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A SPECTRAL STUDY OF THE MAXIMUM REVERSE VERTEX ECCENTRICITY MATRIX OF GRAPHS

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Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The reverse eccentricity of vertex v_i in a simple connected graph G is defined as $D+1-e(v_i)$, where $e(v_i)$ denotes the eccentricity of the vertex v_i and D is the diameter of G. In this study, we introduce a new matrix associated with the graph G, called the maximum reverse vertex eccentricity matrix, denoted by $M_R^{'}(G)$. For any two vertices, the corresponding entry of $M_R^{'}(G)$ is the maximum of their reverse eccentricities if they are adjacent, and zero otherwise. First, we establish some basic properties of the maximum reverse vertex eccentricity matrix. We also derive bounds for the spectral parameters, such as the trace, determinant, spectral radius, and energy of this matrix. Furthermore, the energy of the maximum reverse vertex eccentricity matrix is derived for several graph classes, including the sunlet graph, triangular book graph, and wheel graph. The maximum reverse vertex eccentricity matrix provides a mathematical framework for the spectral energy analysis of graphs, hence supporting the SDG of quality education.

1. Introduction

In spectral graph theory, the concept of graph energy has emerged as one of the interesting directions due to its applications in molecular chemistry, as seen in, for instance [1, 2]. Recently, various types of matrices and the corresponding energies of graphs have been introduced. For a simple connected graph G of order n, Ivan Gutman [3] defined the energy as the sum of absolute values of the eigenvalues of the adjacency matrix of the corresponding graph. Moreover, the spectral study of graph matrices plays a significant role in graph energy analysis.

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1.1. **Terminology and notation.** Let G be an undirected graph on a finite, nonempty set of vertices V and the prescribed collection E of unordered pairs of vertices called edges. Two vertices are said to be adjacent if they share an edge. We denote $v_i \sim v_j$ if the vertex v_i is adjacent to v_j. The degree $\deg(v_j)$ represents the number of edges incident on the vertex v_j. The minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$ are the smallest and largest degrees of the vertices in G, respectively. The distance $d(v_i, v_j)$ is the length of the shortest $v_i - v_j$ path in G. The eccentricity of a vertex $v_i \in V$, denoted by $e(v_i)$, is the maximum distance between v_i and any other vertex in G. The maximum and minimum eccentricity of the graph G are called the diameter $(D(G))$ and radius $(r(G))$ of the graph G, respectively. A graph G is said to be self-centered if $D(G) = r(G)$. The center of a graph G, denoted by $C(G)$ is the set $C(G) = \{v_i \in V(G) : e(v_i) = r(G)\}$. The adjacency matrix $A(G)$ of a graph G is known to be a $\{0, 1\}$ matrix whose rows and columns are indexed by the vertices of the graph G, and the ij^{th} entry is one if $v_i \sim v_j$, and zero otherwise. For other notations and terminology, we follow [4].

1.2. **Background, motivation and methodology.** In the literature, Kulli [5] introduced the concept of reverse vertex degree C_{v_i} of a vertex v_i, as $C_{v_i} = \Delta(G) - \deg(v_i) + 1$. Recently, many ideas have been introduced using the reverse vertex degree and found in the literature [6, 7]. In [8] Chandrashekar Adiga and M. Smitha have defined the maximum degree matrix $M(G)$ in which ij^{th} entry is $\max\{\deg(v_i), \deg(v_j)\}$ if $v_i \sim v_j$, zero otherwise. Using these ideas, the author Gowtham K. J. [9] introduced the concept of the maximum reverse degree matrix and denoted it as $M_R(G) = (m_{ij})$, where

$$(m_{ij}) = \begin{cases} \max\{C_{v_i}, C_{v_j}\}, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases}$$

The new ideas and various applications of the matrix $M_R(G)$ are found in the literature [10].

Motivated by the concept of the maximum reverse degree matrix and its vast applications, we introduce the idea of the maximum reverse vertex eccentricity matrix $M'_R(G)$ defined by,

$$M'_R(G) = (m'_{ij}) = \begin{cases} \max\{C'_{v_i}, C'_{v_j}\}, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $C'_{v_i} = D(G) - e(v_i) + 1$ is called the reverse eccentricity of the vertex v_i. The characteristic polynomial of the maximum reverse vertex eccentricity matrix is denoted by $\phi_p\{M'_R(G, \gamma)\} = \det(\gamma I - M'_R)$. The spectrum $\sigma(M'_R)$ of $M'_R(G)$ is the set of all eigenvalues of the matrix $M'_R(G)$. Suppose $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the eigenvalues of $M'_R(G)$ with respective multiplicities $m_1, m_2, \ldots, m_k$, then the spectrum can be represented as

$$\sigma(M'_R) = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}.$$

The largest eigenvalue γ_1 is called the M'_R-spectral radius. The absolute sum of all eigenvalues of $M'_R(G)$ is denoted by $\mathcal{E}M'_R(G) = \sum_{i=1}^{n} |\gamma_i|$ and is called the M'_R-energy.

To obtain bounds for the parameters associated with the matrix $M'_R(G)$, we make use of classical inequalities from mathematical analysis. Spectral algebraic techniques are used to calculate the eigenvalues and energy of $M'_R(G)$ for each type of graph. Moreover, we compare the matrix $M'_R(G)$ with the classical adjacency matrix $A(G)$.

Furthermore, in this section, some preliminary keywords, notations, and results are presented, which are used in the subsequent part of the article.

Definition 1.1. A star graph $K_{1,n-1}$ is a tree on n vertices with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1.

Definition 1.2. A wheel graph, denoted by $W_{1,n}$, is a graph formed by connecting a single central vertex to all vertices of a cycle graph C_n. The number of vertices in a wheel graph is $n+1$.

Definition 1.3. The triangular book graph B_n^3 is a planar undirected graph with $n+2$ vertices and $2n+1$ edges constructed by n triangles sharing a common edge.

Definition 1.4. A bi-star graph, denoted by $B_{m,n}$, is the graph obtained by joining the central vertices of two stars $K_{1,m-1}$ and $K_{1,n-1}$ by an edge.

Definition 1.5 ([14])**.** A sunlet graph S_n of order $2n$ is a cycle graph with a pendant edge attached to each vertex of C_n. The sunlet graph is a corona product of C_n and K_1.

These are some of the known results available in the literature, which have been used to prove the main results in this article.

Theorem 1.1 ([11])**.** *Let k be a non-negative integer. Then, $(A(G)^k)_{xy}$ is the number of walks of length k from x to y. In particular, $(A(G)^2)_{xx}$ is the degree of the vertex x, and* $\mathrm{trace}(A(G)^2)$ *equals twice the number of edges of G. Similarly,* $\mathrm{trace}(A(G)^3)$ *is six times the number of triangles in G.*

Lemma 1.1 ([12])**.** *The Cauchy-Schwarz inequality: If $(a_1, a_2, \dots, a_p)$ and $(b_1, b_2, \dots, b_p)$ are real p-vectors, then*

$$\sum_{i=1}^{p}(a_i b_i)^2 \leq \sum_{i=1}^{p} a_i^2 \sum_{i=1}^{p} b_i^2.$$

Lemma 1.2. *Suppose a_i and b_i are non-negative real numbers for $1 \leq i \leq n$. Then,*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

where $M_1 = \max_{1\leq i\leq n} a_i$, $M_2 = \max_{1\leq i\leq n} b_i$ and $m_1 = \min_{1\leq i\leq n} a_i$, $m_2 = \min_{1\leq i\leq n} b_i$.

Lemma 1.3 ([13])**.** *Let* $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ *be a symmetric matrix partitioned into blocks. Then, the eigenvalues of A are the eigenvalues of the matrices $A_0 + A_1$ and $A_0 - A_1$.*

Lemma 1.4 ([4])**.** *Cauchy interlacing theorem: Suppose $A \in \mathbf{R}^{n\times n}$ is a symmetric matrix. Let $B \in \mathbf{R}^{m\times m}$ with $m < n$ be a principal submatrix of A. Suppose A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and B has eigenvalues $\beta_1 \leq \cdots \leq \beta_m$. Then, $\lambda_k \leq \beta_k \leq \lambda_{k+n-m}$ for $k = 1, \dots m$, and if $m = n-1$, then $\lambda_1 \leq \beta_1 \leq \lambda_2 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \lambda_n$.*

Lemma 1.5 ([15])**.** *Let $C \in M^{m\times n}$ and $q = \min\{n, m\}, \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_q$ be the ordered singular values of C. Define the Hermitian matrix* $H = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$. *The ordered eigenvalues of H are $-\alpha_1 \leq \cdots \leq -\alpha_q \leq 0 = 0 \cdots = 0 \leq \alpha_q \leq \dots \alpha_1$.*

Lemma 1.6 ([15])**.** *Let $A \subset M^{n\times n}$ be a Hermitian matrix and let eigenvalues of A be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, $\lambda_n y^* y \leq y^* A y \leq \lambda_1 y^* y$ for all $y \in C^n$.*

Definition 1.6. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix, where A and D are square matrices. Then, the following hold.

(a) If D is invertible, then the Schur complement of D in M is $A - BD^{-1}C$.
(b) If A is invertible, then the Schur complement of A in M is $D - BA^{-1}C$.

Lemma 1.7 ([16])**.** *Schur's formula: Let M be a block matrix, where A and D are invertible matrices. Then,*

(a) $\det(M) = \det(A)\det(D - BA^{-1}C)$,
(b) $\det(M) = \det(D)\det(A - BD^{-1}C)$.

A partition of a square matrix A is said to be equitable if all the blocks of the partitioned matrix have constant row sums and each of the diagonal blocks is of square order. A quotient matrix Q of a square matrix A corresponding to an equitable partition is a matrix whose entries are the constant row sums of the corresponding blocks of A. Quotient matrices are useful in finding some eigenvalues of block matrices, which are comparatively larger. In the theory of graph spectra, equitable partitions play a crucial role, primarily due to the following result.

Lemma 1.8 ([17])**.** *Let A be a real symmetric matrix with a quotient matrix Q. Then, the characteristic polynomial of Q divides the characteristic polynomial of A.*

The contents of the article are organized as follows. After a brief introduction and preliminary results given in Section 1, the next section starts with the main results. Section 2 contains some elementary properties of the maximum reverse vertex eccentricity matrix of a graph. In Section 3, we derive some bounds for M'_R-spectral radius and M'_R-energy. Lastly, in Section 4, the maximum reverse vertex eccentricity energy of some standard graphs is obtained. The maximum reverse vertex eccentricity matrix provides a novel mathematical framework for the spectral energy analysis of graphs, thereby supporting SDG 4: quality education through the advancement of research and learning in spectral graph theory.

2. Elementary Properties of Maximum Reverse Vertex Eccentricity Matrix

In this section, we obtain some elementary properties of the maximum reverse vertex eccentricity matrix of G. In the matrix $M'_R(G)$, for $i \neq j$, the principal submatrix of order 2×2 corresponding to the i^{th} and j^{th} rows and columns, becomes the zero matrix if $v_i \nsim v_j$ in G. If $v_i \sim v_j$, then the submatrix takes the form

$$\begin{pmatrix} 0 & \max\{C'_{v_i}, C'_{v_j}\} \\ \max\{C'_{v_i}, C'_{v_j}\} & 0 \end{pmatrix}$$

the determinant of which lies between $2m$ and $(D + r - 1)^2$. Similarly, the principal submatrix of order 3×3, formed by any three distinct rows and columns i, j, k is non-singular if and only if the corresponding vertices v_i, v_j, v_k constitute a triangle in G. In that case, the submatrix is given by

$$\begin{pmatrix} 0 & \max\{C'_{v_i}, C'_{v_j}\} & \max\{C'_{v_i}, C'_{v_k}\} \\ \max\{C'_{v_j}, C'_{v_i}\} & 0 & \max\{C'_{v_i}, C'_{v_k}\} \\ \max\{C'_{v_k}, C'_{v_i}\} & \max\{C'_{v_k}, C'_{v_j}\} & 0 \end{pmatrix}.$$

We omit the proof of propositions whenever they are straightforward, since the results are similar to those of classical adjacency matrices.

Proposition 2.1. *A graph G is bipartite if and only if $\sigma(M'_R)$ of G is symmetric about the origin.*

Proof. Let G be a bipartite graph with the partition $V(G) = V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$. Then, the vertices of G can be labeled in such a way that the matrix $M'_R(G)$ can be written as

$$M'_R(G) = \begin{pmatrix} 0_{p\times p} & B_{p\times q} \\ B^T_{q\times p} & 0_{q\times q} \end{pmatrix},$$

where B is a non-negative matrix of order $p \times q$. Then, by Lemma 1.5, $\sigma(M'_R)$ is symmetric about the origin.

Conversely, suppose $\sigma(M'_R)$ is symmetric about the origin, then for all the odd integers k, the trace$(M'_R(G)^k) = 0$. The matrix $M'_R(G)$ has a non-zero entry if and only if the corresponding entry in the classical adjacency matrix $A(G)$ is non-zero. That is, trace$(M'_R(G)^k) > 0$ if and only if trace$(A(G)^k) > 0$. Suppose G has an odd cycle of length x, by Theorem 1.1, trace$(A(G)^x) > 0$, which in turn means that trace$(M'_R(G)^k) > 0$, a contradiction. Thus, G has no cycles of odd length and hence is bipartite. $\square$

Thus, for a bipartite graph G, if γ is an eigenvalue of $M'_R(G)$, then $-\gamma$ is also an eigenvalue of $M'_R(G)$.

Proposition 2.2. *Let G be a connected graph with diameter D. Suppose $M'_R(G)$ has k distinct eigenvalues, then $k > D$.*

Proof. The proof follows from the fact that the matrices $M'_R(G), M'_R(G)^2, M'_R(G)^3, \dots, M'_R(G)^D$ are linearly independent, and the degree of the minimal polynomial of $M'_R(G)$ is k, and it must exceed D. □

Proposition 2.3. *For any connected graph G of order $n \geq 2$, $\sigma(M'_R)$ of G contains at least two distinct eigenvalues.*

Proof. Suppose all the eigenvalues of the matrix $M'_R(G)$ are identical, that is, $\gamma_1 = \gamma_2 = \cdots = \gamma_n$, then since $\sum\limits_{i=1}^{n} \gamma_i = 0$, it is clear that each of γ_i is zero. This implies $M'_R(G) = 0$, which is impossible as the eccentricity of every vertex in G is non-zero. □

Proposition 2.4. *Let G be a connected graph of order n with diameter D and radius r. Suppose $C(G) = \{v_1, v_2, \dots, v_k\}$ is the center of G. Then,*

(i) $\mathrm{trace}(M'_R) = 0$,

(ii) $\mathrm{trace}((M'_R)^2) = 2 \sum\limits_{(v_i,v_j)\in E} \max\{C'_{v_i}, C'_{v_j}\}^2 \geq 2\left((D-r+1)^2 \sum\limits_{i=1}^{k} \deg(v_i) + P\right)$,

(iii) $\mathrm{trace}\,((M'_R)^3) = 6 \sum\limits_{(v_i,v_j,v_k)\in S} \max\{C'_{v_i}, C'_{v_j}\} \max\{C'_{v_j}, C'_{v_k}\} \max\{C'_{v_i}, C'_{v_k}\}$,

where P is the number of edges having none of its end vertices in $C(G)$ and E, S are the set of all edges and the set of all triangles in G, respectively.

Proof. Let $M'_R = (m_{ij})$. The first result (i) is obvious from the definition.

(ii) We know that $({M'_R}^2)_{ij} = \sum\limits_{v_k \sim v_i, v_j} m_{ik} m_{kj}$. Thus,

$$\mathrm{trace}({M'_R}^2) = \sum_{i=1}^{n} \sum_{v_j \sim v_i} m_{ij} m_{ji} = \sum_{i=1}^{n} \sum_{v_j \sim v_j} m_{ij}^2 = 2 \sum_{v_i \sim v_j} \max\{C_{e(v_i)}, C_{e(v_j)}\}^2.$$

Note that $1 \leq C'_{v_i} \leq D - r + 1$ and $C'_{v_i} = D - r + 1$ if and only if v_i is a center of G. Suppose $e = v_i v_j$ is an edge and either of v_i, v_j is a center, then $\max\{C'_{v_i}, C'_{v_j}\} = D - r + 1$. Thus,

$$\mathrm{trace}({M'_R}^2) = 2 \sum_{v_i \sim v_j} \max\{C'_{v_i}, C'_{v_j}\}^2 \geq 2\left((D - R + 1)^2 \sum_{i=1}^{k} \deg(v_i) + P\right),$$

where P is the number of edges having none of its end vertices in $C(G)$.

Similarly, one can obtain the expression for $\mathrm{trace}((M'_R)^3)$. □

Theorem 2.1. *$M'_R(G) = A(G)$ if and only if the graph G is self-centered.*

Proof. If G is self-centered, then $C'_{v_i} = 1$ and hence $M'_R(G) = A(G)$. Conversely, $M'_R(G) = A(G)$ implies that $\max\{C'_{v_i}, C'_{v_j}\} = 1$ for every pair of adjacent vertices v_i, v_j. This is possible only when $C'_{v_i} = 1$ for all $v_i \in V(G)$. But $C'_{v_i} = 1$ if and only if $e(v_i) = D(G)$ for all $v_i \in V(G)$. Thus, G is self-centred. □

Theorem 2.2. *Let G be a graph with radius r and diameter D. Then,*

$$M'_R(G) \leq (D - r + 1)A(G).$$

Proof. For any two adjacent vertices v_i and v_j $(1 \leq i, j \leq n)$, we have $\max\{C'_{v_i}, C'_{v_j}\} = \max\{D - e(v_i) + 1, D - e(v_j) + 1\} \leq D - r + 1$ as $e(v_i) \geq r$. Thus, $M'_R(G) \leq (D - r + 1)A(G)$. □

3. Bounds for M'_R-Spectral Radius and M'_R-Energy

In this section, some bounds for $M'_R(G)$-energy and $M'_R(G)$-spectral radius are derived using classical matrix inequalities such as the Cauchy-Schwarz inequality, AM-GM inequality, etc.

Theorem 3.1. *Let G be a connected graph of order n having diameter D and radius r. Let γ_1 and γ_n be the largest and smallest eigenvalues of $M'_R(G)$, respectively. Then,*

i. $\gamma_1 \geq D - r + 1$,
ii. $\gamma_n \leq -(D - r + 1)$.

Proof. Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ be the eigenvalues of the matrix $M'_R(G)$. Since r is the radius, there exists at least one pair of vertices v_i, v_j such that $d(v_i, v_j) = r$. The principal submatrix of $M'_R(G)$ corresponding to these vertices is given by

$$B = \begin{pmatrix} 0 & D - r + 1 \\ D - r + 1 & 0 \end{pmatrix}.$$

Note that $\beta_1 \geq \beta_2$ are the two eigenvalues of B, where $\beta_1 = D - r + 1$ and $\beta_2 = -(D - r + 1)$. By Theorem 1.4, $\gamma_n \leq \beta_2$ and $\beta_1 \leq \gamma_1$. On substituting β_1, β_2, we get the inequality. □

Theorem 3.2. *Let G be a graph of order n and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ be the eigenvalues of $M'_R(G)$. Then,*

$$\gamma_n \leq \frac{2 \sum_{v \sim v_j} \max\{C'_{v_i}, C'_{v_j}\}}{n} \leq \gamma_1.$$

Proof. From Lemma 1.6, $\gamma_n y^T y \leq y^T M'_R y \leq \gamma_1 y^T y$ for all vectors $y \in \mathbf{R}^n$. Let $y = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \end{pmatrix}^T$. Then, it is true that

$$y^T M'_R y = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} = 2 \sum_{v_i \sim v_j} \max\{C'_{v_i}, C'_{v_j}\}.$$

Thus, $\gamma_n y^T y \leq y^T M'_R y \leq \gamma_1 y^T y$ implies that

$$\begin{aligned}\gamma_n y^T y \leq& 2\sum_{v_i\sim v_j} \max\{C'_{v_i}, C'_{v_j}\} \leq \gamma_1 y y^T,\\ \gamma_n n \leq& 2\sum_{v_i\sim v_j} \max\{C_{e(v_i)}, C_{e(v_j)}\} \leq \gamma_1 n,\\ \gamma_n \leq& \frac{2\sum\limits_{v_i\sim v_j} \max\{C_{e(v_i)}, C_{e(v_j)}\}}{n} \leq \gamma_1.\end{aligned}$$

□

Theorem 3.3. *Let γ_1 be the M'_R-spectral radius of a graph G. Then, $\mathcal{E}M'_R(G) \geq 2\gamma_1$.*

Proof. We note that M'_R-energy is given by

$$\mathcal{E}M'_R(G) = \sum_{i=1}^{n} |\gamma_i| = |\gamma_1| + \sum_{i=2}^{n} |\gamma_i| \geq |\gamma_1| + |\sum_{i=2}^{n} \gamma_i|.$$

Since $\sum\limits_{i=1}^{n} \gamma_i = 0$, $\gamma_1 = -\sum\limits_{i=2}^{n} \gamma_i$ and $|\gamma_1| = \left|\sum\limits_{i=2}^{n} \gamma_i\right|$, implies

$$\mathcal{E}M'_R(G) \geq |\gamma_1| + |\gamma_1| = 2|\gamma_1|.$$

□

Theorem 3.4. *For any connected graph G of order n,*

$$\sqrt{2\mathcal{K} + n(n-1)\mathcal{P}} \leq \mathcal{E}M'_R(G) \leq \sqrt{2n\mathcal{K}},$$

where $\mathcal{K} = \sum\limits_{v_i\sim v_j} \max\{C'_{v_i}, C'_{v_j}\}^2$ and $\mathcal{P} = \prod\limits_{i=1}^{n} (|\gamma_i||\gamma_j|)^{\frac{1}{n(n-1)}}$.

Proof. Since trace $((M'_R)^2) = \sum\limits_{i=1}^{n} \gamma_i^2$, by Theorem 2.4,

$$\sum_{i=1}^{n} \gamma_i^2 = 2 \sum_{(v_i,v_j)\in E} \max\{C'_{v_i}, C'_{v_j}\}^2. \tag{3.1}$$

Using Cauchy-Schwarz inequality (1.1) and (3.1),

$$\mathcal{E}M'_R(G)^2 = \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 \leq n\sum_{i=1}^{n} |\gamma_i|^2 = n\Big(2\sum_{v_i\sim v_j} \max\{C_{e(v_i)}, C_{e(v_j)}\}^2\Big).$$

Thus, $\mathcal{E}M'_R(G)^2 \leq \sqrt{2n \sum\limits_{v_i\sim v_j} \max\{C_{e(v_i)}, C_{e(v_j)}\}^2} = \sqrt{2n\mathcal{K}}$.

Now, using the arithmetic and geometric mean inequality, we obtain

$$\begin{aligned}\frac{1}{n(n-1)} \sum_{i\neq j} |\gamma_i||\gamma_j| \geq& \prod_{i\neq j} \Big(|\gamma_i||\gamma_j|\Big)^{\frac{1}{n(n-1)}},\\ \sum_{i\neq j} |\gamma_i||\gamma_j| \geq& n(n-1) \prod_{i\neq j} \Big(|\gamma_i||\gamma_j|\Big)^{\frac{1}{n(n-1)}}.\end{aligned}$$

Thus,

$$\sum_{i\neq j}|\gamma_i|\cdot|\gamma_j|\geq n(n-1)\mathcal{P},\quad \text{where } \mathcal{P}=\prod_{i\neq j}\Big(|\gamma_i|\cdot|\gamma_j|\Big)^{\frac{1}{n(n-1)}}. \tag{3.2}$$

Now, consider $\mathcal{E}M'_R(G)^2=\Big(\sum\limits_{i=1}^{n}|\gamma_i|\Big)^2=\sum\limits_{i=1}^{n}|\gamma_i|^2+\sum\limits_{i\neq j}|\gamma_i|\cdot|\gamma_j|$. By (3.1) and (3.2),

$$\Big(\mathcal{E}M^e_R(G)\Big)^2\geq 2\sum_{v_i\sim v_j}\max\{C_{e(v_i)},C_{e(v_j)}\}^2+n(n-1)\mathcal{P}=\sqrt{2\mathcal{K}+n(n-1)\mathcal{P}}.$$

□

Theorem 3.5. *Let G be a graph of order n and size m, then*

$$\mathcal{E}M'_R(G)\geq\frac{2\sqrt{n\mathcal{K}}\sqrt{|\gamma^*_l|\cdot|\gamma^*_s|}}{|\gamma^*_l|+|\gamma^*_s|}$$

where $|\gamma^*_l|=\max\limits_{1\leq i\leq n}|\gamma_i|$, $|\gamma^*_s|=\min\limits_{1\leq i\leq n}|\gamma_i|$ *and* $\mathcal{K}=\sum\limits_{v_i\sim v_j}\max\{C'_{v_i},C'_{v_j}\}^2$.

Proof. Let $\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_n$ be the eigenvalues of $M'_R(G)$. On substituting $a_i=|\gamma_i|$ and $b_i=1$ for $1\leq i\leq n$, in Lemma 1.2, we get

$$\left(\sum_{i=1}^{n}|\gamma_i|^2\right)\left(\sum_{i=1}^{n}1\right)\leq\frac{1}{4}\left(\sqrt{\frac{|\gamma^*_1|}{|\gamma^*_i|}}+\sqrt{\frac{|\gamma^*_s|}{|\gamma^*_1|}}\right)^2\left(\sum_{i=1}^{n}|\gamma_i|\right)^2,$$

where $|\gamma^*_l|=\max\limits_{1\leq i\leq n}|\gamma_i|$ and $|\gamma^*_s|=\min\limits_{1\leq i\leq n}|\gamma_i|$,

$$n\mathcal{K}=\frac{1}{4}\left(\sqrt{\frac{|\gamma^*_1|}{|\gamma^*_i|}}+\sqrt{\frac{|\gamma^*_s|}{|\gamma^*_1|}}\right)^2\mathcal{E}M'_R(G)^2$$

and

$$2\sqrt{n\mathcal{K}}\leq\frac{|\gamma^*_l||\gamma^*_s|}{\sqrt{|\gamma^*_l||\gamma^*_s|}}\ \mathcal{E}M'_R(G).$$

On rearranging, the result holds. □

4. Maximum Reverse Vertex Eccentricity Energy of Some Standard Graphs

In this section, we compute $M'_R(G)$-energy for several standard graph families such as wheel graph, sunlet graph, star graph, bi-star graph, and triangular book graphs.

Theorem 4.1. *Let $K_{1,n-1}$ be a star graph of order n. Then,* $\mathcal{E}M'_R(K_{1,n-1})=4\sqrt{n-1}$.

Proof. The maximum reverse vertex eccentricity matrix of the star graph can be written as

$$M'_R(K_{1,n-1})=\begin{pmatrix}0_{1\times1} & \mathbf{2}_{n-1\times1}\\ \mathbf{2}_{1\times n-1} & 0_{n-1\times n-1}\end{pmatrix}.$$

The block **2** above represents a matrix in which every entry is 2.

Since $\text{rank}(M'_R(K_{1,n-1})) = 2$, it has only two non-zero eigenvalues. Thus, 0 is an eigenvalue with multiplicity $n-2$. Further, the above block partition of the matrix forms an equitable partition. Thus, the quotient matrix can be written as

$$Q(M'_R(K_{1,n-1}) = \begin{pmatrix} 0 & 2(n-1) \\ 2 & 0 \end{pmatrix},$$

whose eigenvalues are $\pm\sqrt{4(n-1)}$. This implies that $\mathcal{E}M'_R(K_{1,n-1}) = 4\sqrt{n-1}$. $\square$

Theorem 4.2. *Let $B_{n,n}$ be a bi-star graph of order $2n$. Then, $\mathcal{E}M'_R(B_{n,n}) = 4\sqrt{4n-3}$.*

Proof. By suitably labeling the vertices of the graph $B_{n,n}$ (where the rows in each of the blocks M_1, M_2 correspond to the vertices inducing the stars $K_{1,n-1}$, the first row of M_1 and M_2 respectively corresponding to the central vertices of star graphs) can be written as,

$$M'_R(B_{n,n}) = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix},$$

where $M_1 = \begin{pmatrix} 0_{1\times 1} & \mathbf{2}_{n-1\times 1} \\ \mathbf{2}_{1\times n-1} & 0_{n-1\times n-1} \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0_{1\times 1} & \mathbf{0}_{n-1\times 1} \\ \mathbf{0}_{1\times n-1} & 0_{n-1\times n-1} \end{pmatrix}$. By Lemma 1.3, the eigenvalues of $M'_R(B_{n,n})$ consist of the eigenvalues of $M_1 + M_2$ and $M_1 - M_2$.

Consider the sum matrix

$$M_1 + M_2 = \begin{pmatrix} 2_{1\times 1} & \mathbf{2}_{n-1\times 1} \\ \mathbf{2}_{1\times n-1} & 0_{n-1\times n-1} \end{pmatrix}.$$

Since $M_1 + M_2$ has rank 2, 0 is its eigenvalue with multiplicity $n-2$. Further, the quotient matrix of $M_1 + M_2$ is given by

$$Q(M_1 + M_2) = \begin{pmatrix} 2 & 2(n-1) \\ 2 & 0 \end{pmatrix},$$

whose eigenvalues can be easily computed as $1 \pm \sqrt{4n-3}$.

Similarly, one can get the eigenvalues of $M_1 - M_2 = \begin{pmatrix} 0_{1\times 1} & \mathbf{2}_{n-1\times 1} \\ \mathbf{2}_{1\times n-1} & 0_{n-1\times n-1} \end{pmatrix}$ as $-1 \pm \sqrt{4n-3}$ with multiplicity one and 0 with the multiplicity $n-2$. Thus, $\mathcal{E}M'_R(B_{n,n} = 4\sqrt{4n-3}$. $\square$

Theorem 4.3. *Let B_n^3 be a triangular book graph of order n, where v_1v_2 is the base of n triangles. Then, $\mathcal{E}M^e_R(B_n^3) = 2 + 2\sqrt{1+8n}$.*

Proof. Since the graph B_n^3 has $n+2$ vertices, the matrix $M'_R(B_n^3)$ is of order $n+2$. Further, $M'_R(B_n^3)$ can be written as the following block matrix

$$M'_R(B_n^3) = \begin{pmatrix} 2I_{2\times 2} & \mathbf{2}_{2\times n} \\ \mathbf{2}_{n\times 2} & 0_{n\times n} \end{pmatrix}.$$

Since $M'_R(B_n^3)$ has rank 3, it follows that the matrix $M'_R(B_n^3)$ has only 3 non-zero eigenvalues. Thus, 0 is an eigenvalue of $M'_R(B_n^3)$ with multiplicity $n-1$. Further, since the above block partition of $M'_R(B_n^3)$ forms an equitable partition, the quotient matrix can be computed as

$$Q(M'_R(B_n^3)) = \begin{pmatrix} 2 & 2n \\ 4 & 0 \end{pmatrix}.$$

The eigenvalues of $Q(M'_R(B_n^3))$ can be computed as $1 \pm \sqrt{1+8n}$. Thus, the $n+1$ eigenvalues of $M'_R(B_n^3$ are 0 and $1 \pm \sqrt{1+8n}$, with respective multiplicities $n-1$ and one. Using the fact that the sum of all eigenvalues of $M'_R(B_n^3)$ is zero, the remaining non-zero eigenvalue with multiplicity one can be computed as -2. Thus, $\mathcal{E}M_R^e(B_n^3) = 2 + 2\sqrt{1+8n}$. $\square$

Theorem 4.4. *Let S_n be a sunlet graph of order $2n$. Then, for all $0 \le i \le n-1$*

$$\sigma(M'_R(S_n)) = \begin{pmatrix} 2k_i + 2\sqrt{k_i^2+1} & 2k_i - 2\sqrt{k_i^2+1} \\ 1 & 1 \end{pmatrix},$$

where $k_i = \cos\left(\frac{2\pi i}{n}\right)$ for $0 \le i \le n-1$.

Proof. The maximum reverse vertex eccentricity matrix of the sunlet graph can be written as

$$M_R^e(S_n) = \begin{pmatrix} 2A(C_n) & 2I \\ 2I & 0 \end{pmatrix},$$

where each of the blocks is of order n and $A(C_n)$ is the adjacency matrix of the cycle graph of order n. The characteristic equation of $M'_R(S_n)$ is given by

$$\det(M'_R(S_n) - \gamma I) = \det \begin{pmatrix} 2A(C_n) - \gamma I & 2I \\ 2I & -\gamma I \end{pmatrix} = 0.$$

Since the matrix $-\gamma I$ is invertible, we use the concept of Schur complement to obtain the spectrum. That is,

$$\begin{aligned} \det(M'_R(S_n) - \gamma I) &= \det(-\gamma I) \det\left[(2A(C_n) - \gamma I) - (2I)(\gamma I)^{-1}(2I)\right] \\ &= (-1)^n \gamma^n \det\left[2A(C_n) - \left(\gamma - \frac{4}{\gamma}\right) I\right]. \end{aligned}$$

This implies, $\gamma = 0$ or $\gamma - \frac{4}{\gamma}$ is an eigenvalue of $2A(C_n)$. Since $\lambda_i = 2\cos\left(\frac{2\pi i}{n}\right)$, $i = 0, 1, \ldots, n-1$ are the eigenvalues of of the adjacency matrix $A(C_n)$ of the cycle C_n, it follows that

$$\gamma_i - \frac{4}{\gamma_i} = 4\cos\left(\frac{2\pi i}{n}\right). \tag{4.1}$$

Suppose $k_i = \cos\left(\frac{2\pi i}{n}\right)$, then (4.1) implies that $\gamma^2 - 4k_i\gamma - 4 = 0$. That is, $\gamma_i = 2k_i \pm 2\sqrt{k_i^2+1}$ for $0 \le i \le n-1$. $\square$

Theorem 4.5. *For a wheel graph $W_{1,n}$ on $n+1$ vertices*

$$\sigma(M'_R(W_{1,n})) = \begin{pmatrix} 2\cos\left(\frac{2\pi i}{n}\right), \ i = 0,1,\dots,n-1, & 1 \pm \sqrt{1+4n} \\ 1 & 1 \end{pmatrix}.$$

Proof. The maximum reverse vertex eccentricity matrix of $W_{1,n}$ can be expressed as

$$M'_R(W_{1,n}) = \begin{pmatrix} 0_{1\times 1} & \mathbf{2}_{1\times n} \\ \mathbf{2}_{n\times 1} & A(C_n)_{n\times n} \end{pmatrix},$$

where $A(C_n)$ is the adjacency matrix of a cycle graph on n vertices. Let γ be an eigenvalue of $M'_R(W_{1,n})$, with the corresponding eigenvector V. Further, let $V = \begin{pmatrix} X_{1\times 1} & Y_{n\times 1} \end{pmatrix}^T$. Then,

$$M'_R V = \gamma V. \tag{4.2}$$

We obtain a pair of a scalar γ and a non-zero vector V satisfying (4.2). Suppose $X = 0$ and $Y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$, then from (4.2), we get

$$2(y_1 + y_2 + \cdots + y_n) = 0, \tag{4.3}$$

$$A(C_n)Y = \gamma Y. \tag{4.4}$$

From (4.3), the components in Y sum up to 0. Further, from (4.4), suppose λ is an eigenvalue of $A(C_n)$ with an eigenvector Y whose components sum up to zero, then $\gamma = \lambda$ is an eigenvalue of $M'_R(G)$. Since $A(C_n)$ has eigenvalues $2\cos\left(\frac{2\pi i}{n}\right)$, $i = 0,1,\dots,n-1$, the $n-1$ eigenvalues of $M'_R(W_{1,n})$ are $\cos\left(\frac{2\pi i}{n}\right)$, $i = 1,\dots,n-1$. The remaining two eigenvalues can be obtained from the quotient matrix. The quotient matrix Q of $M'_R(W_{1,n})$ is given by

$$Q(M'_R(W_{1,n}) = \begin{pmatrix} 0 & 2n \\ 2 & 2 \end{pmatrix}.$$

The eigenvalues of $Q(M'_R(W_{1,n})$ can be computed as $1 \pm \sqrt{1+4n}$. By Lemma 1.8, $1 \pm \sqrt{1+4n}$ are eigenvalues of $M'_R(W_{1,n})$. □

5. Conclusion

Graph energy plays a significant role in diverse fields such as chemistry, physics, and mathematics, with applications in crystallography, protein sequence analysis, and spacecraft design, among others. In recent years, various types of graph matrices and corresponding energies have been introduced with the intention of refining the existing known results and correlations, in order to enhance their applicability. In this paper, we introduced a matrix called the maximum reverse vertex eccentricity matrix of a graph G. First, we list some properties of this matrix, and then the bounds for the energy of the maximum reverse eccentricity matrix are obtained. Finally, the generalized expression for the maximum reverse eccentricity energy of some classes of graphs, such as the star graph, bi-star, triangular book graph, sunlet graph, and wheel graph, is derived.

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