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CRITICAL POINT APPROACHES FOR A CLASS OF DIFFERENTIAL EQUATIONS WITH STURM-LIOUVILLE TYPE NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. A class of p-Laplacian equations with Sturm-Liouville type nonhomogeneous boundary value problem with nonlinear derivative depending on two control parameters is investigated. Existence and multiplicity of solutions are discussed by means of variational methods and critical point theory. Two examples supporting our theoretical results are also presented.

1. Introduction

Various generalizations of classical Sturm-Liouville problems for ordinary linear differential equations have attracted a lot of attention because of appearance of new important applications in physical sciences and applied mathematics. Sturm-Liouville boundary value problems have received a lot of attention in recent years. There have been many papers studying the existence of solutions for boundary value problems, for a small sample of recent work, we refer the reader to [1,7,8,11,13,16–18] that authors have studied the existence of solutions of Sturm-Liouville boundary value problem by using critical point theorem and fixed point theorem. For example, Bonanno and Riccobono in [8] have established the existence of multiple solutions for the second order Sturm-Liouville boundary value problem

$$\begin{cases} (\rho\phi_p(x'))' + s\phi_p(x) = \lambda f(t,x), & t \in [a,b], \\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B, \end{cases}$$

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where $p > 1$, $\phi_p(x) = |x|^{p-2}x$, $\rho, s \in L^{\infty([a,b])}$ with $\text{essinf}_{[a,b]}\rho > 0$ and $\text{essinf}_{[a,b]}s > 0$, $A, B \in \mathbb{R}$, $\alpha, \beta, \gamma, \sigma > 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and λ is a positive real parameter. In [18] Tian and Ge, applying a three critical point theorem due to Averna and Bonanno discussed the existence of three solutions for a Sturm-Liouville boundary value problem depending upon the parameter λ, while in [17] using lower and upper solutions approach and variational methods they proved the existence of multiple solutions for second order Sturm-Liouville boundary value problem

$$\begin{cases} -Lu = f(x,u), & x \in [0,1], \\ R_1(u) = 0,\ R_2(u) = 0, \end{cases}$$

where $Lu = (p(x)u')' - q(x)u$ is a Sturm-Liouville operator $R_1(u) = \alpha u'(0) - \beta u(0)$, $R_2(u) = \gamma u'(1) + \sigma u(1)$. In [13] using critical point theory and Ricceri's variational principle, the existence of infinitely many classical solutions to a boundary value system with Sturm-Liouville boundary conditions was obtained.

In the present paper, we investigate the existence of solutions for the Sturm-Liouville type nonhomogeneous boundary value problem

(1.1)
$$\begin{cases} -(\phi_p(u'))' = \left(\lambda f(x,u(x)) + \int_0^{u'(x)} \frac{\partial}{\partial x}\left(\frac{(p-1)|\tau|^{p-2}}{h(x,\tau)}\right) d\tau\right) h(x,u'(x)), & x \in (a,b), \\ \alpha u(a) - \beta u'(a) = A, \quad \gamma u(b) + \sigma u'(b) = B, \end{cases}$$

where $p > 1$, $\phi_p(t) = |t|^{p-1}t$, $\lambda > 0$, is a parameter, $\alpha, \gamma, \beta, \sigma > 0$ and A, B are arbitrary constants. The function $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions

- (i) $0 < m := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h(x,t) \leq M := \sup_{(x,t)\in[a,b]\times\mathbb{R}} h(x,t)$;
- (ii) the function $t \to h(x,t)$ is continuous for all $x \in [a,b]$ and the function $x \to h(x,t)$ is in $C^1([a,b])$ for all $t \in \mathbb{R}$.

We also assume that the function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

In [14] Sun et al. established the new criteria for the existence of infinitely many solutions for a class of one-dimensional p-Laplacian equations with Sturm-Liouville type nonhomogeneous boundary problem (1.1) with the perturbation term $\mu g(x, u(x))$.

We also refer the interested reader to the papers [3, 12] in which using variational methods and critical point theory, the existence of solutions for boundary value problems with nonlinear derivative dependence have been discussed. A second-order impulsive differential inclusion with Sturm-Liouville boundary conditions is studied. By using a nonsmooth version of a three critical point theorem of Ricceri, the existence of three solutions is obtained in [15]. In [4] utilizing variational methods the existence of at least one weak solution for elliptic problems on the real line was discused.

Here, we study the existence of multiple solutions for the problem (1.1). In Theorem 3.1 we prove the existence of at least two solutions for the problem (1.1). As a special case of Theorem 3.1, we investigate the existence of at least two solutions, when $w(x) = d$, that d is a constant; see Corollary 3.1. In Theorem 3.2 we show that the problem (1.1) has at least three solutions. We also show that for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero if $f(x,0)=0$ for all $x\in[a,b]$, the solutions are nontrivial; see Remark 3.1. Moreover, we deduce the existence of solutions for small positive values of the parameter λ such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero; see Remark 3.2. Finally, we give two examples to show the application of our results.

2. Preliminaries

Let X be a real Banach space and for two functions $\Phi,\Psi : X\to\mathbb{R}$ for all $r,r_1,r_2>\inf_X\Psi$, with $r_1<r_2$ we define the following functions

$$\varphi_1(r)=\inf_{u\in\Psi^{-1}(]-\infty,r[)}\frac{\Phi(u)-\inf_{u\in\overline{\Psi^{-1}(]-\infty,r[)}^{\omega}}\Phi(u)}{r-\Psi(u)}, \tag{2.1}$$

$$\varphi_2(r_1,r_2)=\inf_{u\in\Psi^{-1}(]-\infty,r_1[)}\sup_{v\in\Psi^{-1}([r_1,r_2[)}\frac{\Phi(u)-\Phi(v)}{\Psi(v)-\Psi(u)}, \tag{2.2}$$

where $\overline{\Psi^{-1}(]-\infty,r[)}^{\omega}$ is the closure $\Psi^{-1}(]-\infty,r[)$ in the weak topology.

Theorem 2.1. ([5, Theorem 1.1.]) *Let X be a reflexive real Banach space, and let $\Phi,\Psi : X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty}\Psi(u)=+\infty$. Assume also that there exist two constants r_1 and r_2 such that*

(a_1) $\inf_X\Psi<r_1<r_2$;
(a_2) $\varphi_1(r_1)<\varphi_2(r_1,r_2)$;
(a_3) $\varphi_1(r_2)<\varphi_2(r_1,r_2)$.

Then, there exists a positive real number σ such that, for each

$$\lambda\in\left]\frac{1}{\varphi_2(r_1,r_2)},\min\left\{\frac{1}{\varphi_1(r_1)},\frac{1}{\varphi_1(r_2)}\right\}\right[,$$

the equation $\Psi'+\lambda\Phi'$ admits at least two solutions whose norms are less than σ.

For all $r_1,r_2,r_3>\inf_X\Psi$ we define

$$\varphi_3(r_1,r_2,r_3)=\inf_{u\in\Psi^{-1}([r_1,r_2[)}\sup_{v\in\Psi^{-1}([r_2,r_3[)}\frac{\Phi(u)-\Phi(v)}{\Psi(v)-\Psi(u)}. \tag{2.3}$$

Clearly, $\varphi_2(r_2,r_3)\le\varphi_3(r_1,r_2,r_3)$.

Theorem 2.2. ([5, Theorem 2.2.]) *Let X be a reflexive real Banach space, and let $\Phi,\Psi : X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty}\Psi(u)=+\infty$. Assume also that there exist two constants r_1,r_3 and r_3 such that*

(b_1) $\inf_X\Psi<r_1<r_2<r_3$;

(b_2) $\max\{\varphi_1(r_1), \varphi_1(r_2), \varphi_1(r_3)\} < \min\{\varphi_2(r_1, r_2), \varphi_3(r_1, r_2, r_3)\}$.

Then there exists a positive real number σ such that for each

$$\lambda \in \left] \max\left\{\frac{1}{\varphi_2(r_1, r_2)}, \frac{1}{\varphi_3(r_1, r_2, r_3)}\right\}, \min\left\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}, \frac{1}{\varphi_1(r_3)}\right\}\right[,$$

the equation $\Psi' + \lambda\Phi' = 0$ admits at least three solutions whose norms are less than σ.

Theorems 2.1 and 2.2 have been used to the existence of multiple solutions for a two point boundary value problem driven by one-dimensional p-Laplacian and a second-order Sturm-Liouville boundary value problem in [5, 16], respectively. The present paper paper is a continuation for the application of the critical point theorems.

Let X be the Sobolev space $W^{1,p}([a, b])$ equipped with norm

$$\|u\| := \left(\int_a^b |u(t)|^p + |u'(t)|^p dt\right)^{\frac{1}{p}}, \quad \text{for all } u \in X.$$

Then, the space $(X, \|.\|)$ is a real reflexive Banach space and $\max\{\|u\|_{L^p}, \|u'\|_{L^p}\} \leq \|u\|$ for each $u \in X$. By the Sobolev embedding theorem (see [9]), X is compactly embedded into $C([a, b])$. We also denote $\|\cdot\|_\infty$ as the usual norm of $L^\infty([a, b])$.

For all $x \in [a, b]$ and $s \in \mathbb{R}$, define the functions

$$J_x(s) = J(x, s) := \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(x, \delta)} d\delta$$

and

$$H_x(s) = H(x, s) := \int_0^s J(x, \tau)d\tau.$$

For any fixed $x \in [a, b]$, the fact that $H''_x(s) = J'_x(s) = \frac{(p-1)|s|^{p-2}}{h(x,s)} \geq 0$ implies that H_x is a strictly convex C^2 function and J_x is a strictly increasing C^1 function. Simple calculation shows that for every $x \in [a, b]$, $s \in \mathbb{R}$,

$$\text{(2.4)} \qquad \frac{|s|^{p-1}}{M} \leq |J(x, s)| \leq \frac{|s|^{p-1}}{m}, \quad \frac{|s|^p}{pM} \leq |H(x, s)| \leq \frac{|s|^p}{pm}.$$

For each $u \in X$, let the functionals $\Psi, \Phi : X \to \mathbb{R}$ be as follows

$$\text{(2.5)} \quad \Psi(u) = \int_a^b H(x, u'(x))dx + \frac{\beta}{\alpha} H\left(a, \frac{\alpha}{\beta}u(a) - \frac{1}{\beta}A\right) + \frac{\sigma}{\gamma} H\left(b, -\frac{\gamma}{\sigma}u(b) + \frac{1}{\sigma}B\right)$$

and

$$\text{(2.6)} \qquad \Phi(u) = \int_a^b F(x, u(x))dx,$$

where

$$F(x, t) := \int_0^t f(x, s)ds, \quad \text{for all } (x, t) \in [a, b] \times \mathbb{R}.$$

In view of (2.4), one has

$$\begin{aligned}&\frac{1}{Mp}\left(\|u'\|_{L^p}^p+\frac{\alpha^{p-1}}{\beta^{p-1}}\left|u(a)-\frac{1}{\alpha}A\right|^p+\frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u(b)-\frac{1}{\gamma}B\right|^p\right)\\ &\leq\Psi(u)\leq\frac{1}{mp}\left(\|u'\|_{L^p}^p+\frac{\alpha^{p-1}}{\beta^{p-1}}\left|u(a)-\frac{1}{\alpha}A\right|^p+\frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u(b)-\frac{1}{\gamma}B\right|^p\right).\end{aligned}\tag{2.7}$$

Lemma 2.1. ([14, Lemma 2.1]) *Assume that $u\in X$ and there exists $r>0$ such that $\Phi(u)\leq r$, then, we have*

$$\|u\|_\infty\leq(Mpr)^{\frac{1}{p}}\left(\left(\frac{\beta}{\alpha}\right)^{\frac{1}{q}}+(b-a)^{\frac{1}{q}}\right)+\frac{1}{\alpha}|A|,$$

where q is the conjugate of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$.

Definition 2.1. We say that u is a classical solution to (1.1) if $u\in C^1([a,b])$, $|u'|^{p-2}u'\in AC^1([a,b])$, $\alpha u(a)-\beta u'(a)=\int_a^b\xi(x)u(x)dx$, $\gamma u(b)-\sigma u'(b)=\int_a^{b\eta(x)}u(x)dx$ and

$$-(\phi_p(u'(x)))'=\left(\lambda f(x,u(x))+\int_0^{u'(x)}\frac{\partial}{\partial x}\left(\frac{(p-1)|\tau|^{p-2}}{h(x,\tau)}d\tau\right)\right)h(x,u'(x)),$$

for almost every complete $x\in[a,b]$, where $AC^1([a,b])$ denotes the space of those functions whose first derivatives along with themselves are absolutely continuous on $[a,b]$.

Definition 2.2. We say that u is a weak solution to (1.1) if $u\in X$ and

$$\begin{aligned}&\int_a^b J(x,u'(x))v'(x)dx+J\left(a,\frac{\alpha}{\beta}u(a)-\frac{1}{\beta}A\right)v(a)-J\left(b,-\frac{\gamma}{\sigma}u(b)+\frac{1}{\sigma}B\right)v(b)\\&-\lambda\int_a^b f(x,u(x))v(x)dx=0,\end{aligned}$$

for any $v\in X$.

Lemma 2.2. ([14, Lemma 2.4]) *Weak solutions of* (1.1) *coincide with classical solutions of* (1.1).

Lemma 2.3. ([14, Lemma 2.5]) *Assume that the functional $\Psi:X\to\mathbb{R}$ is defined by* (2.5). *Then Ψ is sequentially weakly lower semicontinuous, continuous, $\lim_{\|u\|\to+\infty}\Psi(u)=+\infty$ and its Gâteaux derivative $u\in X$ is the functional $\Psi'(u)$ given by*

$$\begin{aligned}\Psi'(u)(v)=&\int_a^b J(x,u'(x))v'(x)dx+J\left(a,\frac{\alpha}{\beta}u(a)-\frac{1}{\beta}A\right)v(a)\\&-J\left(b,-\frac{\gamma}{\sigma}u(b)+\frac{1}{\sigma}B\right)v(b),\end{aligned}$$

for every $v\in X$.

Remark 2.1. If $u \in X$ is a critical point of $I_\lambda = \Psi + \lambda\Phi$ in view of Definition 2.2, then, u is a classical solution of the problem (1.1).

3. Main Results

For any $\nu > 0$, we define

$$Q(\nu) := \left\{ t \in \mathbb{R} : |t| \leq \nu\left(\left(\frac{\beta}{\alpha}\right)^{\frac{1}{q}} + (b-a)^{\frac{1}{q}}\right) + \frac{1}{\alpha}|A| \right\}.$$

We formulate our first main result as an application of Theorem 2.1 as follows.

Theorem 3.1. *Assume there exist two positive constants $c_1 < c_2$ and a function $w \in X$ such that*

(A_1) $c_1^p \leq K_w \leq \frac{m}{M}c_2^p$, *where*

$$K_w := \left(\|w'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}\left|w(a) - \frac{1}{\alpha}A\right|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}\left|w(b) - \frac{1}{\gamma}B\right|^p\right);$$

(A_2) $A_{c_i}Mp < \dfrac{\int_a^b F(x,w(x))dx - \int_a^b \sup_{t\in Q(c_1)} F(x,t)dx}{\Psi(w)}$ *for* $i = 1, 2$.

Then, for each

$$\lambda \in \left] \frac{\Psi(w)}{\int_a^b F(x,w(x))dx - \int_a^b \sup_{t\in Q(c_1)} F(x,t)dx}, \frac{\min\{\frac{1}{Ac_1}, \frac{1}{Ac_2}\}}{Mp} \right[,$$

the problem (1.1) *has at least two classical solutions whose norms in* $C([a,b])$ *are less than* c_2 *where* $A_{c_i} = \frac{1}{c_i^p}\int_a^b \sup_{t\in Q(c_i)} F(x,t)dx$.

Proof. Let Ψ, Φ be as given by (2.5) and (2.6), respectively. By Lemma 2.3 we observe that $\Psi, \Phi : X \to \mathbb{R}$ are two sequentially weakly lower semicontinuous and Gâteaux differentiable functions and Ψ is continuous and satisfies $\lim_{\|u\|\to+\infty} \Psi(u) = +\infty$. We want to obtain at least two critical points of $I_\lambda = \Psi + \lambda\Phi$ by applying Theorem 2.1. It remains to verify condition (a_1), (a_2) and (a_3) in Theorem 2.1. Let $r_i = \frac{c_i^p}{Mp}$, $i = 1, 2$. By (2.7) and (A_1) we have

$$r_1 < \frac{1}{Mp}K_w \leq \Psi(w) \leq \frac{1}{mp}K_w < r_2.$$

It is easy to see that (a_1) holds since $r_1, r_2 > 0$. Now we will show that (a_2) in Theorem 2.1 is satisfied. Taking into account that the function $u \equiv 0$ on $[a,b]$ obviously belongs to $\Psi^{-1}(]-\infty, r[)$ and that $\Psi(0) = \Phi(0) = 0$, we get

$$\varphi_1(r) = \inf_{u\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{u\in\overline{\Psi^{-1}(]-\infty,r[)}^w} \Phi(x)}{r - \Psi(u)} \leq -\frac{1}{r} \inf_{u\in\overline{\Psi^{-1}(]-\infty,r[)}^w} \Phi(u). \tag{3.1}$$

Noticing $\overline{\Psi^{-1}(]-\infty,r[)}^{\omega}=\Psi^{-1}(]-\infty,r[)$ by Lemma 2.1 we obtain

$$\begin{aligned}\Psi^{-1}(-\infty,r)&=\{u\in X:\Psi(u)<r\}\\&\subseteq\left\{u\in X:\|u\|_\infty\le(Mpr)^{\frac{1}{p}}\left(\left(\frac{\beta}{\alpha}\right)^{\frac{1}{q}}+(b-a)^{\frac{1}{q}}\right)+\frac{1}{\alpha}|A|\right\}\\&=\left\{u\in X:\max_{x\in[a,b]}|u(x)|\in Q(c)\right\}.\end{aligned}$$

Then

$$\begin{aligned}\varphi_1(r)&\le\frac{\sup_{u\in\Psi^{-1}(-\infty,r)}\int_a^b F(x,u(x))dx}{r}\\&\le\frac{\int_a^b\sup_{t\in Q(c)}F(x,t)dx}{r},\end{aligned}$$

and therefore, we have

$$\varphi_1(r_i)\le\frac{Mp}{c_i^p}\int_a^b\sup_{t\in Q(c_i)}F(x,t)dx,\quad i=1,2.$$

On the one hand, by Lemma 2.1 and $r_1\le\Psi(w)\le r_2$ we have

$$\begin{aligned}\varphi_2(r_1,r_2)&=\inf_{u\in\Psi^{-1}(]-\infty,r_1[)}\sup_{v\in\Psi^{-1}([r_1,r_2[)}\frac{\Phi(u)-\Phi(v)}{\Psi(v)-\Psi(u)}\\&\ge\inf_{u\in\Psi^{-1}(]-\infty,r_1[)}\frac{\Phi(u)-\Phi(w)}{\Psi(w)-\Psi(u)}\\&\ge\inf_{u\in\Psi^{-1}(]-\infty,r_1[)}\frac{1}{\Psi(w)-\Psi(u)}\left(\int_a^b F(x,w(x))dx-\int_a^b F(x,u(x))dx\right)\\&\ge\frac{\int_a^b F(x,w(x))dx-\int_a^b\sup_{t\in Q(c_1)}F(x,t)dx}{\Psi(w)-\Psi(u)}.\end{aligned}$$

By (A_2) we have that $\int_a^b F(x,w(x))dx-\int_a^b F(x,u(x))dx>0$, so

$$\varphi_2(r_1,r_2)\ge\frac{\int_a^b F(x,w(x))dx-\int_a^b\sup_{t\in Q(c_1)}F(x,t)dx}{\Psi(w)}.$$

Then, from (A_2), (a_2) and (a_3) in Theorem 2.1 are fulfilled. By choosing $\sigma=r_2$, the conclusion follows. Therefore, it follows that the functional I_λ has two critical points which are the weak solutions of the problem (1.1), and since from Lemma 2.3 the weak solutions coincide with the classical solutions, we have the desired result. □

In Theorem 3.1, the condition (A_2) is related to the function $w\in W^{1,p}$. A different function $w\in W^{1,p}$ would lead to a different condition, which is similar to (A_2). For example, we let $w(x)=d$ where d is a constant. We have the following result.

Corollary 3.1. *Assume there exist three positive constants c_1,d,c_2 such that*

(A_1') $c_1^p < K_d < \frac{m}{M} c_2^p$, *where*

$$K_d := \left(\frac{\alpha}{\beta}\right)^{p-1} \left|d - \frac{1}{\alpha}A\right|^p + \left(\frac{\gamma}{\sigma}\right)^{p-1} \left|d - \frac{1}{\gamma}B\right|^p;$$

(A_2') $A_{c_i} \frac{M}{m} < \frac{B(d,c_1)}{K_d}$,

where A_{c_i} *is defined in Theorem* 3.1 *and*

$$B(d, c_1) = \int_a^b F(x, d)dx - \int_a^b \sup_{t \in Q(c_1)} F(x, t)dx.$$

Then, for every

$$\lambda \in \left] \frac{K_d}{mpB(d, c_1)}, \frac{\min\{\frac{1}{Ac_1}, \frac{1}{Ac_2}\}}{Mp} \right[,$$

the problem (1.1) *has at least two classical solutions whose norms in* $C([a, b])$ *are less than* c_2.

Next, we state our second main result as an application of Theorem 2.2 as follows.

Theorem 3.2. *Assume that there exist five constants* c_1, d_1, c_2, d_2, c_3 *with*

$$c_i^p < K_{d_i} \le c_{i+1}^p \frac{m}{M}, \quad i = 1, 2,$$

such that

$$\frac{M}{m} A^*(c_1, c_2, c_3) \le B^*_{c_1,c_2}(d_1, d_2), \tag{3.2}$$

where

$$A^*(c_1, c_2, c_3) = \max\{A_{c_i} : i = 1, 2, 3\}$$

and

$$B^*_{c_1,c_2}(d_1, d_2) = \min\left\{\frac{B(d_1, c_1)}{K_{d_1}}, \frac{B(d_2, c_2)}{K_{d_2}}\right\}.$$

Then, for each

$$\lambda \in \left] \frac{1}{mpB^*_{c_1,c_2}(d_1, d_2)}, \frac{1}{Mp\ A^*(c_1, c_2, c_3)} \right[,$$

the problem (1.1) *admits at least three classical solutions whose norms in* $C([a, b])$ *are less than* c_3.

Proof. Take the Banach space X and the functionals Ψ, Φ on X are defined by (2.5) and (2.6). Let $r_i = \frac{c_i^p}{Mp}$ and $w_1 = d_1$, $w_2 = d_2$. By the same arguing as given in the proof of Theorem 3.1 one has

$$r_1 < \Psi(w_1) < r_2 < \Psi(w_2) < r_3,$$

$$\varphi_2(r_1, r_2) \ge \frac{mp}{K_{d_1}} B(d_1, c_1),$$

$$\varphi_2(r_1, r_2, r_3) \ge \varphi_2(r_2, r_3) \ge \frac{mp}{K_{d_2}} B(d_2, c_2)$$

and

$$\varphi(r_i) \leq Mp\ A_{c_i}, \quad i = 1, 2, 3.$$

Therefore, taking into account (3.2), there exist at least three classical solutions. Not taking into account the zero solution, there are at least three nonzero classical solutions whose norms in $C([a,b])$ are less than c_3. Then, taking into account the fact that the weak solutions of the problem (1.1) are exactly critical points of the functional I_λ, also by using Lemma 2.3, we know the weak solutions coincide with the classical solutions, so we have the desired conclusion. □

Remark 3.1. If $f(x,0) \neq 0$ for some $x \in [a,b]$, then the ensured solutions in Theorem 3.1 are non-trivial. On the other hand, the non-triviality of the solution can be achieved also in $f(x,0) = 0$ for some $x \in [a,b]$, requiring the extra condition at zero, and there are a non-empty open set $D \subseteq (a,b)$ and $B \subset D$ such that

$$\limsup_{\xi \to 0^+} \frac{\inf_{x \in B} F(x,\xi)}{|\xi|^p} = +\infty$$

and

$$\liminf_{\xi \to 0^+} \frac{\inf_{x \in D} F(x,\xi)}{|\xi|^p} > -\infty.$$

Indeed, let $0 < \overline{\lambda} < \lambda^*$, where

$$\lambda^* = \frac{\min\left\{\frac{1}{Ac_1}, \frac{1}{Ac_2}\right\}}{Mp}.$$

Let Φ and Ψ be as given in (2.5) and (2.6), respectively. Due to Corollary 3.1 for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \overline{\lambda}\right)$ there exists a critical point of $I_\lambda = \Psi + \lambda\Phi$ such that $u_\lambda \in \Psi^{-1}(-\infty, r)$, where $r_\lambda = \frac{c_\lambda^p}{Mp}$. In particular, u_λ is a global minimum of the restriction of I_λ to $\Psi^{-1}(-\infty, r)$. We will prove that u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\| \to 0^+} \frac{\Phi(u)}{\Psi(u)} = +\infty. \tag{3.3}$$

Thanks to our assumptions at zero, we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that for every $\xi \in [0, \sigma]$

$$\lim_{\xi \to 0^+} \frac{\inf_{x \in B} F(x,\xi_n)}{|\xi_n|^p} = +\infty \tag{3.4}$$

and

$$\inf_{x \in D} F(x,\xi) > \kappa|\xi|^p.$$

We consider a set $G \subset B$ of positive measure and a function $v \in X$ such that

(k_1) $v(t) \in [0,1]$ for every $t \in (a,b)$;
(k_2) $v(t) = 1$ for every $t \in G$;
(k_3) $v(t) = 0$ for every $t \in (a,b) \backslash D$.

Finally, fix $M > 0$ and consider a real positive number η with

$$M < \frac{mp\eta \text{ meas}(G) + mp\kappa \int_{D\setminus G} |v(t)|dt}{K_u},$$

where

$$K_u = \frac{1}{mp}\left(\|u'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}\left|u(a) - \frac{1}{\alpha}A\right|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u(b) - \frac{1}{\gamma}B\right|^p \right).$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$\inf_{x\in B} F(x, \xi_n) \geq \kappa|\xi_n|^p,$$

for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \leq \xi_n v(t) < \sigma$ for n large enough), one has

$$\begin{aligned} \frac{\Phi(\xi_n v)}{\Psi(\xi_n v)} =& \frac{\int_G F(t, \xi_n)dt + \int_{D\setminus G} F(t, \xi_n v(t))dt}{\Psi(\xi_n v)} \\ >& \frac{mp\eta \text{ meas}(G) + mp\kappa \int_{D\setminus G} |v(t)|dt}{K_u} > M. \end{aligned}$$

Since M could be arbitrarily large, it yields

$$\lim_{n\to\infty} \frac{\Phi(\xi_n v)}{\Psi(\xi_n v)} = +\infty$$

from which (3.3) clearly follows. Hence, there exists $\{\omega_n\} \subset X$ strongly converging to zero such that, $\omega_n \in \Psi^{-1}(-\infty, r)$ and

$$I_\lambda(\omega_n) = \Psi(\omega_n) + \lambda\Phi(\omega_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Psi^{-1}(-\infty, r)$, we conclude that

$$I_\lambda(u_\lambda) < 0. \tag{3.5}$$

Remark 3.2. From (3.5) we easily observe that the map

$$\left(\frac{K_d}{mpB(d, c_1)}, \lambda^*\right) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.6}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, bearing in mind that Ψ is coercive and for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$ the solution $u_\lambda \in \Psi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \leq L$ for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Then, there exists a positive constant N such that

$$\left|\int_a^b f(x, u(x))v(x)dx\right| \leq N\|u_\lambda\| \leq NL, \tag{3.7}$$

for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in X$ and every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is

$$\Psi'(u_\lambda)(u_\lambda) = -\lambda \int_a^b f(x, u_\lambda(x))u_\lambda(x)dx,$$

for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Then, it follows

$$\begin{aligned}
0 &\leq \frac{1}{Mp}\left(\|u'_\lambda\|^p_{L^p} + \frac{\alpha^{p-1}}{\beta^{p-1}}\left|u_\lambda(a) - \frac{1}{\alpha}A\right|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u_\lambda(b) - \frac{1}{\gamma}B\right|^p\right) \\
&\leq \Psi'(u_\lambda)(u_\lambda) \\
&= -\lambda \int_a^b f(x, u_\lambda(x))u_\lambda(x)dx,
\end{aligned}$$

for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Letting $\lambda \to 0^+$ by (3.7), we get

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. Finally, we have to show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. We see that for any $u \in X$ one has

$$I_\lambda = \lambda\left(\frac{\Psi(u)}{\lambda} + \Phi(u)\right). \tag{3.8}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Psi(-\infty, r)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \left(\frac{\Psi(u_{\lambda_i})}{\lambda_i} + \Phi(u_{\lambda_i})\right) = \inf_{v \in \Psi^{-1}(-\infty,r)}\left(\frac{\Psi(v)}{\lambda_i} + \Phi(v)\right),$$

for every $i = 1, 2$. Clearly, (3.6) together with (3.8) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad \text{for } i = 1, 2. \tag{3.9}$$

Moreover

$$m_{\lambda_2} < m_{\lambda_1}, \tag{3.10}$$

due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.8)–(3.10) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1},$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. The arbitrariness of $\lambda < \lambda^*$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$.

We now present the following example to illustrate Corollary 3.1.

Example 3.1. Let $a = 0$, $b = 1$, $\alpha = \beta = 1$, $\gamma = 1$, $\sigma = 2$, $A = 0$, $B = 10$, $p = 2$, $h(x,t) = 1+x+|\sin t|$ for every $(x,t) \in [0,1]\times\mathbb{R}$ and $f(x,t) = \frac{1}{10^6}\Big(t^9e^{-t}(10-t)\sin x\Big)$ for every $t \in \mathbb{R}$. By the expression of f, we have $F(x,t) = \frac{1}{10^6}\Big(t^{10}e^{-t}\sin x\Big)$ for every $t \in \mathbb{R}$. We observe that $m = 1$, and $M = 3$. Choosing $d = 10$, $c_1 = \frac{1}{10}$, $c_2 = 10^2$, since $Q(c_1) = \frac{2}{10}$, $Q(c_2) = 2\times 10^2$, $K_d = 10^2$, we see that all conditions in Corollary 3.1 are satisfied. Therefore, taking Remark 3.2 it follows that for each

$$\lambda \in \left]\frac{10^2}{9180e^{-10}}, \frac{2^{13}\times 57375e^{-200}}{6}\right[,$$

the problem

$$\begin{cases} -(\phi_p(u'))' = \left(\lambda f(u) + \displaystyle\int_0^{u'(x)} \frac{\partial}{\partial x}\Big(\frac{(p-1)|\tau|^{p-2}}{1+x+|\sin\tau|}\Big)d\tau\right) & \\ \times(1+x+|\sin u'(x)|), & x\in(0,1), \\ u(0)-u'(0) = 0, \quad u(1)+2u'(1) = 10, & \end{cases}$$

has at least two nontrivial solutions $u_{1\lambda}$ and $u_{2\lambda}$ in X such that

$$\lim_{\lambda\to 0^+} \|u_{i\lambda}\| = 0$$

and the real function

$$\begin{aligned} \lambda \to \int_a^b H(x, u_{i\lambda}'(x))dx + H\Big(0, u_{i\lambda}(0)\Big) + \frac{\sigma}{\gamma}H\Big(1, -\frac{1}{2}u_{i\lambda}(1) + \frac{1}{2}10\Big) \\ + \frac{\lambda}{10^6}\int_0^1 t^{10}e^{-t}\sin u_{i\lambda}(x)dx, \end{aligned}$$

for $i = 1, 2$.

References

[1] R. P. Agarwal, H. L. Hong and C. C. Yeh, *The existence of positive solutions for the Sturm-Liouville boundary value problems*, Comput. Math. Appl. **35**(9) (1998), 89–96. http://scholarbank.nus.edu.sg/handle/10635/104293

[2] D. Averna and G. Bonanno, *A three critical points theorem and its applications to the ordinary Dirichlet problem*, Topol. Methods Nonlinear Anal. **22**(1) (2003), 93–104.

[3] D. Averna and G. Bonanno, *Three solutions for a quasilinear two-point boundary value problem involving the one-dimensional p-Laplacian*, Proc. Edinb. Math. Soc. **47** (2004), 257–270. https://doi.org/10.1017/S0013091502000767

[4] M. Bohner, G. Caristi, S. Heidarkhani and S. Moradi, *A critical point approach to boundary value problems on the real line*, Appl. Math. Lett. **76** (2018), 5 pages. https://doi.org/10.1016/j.aml.2017.08.017

[5] G. Bonanno, *Multiple critical points theorems without the Palais-Smale condition*, J. Math. Anal. Appl. **299**(2) (2004), 600–614. https://doi.org/10.1016/j.jmaa.2004.06.034

[6] G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differ. Equ. **244**(12) (2008), 3031–3059.

[7] G. Bonanno and G. D'Aguì, *A Neumann boundary value problem for the Sturm-Liouville equation*, Appl. Math. Comput. **208**(2) (2009), 318–327. https://doi.org/10.1016/j.amc.2008.12.029

[8] G. Bonanno and G. Riccobono, *Multiplicity results for Sturm-Liouville boundary value problems*, Appl. Math. Comput. **210**(2) (2009), 294–297. https://doi.org/10.1016/j.amc.2008.12.081

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[10] L. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equation*, Proc. Amer. Math. Soc. **120** (1994), 743–748. https://doi.org/10.1090/S0002-9939-1994-1204373-9

[11] W. Ge and J. Ren, *New existence theorems of positive solutions for Sturm-Liouville boundary value problems*, Appl. Math. Comput. **148**(3) (2004), 631–644.

[12] J. R. Graef, S. Heidarkhani and L. Kong, *A critical points approach for the existence of multiple solutions of a Dirichlet quasilinear system*, J. Math. Anal. Appl. **388**(2) (2012), 1268–1278. https://doi.org/10.1016/j.jmaa.2011.11.019

[13] J. R. Graef, S. Heidarkhani and L. Kong, *Infinitely many solutions for systems of Sturm-Liouville boundary value problems*, Results Math. **66** (2014), 327–341.

[14] F. Sun, L. Liu and Y. Wu, *Infinitely many solutions for a class of p-Laplacian equation with Sturm-Liouville type nonhomogeneous boundary conditions*, J. Nonlinear Sci. Appl. **10**(11) (2017), 6020–6034. http://dx.doi.org/10.22436/jnsa.010.11.37

[15] Y. Tian, J. R. Graef, L. Kong and M. Wang, *Three solutions for second-order impulsive differential inclusions with Sturm- Liouville boundary conditions via nonsmooth critical point theory*, Topol. Methods Nonlinear Anal. **47**(1) (2016), 17 pages. https://doi.org/10.12775/TMNA.2015.089

[16] Y. Tian and W. Ge, *Multiple solutions for a second-order Sturm-Liouville boundary value problem*, Taiwanese J. Math. **11**(4) (2007), 975–988. https://doi.org/10.11650/twjm/1500404796

[17] Yu. Tian and W. Ge, *Multiple solutions of Sturm-Liouville boundary value problem via lower and upper solutions and variational methods*, Nonlinear Anal. **74**(17) (2011), 6733–6746. https://doi.org/10.1016/j.na.2011.06.053

[18] Y. Tian and W. Ge, *Second-order Sturm-Liouville boundary value problem involving the one-dimensional p-Laplacian*, Rocky Mountain J. Math. **38**(1) (2008), 309–327. https://doi.org/10.1216/RMJ-2008-38-1-309

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