot y)) \cdot (y \cdot x) \in J$ by (J1). It follows from (SIJa) that $y \cdot (y \cdot x) \in J$. This means that J is a sub-implicative ideal in $\mathfrak{A}$. □

The converse of Proposition 3.2 need not be valid as the following example shows.

Example 3.8. Let $\mathfrak{A} = (A, \cdot, 0)$ be BI-algebra as in Example 3.1. The ideal $J_1 = \{0, a\}$ is not a sub-implicative ideal in $\mathfrak{A}$ because, for example, we have $(a \cdot (a \cdot c)) \cdot (c \cdot a) = (a \cdot b) \cdot b = a \cdot b = a \in J_1$ but $c \cdot (c \cdot a) = c \cdot b = c \notin J_1$. □

However, the following holds.

Example 3.9. Let $A = \{1, a, b, c\}$ be as in Example 3.2. Then, $\mathfrak{A} = (A, \cdot, 0)$ is a right distributive BI-algebra. The ideals $J_0 = \{0\}$, $J_1 = \{0, a\}$ and $J_2 = \{0, b\}$ are sub-implicative ideals in $\mathfrak{A}$.

At the end of this subsection, we show a sufficient condition for an ideal in a right distributive BI-algebra to be a sub-implicative ideal.

Theorem 3.9. *Let J be an ideal in a right distributive BI-algebra $\mathfrak{A} = (A, \cdot, 0)$. Then, J is a sub-implicative ideal in $\mathfrak{A}$ if it satisfies*
(SIJb) $(\forall x, y, z \in A)(((x \cdot y) \cdot (y \cdot z) \in J \wedge x \in J) \Rightarrow y \in J)$.

Proof. (i) Let the ideal J satisfy the condition (SIJb) and let $x, y \in A$ be such that $(x \cdot (x \cdot y)) \cdot (y \cdot x) \in J$. We gave:

$$\begin{aligned} J \ni (x \cdot (x \cdot y)) \cdot (y \cdot x) &= ((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot 0 \quad \text{by (M)} \\ &= ((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot ((y \cdot (y \cdot x)) \cdot (y \cdot (y \cdot x))) \quad \text{according to (Re)} \\ &\succcurlyeq (((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot (y \cdot (y \cdot x))) \cdot (y \cdot (y \cdot x)) \quad \text{according to (6).} \end{aligned}$$

From here, according to (J2), we get

$$(((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot (y \cdot (y \cdot x))) \cdot (y \cdot (y \cdot x)) \in J.$$

If we introduce substitutions $u = (x \cdot (x \cdot y)) \cdot (y \cdot x)$, $v = y \cdot (y \cdot x)$ and $z = 0$, the previous formula takes the following form $(u \cdot v) \cdot (v \cdot z) \in J$. From here, according to (SIJb), it follows that $y \cdot (y \cdot x) = v \in J$ since $(x \cdot (x \cdot y)) \cdot (y \cdot x) = u \in J$. This shows that J is a sub-implicative ideal in $\mathfrak{A}$. □

Remark 3.3. It can be shown that in right distributive BI-algebras

$$(\forall x, y \in A)(y \cdot (y \cdot x) \preccurlyeq x)$$

holds. Indeed, for arbitrary $x, y \in A$, according to (Re), $(y \cdot x) \cdot (y \cdot x) = 0$ holds. Hence, in accordance with Proposition 2.1 (iii), we get $(y \cdot x) \cdot ((y \cdot x) \cdot x) = 0$. From here we get $(y \cdot (y \cdot x)) \cdot x = 0$. since $\mathfrak{A}$ is a right distributive BI-algebra. This means $y \cdot (y \cdot x) \preccurlyeq x$. However, the equality $y \cdot (y \cdot x) = x$ does not hold for $y = 0$ and arbitrary $x \in A \backslash \{0\}$. This implies that the following demonstration is not acceptable because it assumes an unrealizable formula as a hypothesis.

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra which, additionally, satisfies the condition
(hyp) $(\forall x, y \in A)(x \cdot (x \cdot y) = y)$.

Then, any ideal in $\mathfrak{A}$ is a sub-implicative ideal in $\mathfrak{A}$.

Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(((x \cdot (x \cdot y)) \cdot (y \cdot x)) \cdot z \in J$ and $z \in J$. Then, $((x \cdot (x \cdot y)) \cdot (y \cdot x) \in J$ by (J1). Thus, $y \cdot (y \cdot x) \in J$ in accordance with hypothesis (hyp). So, J is a sub-implicative ideal in $\mathfrak{A}$.

Acknowledgement The author thanks to colleague Arsham Borumand Saeid, Kerman, Iran, for patient reading, commenting and helpful suggestions on a first draft of this paper. Also, the author thanks the reviewer(s) for his/her/their helpful suggestions.

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