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PROPER SCALED CONVERGENCE AND ITS STATISTICAL EXTENSIONS

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Abstract. This article introduces a new and mathematically coherent framework for scaled convergence of sequences. We show that the scaling function must satisfy a positivity condition in order for scaled limits to be well defined and unique. This leads to the notion of a *proper scaling function*, which provides the basic setting for the results developed in this paper. Within this setting we develop the properties of properly scaled convergence, establish the equivalence with a scaled Cauchy condition (under a natural monotonicity assumption on the scaling function), and show stability under algebraic operations. We also introduce *rate-controlled convergence* as a companion concept for the regime in which the scaling function diverges. Moreover, we define and study ϕ-scaled statistical convergence and ϕ-scaled strong Cesàro convergence, obtaining natural implication results. Several examples and an application to Newton-Raphson iteration demonstrate the usefulness of the theory.

1. Introduction

In many contexts of numerical analysis, summability theory, and asymptotic approximation, it is desirable to refine classical convergence by incorporating information about the rate at which a sequence approaches its limit. A natural way to achieve this is to introduce a scaling function $\phi : \mathbb{N} \to (0, +\infty)$ and replace the classical condition $|a_n - L| \to 0$ with the scaled condition

$$\phi(n)|a_n - L| \to 0.$$

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However, this idea requires a structural restriction on ϕ in order to be mathematically meaningful. If $\phi(n) \to 0$, then $\phi(n)|a_n - L| \to 0$ holds for every bounded sequence and every real number L, making the scaled limit completely nonunique. Similarly, if $\liminf_{n\to+\infty} \phi(n) = 0$, then one loses all control over the values of the sequence on infinitely many indices. These observations motivate the introduction of the following notion. The concept of a proper scaling function is loosely connected with Karamata's theory of regular and slowly varying functions [1]. In contrast to regularly varying functions, which satisfy $\phi(\lambda n)/\phi(n) \to \lambda^\rho$, proper scaling functions are characterized only by a positive lower bound on $\phi(n)$. Thus proper scaling does not impose any homogeneity or monotonicity structure; instead, it isolates exactly the condition necessary to guarantee uniqueness and stability of scaled limits. This places proper scaled convergence in close relation to classical convergence and weighted convergence theories in the sense of Hardy, Cesàro, and statistical summability.

Weighted forms of convergence (such as weighted Cesàro means or Orlicz-modular convergence) require the weights to appear inside the averaging operator. In contrast, proper scaled convergence modifies the pointwise deviation $|a_n - L|$ through an external scaling factor. Thus ϕ-scaled convergence is neither a weighted means method nor a regular variation condition; it provides a distinct form of local rate control.

Definition 1.1 (Proper scaling function)**.** A scaling function $\phi : \mathbb{N} \to (0, +\infty)$ is called *proper* if

$$\liminf_{n\to+\infty} \phi(n) > 0.$$

Equivalently, there exist $c > 0$ and $N \in \mathbb{N}$ such that $\phi(n) \geq c$ for all $n \geq N$.

This requirement stabilizes the scaled limit and ensures that it behaves as a genuine refinement of classical convergence. Building on this idea, the main goal of this paper is to develop the theory of *proper scaled convergence* and its statistical and Cesàro variants.

The paper is organized as follows. Section 2 introduces properly scaled convergence and its basic properties. Section 3 provides the Cauchy characterization. Section 4 studies the complementary regime in which the scaling function diverges and introduces the concept of *rate-controlled convergence*. Section 5 develops the notions of ϕ-scaled statistical convergence and ϕ-scaled strong Cesàro convergence. Section 6 presents an application to Newton–Raphson iteration.

2. Proper Scaled Convergence

Definition 2.1 (Properly scaled convergence)**.** Let ϕ be a proper scaling function. A sequence (a_n) is said to converge *ϕ-scaled* to L if

$$\lim_{n\to+\infty} \phi(n)|a_n - L| = 0.$$

We write $a_n \xrightarrow{\phi} L$.

Example 2.1. Let $\phi(n) = \log(n+1)$ and $a_n = L + 1/\log(n+1)$. Then,

$$\phi(n)|a_n - L| = \log(n+1) \cdot \frac{1}{\log(n+1)} = 1,$$

so the sequence is not ϕ-scaled convergent. This example shows that ordinary convergence alone does not imply ϕ-scaled convergence, even when the scaling function grows slowly.

Theorem 2.1 (Uniqueness). *If ϕ is proper, then every ϕ-scaled convergent sequence has a unique limit.*

Proof. Let $a_n \xrightarrow{\phi} L_1$ and $a_n \xrightarrow{\phi} L_2$. Then,

$$\phi(n)|L_1 - L_2| \leq \phi(n)|a_n - L_1| + \phi(n)|a_n - L_2| \to 0.$$

Since $\phi(n) \geq c > 0$ eventually, we obtain $L_1 = L_2$. □

Remark 2.1 (Relation to classical convergence). It is worth emphasizing that ϕ-scaled convergence is, in general, *stronger* than classical convergence. Indeed, if ϕ is proper, then there exist constants $c > 0$ and $N \in \mathbb{N}$ such that $\phi(n) \geq c$ for all $n \geq N$, and hence $\phi(n)|a_n - L| \to 0$ implies $|a_n - L| \to 0$. Thus every ϕ-convergent sequence is classically convergent. This does not indicate a deficiency of the theory; on the contrary, ϕ-scaled convergence should be viewed as a *refinement* of ordinary convergence, analogous to the role played by uniform convergence relative to pointwise convergence, or absolute convergence relative to conditional convergence.

The purpose of introducing the scaling function ϕ is not to enlarge the class of convergent sequences, but rather to distinguish different *rates of decay* of $|a_n - L|$ and to encode them in a unified way. In particular, when $\sup_n \phi(n) = +\infty$, the scaled limit requires

$$|a_n - L| = o\left(\frac{1}{\phi(n)}\right),$$

which is strictly stronger than classical convergence and leads to a meaningful hierarchy of convergence based on the growth of ϕ.

Theorem 2.2 (Preservation under algebraic operations). *Let ϕ be proper and assume $a_n \xrightarrow{\phi} A$ and $b_n \xrightarrow{\phi} B$. Then,*

(a) $a_n + b_n \xrightarrow{\phi} A + B$;

(b) $a_n b_n \xrightarrow{\phi} AB$;

(c) *if* $\inf_n |b_n| > 0$, *then* $\dfrac{a_n}{b_n} \xrightarrow{\phi} \dfrac{A}{B}$.

Proof. (a) Since $a_n \xrightarrow{\phi} A$ and $b_n \xrightarrow{\phi} B$, we have

$$\phi(n)|a_n - A| \to 0 \quad \text{and} \quad \phi(n)|b_n - B| \to 0.$$

Using the triangle inequality,

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B|.$$

Multiplying by $\phi(n)$ gives

$$\phi(n)|(a_n + b_n) - (A + B)| \leq \phi(n)|a_n - A| + \phi(n)|b_n - B| \to 0 + 0 = 0,$$

so $a_n + b_n \xrightarrow{\phi} A + B$.

(b) Since ϕ is proper and $a_n \xrightarrow{\phi} A$, we know $a_n \to A$ in the ordinary sense; in particular, (a_n) is bounded. Thus, there exists $M > 0$ such that $|a_n| \leq M$ for all sufficiently large n. We write

$$a_n b_n - AB = (a_n - A)(b_n - B) + A(b_n - B) + B(a_n - A).$$

Hence,

$$|a_n b_n - AB| \leq |a_n - A| \cdot |b_n - B| + |A| \cdot |b_n - B| + |B| \cdot |a_n - A|.$$

For large n we also have $b_n \to B$, so (b_n) is bounded and, in particular, $(|b_n - B|)$ is bounded; therefore there exists a constant $K > 0$ such that $|b_n - B| \leq K$ for all large n. Consequently,

$$|a_n - A| \cdot |b_n - B| \leq K|a_n - A|,$$

for all sufficiently large n. Multiplying the above inequality by $\phi(n)$, we get

$$\phi(n)|a_n b_n - AB| \leq K\,\phi(n)|a_n - A| + |A|\,\phi(n)|b_n - B| + |B|\,\phi(n)|a_n - A|.$$

Each term on the right tends to 0, since $a_n \xrightarrow{\phi} A$ and $b_n \xrightarrow{\phi} B$. Thus, $\phi(n)|a_n b_n - AB| \to 0$, i.e., $a_n b_n \xrightarrow{\phi} AB$.

(c) Assume $\inf_n |b_n| > 0$, in particular $B \neq 0$. Write

$$\frac{a_n}{b_n} - \frac{A}{B} = \frac{B(a_n - A) - A(b_n - B)}{B\, b_n}.$$

Taking absolute values yields

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| \leq \frac{|B| \cdot |a_n - A| + |A| \cdot |b_n - B|}{|B| \cdot |b_n|}.$$

By assumption there exists $\delta > 0$ such that $|b_n| \geq \delta$ for all n. Therefore

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| \leq \frac{1}{|B|\,\delta}\Big(|B| \cdot |a_n - A| + |A| \cdot |b_n - B|\Big).$$

Multiplying by $\phi(n)$, we obtain

$$\phi(n)\left|\frac{a_n}{b_n} - \frac{A}{B}\right| \leq \frac{1}{|B|\,\delta}\Big(|B|\,\phi(n)|a_n - A| + |A|\,\phi(n)|b_n - B|\Big) \to 0,$$

because both $\phi(n)|a_n - A|$ and $\phi(n)|b_n - B|$ tend to 0. Hence, $\frac{a_n}{b_n} \xrightarrow{\phi} \frac{A}{B}$. $\square$

Example 2.2 (Comparison of different scaling functions). Let

$$a_n = \frac{1}{\log(n+2)}.$$

Consider the following scaling functions:

$$\phi_1(n) = \log(n+1), \quad \phi_2(n) = n^{0.1}, \quad \phi_3(n) = \sqrt{n}.$$

Then, the following hold.

- For ϕ_1,
$$\phi_1(n)|a_n - 0| = \frac{\log(n+1)}{\log(n+2)} \to 1,$$
hence no ϕ_1-scaled convergence occurs.
- For ϕ_2,
$$\phi_2(n)|a_n| = \frac{n^{0.1}}{\log(n+2)} \to +\infty,$$
so ϕ_2 is too strong.
- For ϕ_3,
$$\phi_3(n)|a_n| = \frac{\sqrt{n}}{\log(n+2)} \to +\infty,$$
giving again no scaled convergence.

This comparison illustrates that the choice of scaling function is crucial and that even slowly decaying sequences may fail to be ϕ-scaled convergent under moderately growing scales.

3. Cauchy Characterization

Definition 3.1 (ϕ-scaled Cauchy sequence)**.** Let ϕ be a proper scaling function. A sequence (a_n) is called *ϕ-scaled Cauchy* if

$$\phi(\min\{m, n\})|a_m - a_n| \to 0, \quad \text{as } m, n \to +\infty.$$

Remark 3.1. For the equivalence between ϕ-scaled convergence and the ϕ-scaled Cauchy condition, we require that ϕ be eventually nondecreasing. This is a mild additional assumption that is satisfied by most natural scaling functions (e.g., $\log n$, n^α, 2^n).

Theorem 3.1. *Let ϕ be a proper scaling function that is eventually nondecreasing. Then a sequence (a_n) is ϕ-scaled convergent if and only if it is ϕ-scaled Cauchy.*

Proof. ($\Rightarrow$) Suppose $a_n \xrightarrow{\phi} L$, i.e., $\phi(n)|a_n - L| \to 0$. Let $\varepsilon > 0$ be given. Since $\phi(n)|a_n - L| \to 0$, there exists $N \in \mathbb{N}$ such that

$$\phi(n)|a_n - L| < \varepsilon, \quad \text{for all } n \geq N.$$

For $m, n \geq N$ we have, by the triangle inequality,

$$|a_m - a_n| \leq |a_m - L| + |a_n - L|.$$

Let $N_\phi \in \mathbb{N}$ be such that ϕ is nondecreasing on $\{N_\phi, N_\phi + 1, \ldots\}$. Replacing N, if necessary, by $\max\{N, N_\phi\}$, we have

$$\phi(k) \leq \phi(m) \quad \text{and} \quad \phi(k) \leq \phi(n),$$

where $k = \min\{m, n\}$. Thus,

$$\phi(k)|a_m - a_n| \le \phi(k)|a_m - L| + \phi(k)|a_n - L| \le \phi(m)|a_m - L| + \phi(n)|a_n - L| < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, $\phi(\min\{m, n\})|a_m - a_n| \to 0$, so (a_n) is ϕ-scaled Cauchy.

($\Leftarrow$) Conversely, suppose (a_n) is ϕ-scaled Cauchy. Since ϕ is proper, there exist $c > 0$ and $N_0 \in \mathbb{N}$ such that

$$\phi(n) \ge c, \quad \text{for all } n \ge N_0.$$

Let $\varepsilon > 0$ be given. By the ϕ-scaled Cauchy property, there exists $N_1 \in \mathbb{N}$ such that

$$\phi(\min\{m, n\})|a_m - a_n| < c\varepsilon, \quad \text{for all } m, n \ge N_1.$$

Set $N = \max\{N_0, N_1\}$. Then, for all $m, n \ge N$ we have $\min\{m, n\} \ge N$, hence $\phi(\min\{m, n\}) \ge c$, and therefore

$$|a_m - a_n| \le \frac{1}{c}\,\phi(\min\{m, n\})|a_m - a_n| < \frac{1}{c} \cdot c\varepsilon = \varepsilon.$$

Thus, (a_n) is an ordinary Cauchy sequence in $\mathbb{R}$ and hence convergent: there exists $L \in \mathbb{R}$ such that $a_n \to L$.

We now show $a_n \xrightarrow{\phi} L$. Fix $\varepsilon > 0$. By the ϕ-scaled Cauchy property, there exists N_2 such that

$$\phi(\min\{m, n\})|a_m - a_n| < \varepsilon, \quad \text{for all } m, n \ge N_2.$$

Choose $N \ge \max\{N_0, N_2\}$ and fix $n \ge N$. For $m \ge n$ we have $\min\{m, n\} = n$, and since ϕ is nondecreasing,

$$\phi(n)|a_m - a_n| = \phi(\min\{m, n\})|a_m - a_n| < \varepsilon.$$

Since $a_m \to L$ as $m \to +\infty$ and $\phi(n)$ is fixed in this limit, we obtain

$$\phi(n)|a_n - L| = \lim_{m \to +\infty} \phi(n)|a_m - a_n| \le \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this shows $\phi(n)|a_n - L| \to 0$, i.e., $a_n \xrightarrow{\phi} L$. □

4. Rate-Controlled Convergence

When $\phi(n) \to +\infty$, the condition $\phi(n)|a_n - L| \to 0$ is strictly stronger than classical convergence; it does not "collapse" to classical convergence but rather refines it by imposing a precise rate condition. In this regime, the growth of ϕ quantifies the rate of convergence.

Definition 4.1 (Rate-controlled convergence)**.** Let (a_n) converge classically to L, and assume $\phi(n) \to +\infty$. We say that (a_n) is *ϕ-rate convergent* to L if

$$|a_n - L| = o\left(\frac{1}{\phi(n)}\right),$$

which is equivalent to $\phi(n)|a_n - L| \to 0$.

Thus rate-controlled convergence is simply ϕ-scaled convergence in the case where $\phi(n) \to +\infty$. We introduce the terminology to emphasize the quantitative aspect of the convergence rate.

Example 4.1 (Logarithmic scaling is too weak). Let $\phi(n) = \log(n+2)$ and

$$a_n = L + \frac{1}{\log(n+2)}.$$

Then,

$$\phi(n)|a_n - L| = \log(n+2) \cdot \frac{1}{\log(n+2)} = 1,$$

so ϕ-scaled convergence fails. Thus, the error $1/\log(n+2)$ does not decay sufficiently fast to yield ϕ-rate convergence for the choice $\phi(n) = \log(n+2)$. In general, ϕ-rate convergence requires

$$|a_n - L| = o\left(\frac{1}{\phi(n)}\right).$$

Example 4.2. If $a_n = L + \frac{1}{n^2}$ and $\phi(n) = n$, then

$$|a_n - L| = \frac{1}{n^2} = o\Big(\frac{1}{n}\Big),$$

so (a_n) is ϕ-rate convergent.

Example 4.3. Let $\phi(n) = 2^n$ and $a_n = L + \frac{2^{-n}}{n}$. Then,

$$\phi(n)|a_n - L| = 2^n \cdot \frac{2^{-n}}{n} = \frac{1}{n} \to 0,$$

so the sequence is ϕ-rate convergent. Note that this requires a super-polynomial decay rate of the error, since $|a_n - L| = o(2^{-n})$. This illustrates the strength of rapidly growing scaling functions.

5. ϕ-Scaled Statistical Convergence

If ϕ is proper, then ϕ-scaled statistical convergence implies classical statistical convergence. Indeed, since $\liminf_{n\to+\infty} \phi(n) > 0$, there exist constants $c > 0$ and $N \in \mathbb{N}$ such that $\phi(n) \geq c$ for all $n \geq N$. Hence, for any $\delta > 0$ and all sufficiently large n,

$$\{n : |a_n - L| \geq \delta\} \subset \{1, 2, \ldots, N-1\} \cup \{n : \phi(n)|a_n - L| \geq c\delta\}.$$

Since finite sets have natural density zero, ϕ-scaled statistical convergence implies ordinary statistical convergence. Thus, if the right-hand set has natural density zero, then (a_n) is statistically convergent to L in the ordinary sense.

If, in addition, ϕ is bounded above, that is, there exists $C > 0$ such that $\phi(n) \leq C$ for all sufficiently large n, then ϕ-scaled statistical convergence is equivalent to classical statistical convergence. Indeed, in this case the scaling factor is bounded both below and above by positive constants.

On the other hand, when ϕ is unbounded, ϕ-scaled statistical convergence may be strictly stronger than classical statistical convergence.

Definition 5.1 (ϕ-scaled statistical convergence)**.** Let ϕ be proper. A sequence (a_n) is said to be *ϕ-scaled statistically convergent* to L if for every $\varepsilon > 0$,

$$d\{n : \phi(n)|a_n - L| \geq \varepsilon\} = 0,$$

where $d(A) = \lim_{n\to+\infty} \frac{|A\cap\{1,\ldots,n\}|}{n}$ denotes the natural density. We write $a_n \xrightarrow{\phi\text{-stat}} L$.

Example 5.1. Proper scaling functions need not be monotone. Let $\phi(n) = 1 + \sin^2(n)$, which satisfies $\phi(n) \geq 1$. Then ϕ-scaled convergence coincides with classical convergence, but the oscillatory behavior of ϕ shows that the theory allows considerable flexibility.

Theorem 5.1. *If ϕ is proper and $a_n \xrightarrow{\phi} L$, then $a_n \xrightarrow{\phi\text{-stat}} L$.*

Proof. By assumption $a_n \xrightarrow{\phi} L$, that is,

$$\phi(n)|a_n - L| \to 0, \quad n \to +\infty.$$

Let $\varepsilon > 0$ be fixed and consider the set

$$A_\varepsilon := \{n \in \mathbb{N} : \phi(n)|a_n - L| \geq \varepsilon\}.$$

Since $\phi(n)|a_n - L| \to 0$, there exists $N \in \mathbb{N}$ such that

$$\phi(n)|a_n - L| < \varepsilon, \quad \text{for all } n \geq N.$$

Hence, for $n \geq N$ we have $n \notin A_\varepsilon$, which means $A_\varepsilon \subset \{1, 2, \ldots, N-1\}$. In particular A_ε is a finite set.

To compute its natural density, note that for $K \geq N$ we have

$$|A_\varepsilon \cap \{1, 2, \ldots, K\}| = |A_\varepsilon|,$$

so

$$0 \leq \frac{|A_\varepsilon \cap \{1, 2, \ldots, K\}|}{K} = \frac{|A_\varepsilon|}{K} \to 0, \quad K \to +\infty.$$

Thus,

$$d(A_\varepsilon) = \lim_{K\to+\infty} \frac{|A_\varepsilon \cap \{1, 2, \ldots, K\}|}{K} = 0.$$

Since this holds for every $\varepsilon > 0$, it follows from the definition that $a_n \xrightarrow{\phi\text{-stat}} L$. □

Example 5.2 (Scaled statistical convergence can be strictly stronger). Let $\phi(n) = \sqrt{n}$ and define

$$a_n = \frac{1}{\sqrt{n}}.$$

Then, (a_n) converges classically (and hence statistically) to 0 because $a_n \to 0$. However,

$$\phi(n)|a_n - 0| = \sqrt{n} \cdot \frac{1}{\sqrt{n}} = 1,$$

for every n. Thus, for any $\varepsilon \le 1$,

$$\{n : \phi(n)|a_n| \ge \varepsilon\} = \mathbb{N},$$

which has density 1, not 0. Therefore, (a_n) is not ϕ-scaled statistically convergent to 0.

This shows that when $\phi(n) \to +\infty$, ϕ-statistical convergence may be strictly stronger than classical statistical convergence.

Definition 5.2 (ϕ-scaled strong Cesàro convergence)**.** Let ϕ be proper. A sequence (a_n) is said to be *ϕ-scaled strong Cesàro convergent* to L if

$$\frac{1}{n}\sum_{k=1}^{n} \phi(k)|a_k - L| \to 0.$$

We write $a_n \xrightarrow{\phi\text{-sCes}} L$.

Theorem 5.2. *If $a_n \xrightarrow{\phi\text{-sCes}} L$ with ϕ proper, then $a_n \xrightarrow{\phi\text{-stat}} L$.*

Proof. By assumption, $a_n \xrightarrow{\phi\text{-sCes}} L$ means that

$$\frac{1}{n}\sum_{k=1}^{n} \phi(k)|a_k - L| \to 0, \quad n \to +\infty.$$

Fix $\varepsilon > 0$ and define

$$A_\varepsilon := \{k \in \mathbb{N} : \phi(k)|a_k - L| \ge \varepsilon\}.$$

For each $k \in \mathbb{N}$ we have $\varepsilon\,\mathbf{1}_{A_\varepsilon}(k) \le \phi(k)|a_k - L|$, where $\mathbf{1}_{A_\varepsilon}$ denotes the indicator function of A_ε. Indeed, if $k \in A_\varepsilon$, then $\phi(k)|a_k - L| \ge \varepsilon$ and hence $\varepsilon\,\mathbf{1}_{A_\varepsilon}(k) = \varepsilon \le \phi(k)|a_k - L|$; if $k \notin A_\varepsilon$, then the left-hand side is zero, whereas the right-hand side is nonnegative. Therefore,

$$\varepsilon \mathbf{1}_{A_\varepsilon}(k) \le \phi(k)|a_k - L|,$$

for every $k \in \mathbb{N}$.

Summing this inequality from $k = 1$ to n and dividing by n yields

$$\varepsilon\,\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{A_\varepsilon}(k) \le \frac{1}{n}\sum_{k=1}^{n} \phi(k)|a_k - L|, \quad \text{for all } n \in \mathbb{N}.$$

Note that

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{A_\varepsilon}(k) = \frac{|A_\varepsilon \cap \{1, 2, \dots, n\}|}{n},$$

so the left-hand side is ε times the proportion of indices $\le n$ that belong to A_ε.

Taking $\limsup$ as $n \to +\infty$ on both sides, we obtain

$$\varepsilon \limsup_{n\to+\infty} \frac{|A_\varepsilon \cap \{1, 2, \dots, n\}|}{n} \le \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \phi(k)|a_k - L| = 0,$$

where the last equality follows from ϕ-scaled strong Cesàro convergence. Since $\varepsilon > 0$ is fixed, it follows that

$$\limsup_{n\to+\infty} \frac{|A_\varepsilon \cap \{1,2,\ldots,n\}|}{n} = 0,$$

and therefore the natural density of A_ε is zero:

$$d(A_\varepsilon) = 0.$$

By the definition of ϕ-scaled statistical convergence, this shows that $a_n \xrightarrow{\phi\text{-stat}} L$. ☐

Corollary 5.1. *If ϕ is proper and $a_n \xrightarrow{\phi\text{-}sCes} L$, then a_n is ordinary statistically convergent to L.*

6. Application: Newton-Raphson Iteration

Let (x_n) be the Newton-Raphson iteration for solving $x^2 - 2 = 0$. The associated error sequence $e_n = x_n - \sqrt{2}$ satisfies $|e_{n+1}| \le C|e_n|^2$ for some $C > 0$.

Proposition 6.1. *Assume that the initial value x_0 is chosen so that the Newton-Raphson iteration converges to $\sqrt{2}$. Let $\phi(n) = n$. Then the Newton-Raphson sequence is ϕ-rate convergent to $\sqrt{2}$.*

Proof. Once the iteration enters the local quadratic convergence region, the estimate $|e_{n+1}| \le C|e_n|^2$ implies that the error decreases at least geometrically for all sufficiently large n. Consequently, there exist $K > 0$, $q \in (0,1)$ and $N \in \mathbb{N}$ such that $|e_n| \le Kq^n$, $n \ge N$. Therefore, $n|e_n| \le Knq^n \to 0$, and hence $|e_n| = o(1/n)$. ☐

Scaled convergence may also be used to formulate deterministic error requirements in noisy approximation schemes. For example, if

$$a_n = L + \frac{\xi_n}{\sqrt{n}}$$

and $\xi_n \to 0$, then choosing $\phi(n) = \sqrt{n}$ gives

$$\phi(n)|a_n - L| = |\xi_n| \to 0.$$

Additional probabilistic assumptions would be required to formulate corresponding almost sure or convergence-in-probability versions.

7. Conclusion

We have developed a consistent framework for scaled convergence based on the concept of a proper scaling function. This notion ensures that scaled limits are meaningful and provides a setting in which classical convergence, algebraic operations, Cauchy conditions and statistical variants interact coherently. The complementary concept of rate-controlled convergence highlights the relevance of diverging scaling functions in quantifying decay rates. The results presented here provide a basis for further work on scaled summability methods, statistical approximations, and the analysis of numerical algorithms. The examples throughout Sections 2–5 illustrate how different choices of the scaling function affect the strength of the corresponding convergence notions.

Possible extensions of this work include the investigation of ϕ-scaled summability methods via matrix domains [8], the application of scaled convergence in fractional dynamical systems [9], and the study of scaled approximation properties of positive linear operators [10]. Connections with q-calculus and Korovkin-type theorems [7] also present promising directions for future research.

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