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A STUDY OF THE SCATTERING PROPERTIES OF EIGENPARAMETER-DEPENDENT MATRIX DIFFERENCE OPERATOR WITH TRANSMISSION CONDITION

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Abstract. In this paper, we set a transmission boundary value problem for a matrix valued difference equation on the semi axis. The main purpose of this study is to examine the properties of scattering solutions and scattering functions of this problem. Firstly, by giving the Jost solution and scattering solutions of this problem, we obtain the Jost function and the scattering function of the problem. We also investigate eigenvalues, spectral singularities, resolvent operator and continuous spectrum of this problem.

1. Introduction

In daily life, boundary value or initial value problems are used in the functional analysis, applied mathematics, spectral analysis and scattering analysis modeling of many problems encountered in the fields of physics, mathematics and engineering. For solving these problems in spectral and scattering theory, operator theory is an important tool. For many years, many scientists have used it to analyse the spectral and scattering properties of differential and difference operators in physics, quantum mechanics and applied mathematics. The Sturm-Liouville operator, which is a one-dimensional Schrödinger operator, has an important one in the literature [23, 25, 27, 31] for this analysis. On the other hand, the state of the process can suddenly change during some physical and chemical events, including natural problems. Both differential equations and difference equations theory could not answer this situation.

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Therefore, a new theory was needed. Sudden and sharp changes can be encountered at same stages of scientific processes. Compared to the whole process, the duration of this sudden and sharp change is negligible, but the functioning of this system still changes. These short-term effects are called impulse effects, and to deal with these effects, the conditions called transmission condition, point interaction, impulsive condition, jump condition and interface condition are applied to the value problem [1, 21, 26, 28, 29]. Non-stationary biological systems such as heart rhythm beats, blood flows, population dynamics; physical phenomena with variable structure such as theoretical physics, atomic physics, radiophysics, phormacokinetics, and many other such as mathematical economy, chemical technology, electrical technology, metallurgy, ecology, industrial robotics, medicine contain impulse effects. Therefore, as a natural response to the developing technology, interest in differential equations with transmission condition has increased and these equations have been the subject of both theoretical and experimental researches. The problems for the differential equation systems with transmission condition were examined in detail by Samoilenko and Perestyuk, Perestyuk et al. and Lakshmikantham et al. and important results were obtained [22, 32, 33]. There are many studies in the literature examining the spectral and scattering analysis of transmission boundary value problems [7, 10–15, 18, 34]. On the other hand, although there are many studies investigating the spectral and scattering theory of various matrix-valued operators without transmission condition [2–5,9,17,30], there are few studies examining the spectral and scattering theory of transmission boundary value problem with matrix coefficients [6, 8, 16]. In this study, our aim is to examine some spectral and scattering properties of a matrix difference operator with transmission conditions. The difference from [8] is that the spectral parameter λ is included in both the matrix coefficient difference equation and the boundary condition. This gives a different perspective to the problem and so this paper becomes the general form of [8].

Let $\mathcal{L}$ denote the matrix difference operator generated in the Hilbert space $l_2(\mathbb{N}, \mathbb{C}^\mu)$ given by

$$l_2(\mathbb{N}, \mathbb{C}^\mu) := \left\{ Y = \{Y_n\}_{n\in\mathbb{N}}, Y_n \in \mathbb{C}^\mu, ||Y||^2 = \sum_{n\in\mathbb{N}} ||Y_n||^2 < +\infty \right\},$$

where $\mathbb{C}^\mu$ is a μ-dimensional ($\mu < \infty$) Euclidian space, $||\cdot||$ denotes the matrix norm in $\mathbb{C}^\mu$. We shall consider that the operator $\mathcal{L}$ is created by the following difference expression

$$Y_{n-1} + D_n Y_n + Y_{n+1} = \lambda Y_n, \quad n \in \mathbb{N}\backslash\{m_0 - 1, m_0, m_0 + 1\}, \tag{1.1}$$

with the boundary condition

$$(\gamma_0 + \gamma_1\lambda) Y_1 + (\nu_0 + \nu_1\lambda) Y_0 = 0, \quad \gamma_0\nu_1 - \gamma_1\nu_0 \neq 0, \tag{1.2}$$

and the transmission conditions

$$\begin{cases} Y_{m_0+1} = \widetilde{K} Y_{m_0-1}, \\ Y_{m_0+2} = \widetilde{M} Y_{m_0-2}, \end{cases} \tag{1.3}$$

where $\lambda = 2\cos z$ is a spectral parameter, for $i = 0, 1$, γ_i, ν_i are real numbers, $D := \{D_n\}_{n\in\mathbb{N}}$ is a selfadjoint matrix acting in $\mathbb{C}^\mu$ satisfying

$$\sum_{n\in\mathbb{N}} n\, ||D_n|| < +\infty, \tag{1.4}$$

and m_0 is an arbitrary natural number. Throughout this paper, we assume that $\widetilde{K}$ and $\widetilde{M}$ are selfadjoint diagonal matrices in $\mathbb{C}^\mu$ such that all eigenvalues of $\widetilde{K}$ and $\widetilde{M}$ are different and nonzero. Since D is a selfadjoint matrix, it is clear that if $Y_n(z)$ is a solution of (1.1), then $Y_n^T(z)$ is a solution of (1.1), where "T" is the transpose operator.

The set of this paper is summarized as follows. In Section 2, we give the basic solutions and properties of equation of (1.1) without the transmission condition. In Section 3, we obtain basic results and theorems for Jost solution, Jost function and scattering function of this problem. In Section 4, we find resolvent operator and Green function of the operator $\mathcal{L}$. We also get the sets of eigenvalues and spectral singularities of this problem. Then, we obtain the asymptotic representation of the Jost function and continuous spectrum of (1.1)–(1.3).

2. Preliminaries And Auxiliary Results

In this section, we first give useful information and results for matrix difference equation with a general boundary condition that we use throughout the study. We remark that Wronskian of any two solutions $U = \{U_n(z)\}$ and $V = \{V_n(z)\}$ of the equation (1.1) is known as

$$W\left[U, V^T\right](n) = V_{n-1}^T U_n - V_n^T U_{n-1}. \tag{2.1}$$

Now, let us define two semi-strips

$$B := \left\{ z \in \mathbb{C} : z = x + iy, y > 0, -\frac{\pi}{2} \le x \le \frac{3\pi}{2} \right\}, \quad B_0 := B \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Assume that $P(z) = \{P_n(z)\}$ and $Q(z) = \{Q_n(z)\}$ are the fundamental solutions of (1.1) for $z \in B_0$ and $n = 0, 1, \ldots, m_0 - 1$, fulfilling the initial conditions

$$P_0(z) = 0, \quad P_1(z) = I,$$
$$Q_0(z) = I, \quad Q_1(z) = 0.$$

The solutions $P_n(z)$ and $Q_n(z)$ are entire functions of z.

Furthermore, for $z \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im} z \ge 0\}$, the bounded solution $E(z) = \{E_n(z)\}$ of (1.1) which is represented by

$$E_n(z) = e^{inz}\left[I + \sum_{m=1}^{+\infty} K_{nm} e^{imz} \right], \quad n = m_0 + 1, m_0 + 2, \ldots,$$

where K_{nm} is expressed in terms of $\{D_n\}$. $E(z)$ is called the Jost solution of the equation (1.1) and provides the following asymptotic equalities for $z \in \overline{\mathbb{C}}_+$ [20]

$$
\begin{aligned}
E_n(z) =& e^{inz}\left[I + o(1)\right], \quad n \to +\infty, \\
E_n(z) =& e^{inz}\left[I + o(1)\right], \quad \operatorname{Im} z \to +\infty. \qquad (2.2)
\end{aligned}
$$

Additionally, equation (1.1) has an unbounded solution, denoted by $\widehat{E}(z) = \left\{\widehat{E}_n(z)\right\}$, which satisfies the following asymptotic equation

$$\widehat{E}_n(z) = e^{-inz}\left[I + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad n \to +\infty.$$

3. Jost Solution, Jost Function and Scattering Matrix

For $z \in B_0$, let us define the following solution of (1.1)–(1.3) by using $P(z)$, $Q(z)$ and $E(z)$

$$J_n(z) = \begin{cases} P_n(z)\theta_1(z) + Q_n(z)\theta_2(z), & \text{if } n \in \{0, 1, \ldots, m_0 - 1\}, \\ E_n(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \ldots\}, \end{cases}$$

here θ_1 and θ_2 are z-dependent coefficients. By the help of (1.3), we can obtain the following equalities

$$\widetilde{K}^{-1} E_{m_0+1}(z) = P_{m_0-1}(z)\theta_1(z) + Q_{m_0-1}(z)\theta_2(z) \qquad (3.1)$$

and

$$\widetilde{M}^{-1} E_{m_0+1}(z) = P_{m_0-2}(z)\theta_1(z) + Q_{m_0-2}(z)\theta_2(z). \qquad (3.2)$$

From (2.1), it can be easily found that $W\left[P(z), P^T(z)\right] = 0$, $W\left[Q(z), Q^T(z)\right] = 0$ and $W\left[P(z), Q^T(z)\right] = I$ for all $z \in \overline{\mathbb{C}}_+$. Using these Wronskian equalities, (3.1) and (3.2), $\theta_1(z)$ and $\theta_2(z)$ must be as follows:

$$
\begin{aligned}
\theta_1(z) &= \widetilde{K}^{-1}\widetilde{M}^{-1}\left[\widetilde{M} Q^T_{m_0-2}(z) E_{m_0+1}(z) - \widetilde{K} Q^T_{m_0-1}(z) E_{m_0+2}(z)\right], \\
\theta_2(z) &= \widetilde{K}^{-1}\widetilde{M}^{-1}\left[\widetilde{K} P^T_{m_0-1}(z) E_{m_0+2}(z) - \widetilde{M} P^T_{m_0-2}(z) E_{m_0+1}(z)\right],
\end{aligned}
$$

respectively. The function $J_n(z)$ is called the Jost solution of (1.1)–(1.3). We define the Jost function of (1.1)–(1.3) by applying the boundary condition (1.2) to the Jost solution $J_n(z)$ of the operator $\mathcal{L}$

$$\widetilde{J}(z) = (\gamma_0 + \gamma_1\lambda)\, J_1(z) + (\nu_0 + \nu_1\lambda)\, J_0(z) = (\gamma_0 + \gamma_1\lambda)\, \theta_1(z) + (\nu_0 + \nu_1\lambda)\, \theta_0(z).$$

It is easily seen that the function $\widetilde{J}$ is analytic in $\mathbb{C}_+$ and continuous up to the real axis.

For $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$, (1.1) has another solution $F(z) := \{F_n(z)\}$ represented by

$$F_n(z) = \begin{cases} \psi_n(z), & \text{if } n \in \{0, 1, \ldots, m_0 - 1\}, \\ E_n(z)\theta_3(z) + E_n(-z)\theta_4(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \ldots\}. \end{cases}$$

By using the transmission conditions (1.3), it is easy to write

$$E_{m_0+1}(z)\theta_3(z) + E_{m_0+1}(-z)\theta_4(z) = \widetilde{K}\psi_{m_0-1}(z) \tag{3.3}$$

and

$$E_{m_0+2}(z)\theta_3(z) + E_{m_0+2}(-z)\theta_4(z) = \widetilde{M}\psi_{m_0-2}(z). \tag{3.4}$$

Since $W\left[E(z), E^T(z)\right] = 0$ and $W\left[E(-z), E^T(z)\right] = -2\sin z$, by making some calculations in equations (3.3) and (3.4), we find

$$\begin{aligned}\theta_3(z) =& -\frac{1}{2i\sin z}\left[\widetilde{K}E^T_{m_0+2}(-z)\psi_{m_0-1}(z) - \widetilde{M}E^T_{m_0+1}(-z)\psi_{m_0-2}(z)\right],\\ \theta_4(z) =& \frac{1}{2i\sin z}\left[\widetilde{K}E^T_{m_0+2}(z)\psi_{m_0-1}(z) - \widetilde{M}E^T_{m_0+1}(z)\psi_{m_0-2}(z)\right],\end{aligned}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$.

Corollary 3.1. *The coefficients θ_3 and θ_4 have the following relation between the Jost function $\widetilde{J}$*

$$\theta_4^T(z) = \theta_3^T(-z) = -\frac{\widetilde{K}\widetilde{M}}{2i\sin z}\widetilde{J}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}. \tag{3.5}$$

Theorem 3.1. *For all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, $\det \widetilde{J}(z) \neq 0$.*

Proof. We assume that there exists a $z_0 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, such that $\det \widetilde{J}(z_0) = 0$. In accordance with (3.5), we get

$$\det \theta_4^T(z_0) = \det \theta_3^T(-z_0) = \frac{1}{4\sin^2 z}\det \widetilde{K} \det \widetilde{M} \det \widetilde{J}(z)$$

and

$$\det \theta_4\left(z_0\right) = \det \theta_3\left(z_0\right) = 0.$$

It follows from that $F_n(z_0) = 0$, that is, F is a trivial solution of (1.1)–(1.3). This gives a contradiction with our assumption, i.e., for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, $\det \widetilde{J}(z) \neq 0$. The proof is completed. □

Theorem 3.1 says that the inverse of the function $\widetilde{J}$ exists and we give the following definition.

Definition 3.1. The matrix function

$$S(z) = \widetilde{J}^{-1}(z)\widetilde{J}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\},$$

is called the scattering matrix of (1.1)–(1.3).

Theorem 3.2. *For all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, the matrix function $S(z)$ satisfies*

$$S(-z) = S^{-1}(z) = S^*(z),$$

and it is an uniter matrix, where "" denotes the adjoint operator.*

Proof. By the help of definition of scattering matrix, for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, we obtain

$$S(-z) = \widetilde{J}^{-1}(-z)\widetilde{J}(z),$$

and it concludes

$$S(z)S(-z) = S(-z)S(z) = I, \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}.$$

From the last equality, we find

$$S(-z) = S^{-1}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}.$$

Now, let us consider the solutions $J_n(z)$, $J_n(-z)$ and $F_n(z)$, to prove $S^*(z) = S(-z)$. Hence, we write

$$\begin{aligned} F_n(z) =& J_n(z)\eta + J_n(-z)\alpha, \\ F_{n+1}(z) =& J_{n+1}(z)\eta + J_{n+1}(-z)\alpha, \end{aligned} \tag{3.6}$$

where η and α are matrices not depending on n. By making some calculations in (3.6), η and α are obtained as follows:

$$\eta = W^{-1}\left[\widetilde{J}(z), \widetilde{J}^*(z)\right]\left\{J^*_{n+1}(z)F_n(z) - J^*_n(z)F_{n+1}(z)\right\}$$

and

$$\alpha = W^{-1}\left[\widetilde{J}(-z), \widetilde{J}^*(-z)\right]\left\{J^*_{n+1}(-z)F_n(z) - J^*_n(-z)F_{n+1}(z)\right\},$$

respectively. Because of the characteristic features of the transmission conditional equations, we find that $W^{-1}[J(z), J^*(z)] = -W^{-1}[J(-z), J^*(-z)]$. Then, letting $n = 0$ in η and α, the following expressions are obtained

$$\eta = W^{-1}[J(z), J^*(z)]\, J^*(z), \quad \alpha = -W^{-1}[J(z), J^*(z)]\, J^*(-z).$$

When we substitute η and α in (3.6), we get

$$F_n(z) = W^{-1}[J(z), J^*(z)]\{J_n(z)J^*(z) - J_n(-z)J^*(-z)\}.$$

By taking $n = 0$ and $n = 1$ in last equation, we find the following equations

$$(\gamma_0 + \gamma_1\lambda) = W^{-1}[J(z), J^*(z)]\{J_0(z)J^*(z) - J_0(-z)J^*(-z)\}, \tag{3.7}$$

$$(\nu_0 + \nu_1\lambda) = -W^{-1}[J(z), J^*(z)]\{J_1(z)J^*(z) - J_1(-z)J^*(-z)\}. \tag{3.8}$$

By making some calculations in (3.7) and (3.8), we obtain

$$\widetilde{J}(z)\widetilde{J}^*(z) = \widetilde{J}(-z)\widetilde{J}^*(-z). \tag{3.9}$$

Using (3.9), we easily find

$$\widetilde{J}^*(z) = \widetilde{J}^{-1}(z)\widetilde{J}(-z)\widetilde{J}^*(-z)$$

and

$$\widetilde{J}^*(z)\left[\widetilde{J}^*(-z)\right]^{-1} = \widetilde{J}^{-1}(z)\widetilde{J}(-z).$$

Finally, it is clear that $S^*S = SS^* = I$, $||S|| = I$, i.e., S is unitary. □

Lemma 3.1. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$*, the following equation holds*

$$W[J(z), F^T(z)](n) = \begin{cases} \widetilde{J}(z), & \text{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ -\widetilde{K}\widetilde{M}\widetilde{J}(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \dots\}. \end{cases}$$

Proof. From (2.1), we obtain

$$W\left[J(z), F^T(z)\right](n) = F_0^T(z)J_1(z) - F_1^T(z)J_0(z),$$

for $n = 0, 1, \dots, m_0 - 1$. Since it is known that $P_0(z) = 0$, $P_1(z) = I$, $Q_0(z) = I$ and $Q_1(z) = 0$, the following Wronskian is easily found

$$W[J(z), F^T(z)](n) = \widetilde{J}(z), \quad n = 0, 1, \dots, m_0 - 1.$$

Similarly, for $n = m_0 + 1, m_0 + 2, \dots$, we find $W[J(z), F^T(z)](n) = 2i \sin z \theta_4^T(z)$. In view of (3.5), the Wronskian can be arranged

$$W[J(z), F^T(z)](n) = -\widetilde{K}\widetilde{M}\widetilde{J}(z), \quad n = m_0 + 1, m_0 + 2, \dots$$

The proof is completed. □

4. Resolvent Operator, Eigenvalues, Spectral Singularities And Continuous Spectrum

In the following, we will define the other solution of (1.1)–(1.3) for all $z \in B_0$

$$G_n(z) = \begin{cases} \psi_n(z), & \text{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ E_n(z)\theta_5(z) + \widehat{E}_n(z)\theta_6(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \dots\}. \end{cases}$$

By using the transmission condition (1.3) to $G_n(z)$, we get

$$E_{m_0+1}(z)\theta_5(z) + \widehat{E}_{m_0+1}(z)\theta_6(z) = \widetilde{K}\psi_{m_0-1}(z),$$
$$E_{m_0+2}(z)\theta_5(z) + \widehat{E}_{m_0+2}(z)\theta_6(z) = \widetilde{M}\psi_{m_0-}(z).$$

To get the coefficients $\theta_5(z)$ and $\theta_6(z)$, we will use same way as finding $\theta_1(z)$ and $\theta_2(z)$. Since

$$W\left[E(z), E^T(z)\right] = 0, \quad W\left[\widehat{E}(z), E^T(z)\right] = -2i \sin z$$

and

$$W\left[\widehat{E}(z), \widehat{E}^T(z)\right] = 0, \quad W\left[E(z), \widehat{E}^T(z)\right] = 2i \sin z,$$

$\theta_5(z)$ and $\theta_6(z)$ must be as follows:

$$\theta_5(z) = \frac{1}{2i \sin z}\left[\widetilde{K}\widehat{E}^T_{m_0+2}(z)\psi_{m_0-1}(z) - \widetilde{M}\widehat{E}^T_{m_0+1}(z)\psi_{m_0-2}(z)\right]$$

and

$$\theta_6(z) = \frac{1}{2i \sin z}\left[\widetilde{K}E^T_{m_0+2}(z)\psi_{m_0-1}(z) - \widetilde{M}E^T_{m_0+1}(z)\psi_{m_0-2}(z)\right].$$

Note that

$$\theta_6(z) = -\frac{\widetilde{K}\widetilde{M}}{2i \sin z}\widetilde{J}^T(z).$$

Similar to Lemma 3.1, the following Wronskian equation is obtained

$$\widetilde{C}(z) := W[J(z), G^T(z)](n) = \begin{cases} \widetilde{J}(z), & \text{if } n \in \{0, 1, \ldots, m_0 - 1\}, \\ -\widetilde{K}\widetilde{M}\widetilde{J}(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \ldots\}, \end{cases}$$

for $z \in B_0$.

Theorem 4.1. *The resolvent operator of* $\mathcal{L}$ *has the representation*

$$(\mathcal{R}_\lambda(\mathcal{L})\varphi)_n := \sum_{k=0}^{\infty} \mathcal{H}_{n,k}(z)\varphi(k), \quad \varphi := \{\varphi_k\} \in l_2\left(\mathbb{N}, \mathbb{C}^h\right),$$

where

$$\mathcal{H}_{n,k} = \begin{cases} J_n(z)\widetilde{C}^{-1}(z)G_k^T(z), & \text{if } k < n, \\ G_n(z)\left[\widetilde{C}^{-1}(z)\right]^T J_k^T(z), & \text{if } k \geq n, \end{cases}$$

is the Green function of $\mathcal{L}$ *for* $z \in B_0$ *and* $k, n \neq m_0$.

Proof. To obtain the resolvent operator and Green function of $\mathcal{L}$, we need to find the solutions of the following equation

$$\nabla(\triangle Y_n) + M_n Y_n - \lambda Y_n = \psi_n, \tag{4.1}$$

where $M_n = 2I_n + D_n$. Using $J(z)$ and $G(z)$, we can write the general solution of (4.1) as

$$Y_n(z) = J_n(z)R_n + G_n(z)T_n,$$

where $R := \{R_n\}_{n\in\mathbb{N}}$ and $T := \{T_n\}_{n\in\mathbb{N}}$ are self-adjoint diagonal matrices in $\mathbb{C}^\mu$. By the help of the method of variation of parameters, the coefficients R and T can be written

$$R_n = R_0 + \sum_{k=1}^{n} \frac{G_k^T(z)\varphi_k(z)}{\widetilde{C}(z)}, \quad T_n = \zeta + \sum_{k=n+1}^{\infty} \frac{J_k^T(z)\varphi_k(z)}{\widetilde{C}^T(z)},$$

where R_0 and ζ are self-adjoint diagonal matrices in $\mathbb{C}^\mu$. Since the solution $Y_n(z)$ in $l_2(\mathbb{N}, \mathbb{C}^\mu)$, ζ is zero. By the help of the boundary condition (1.2), we find that R_0 is equal to zero. It completes the proof of Theorem 4.1. □

Now, from Theorem 4.1, we define the sets of eigenvalues and spectral singularities of $\mathcal{L}$ as follows:

$$\sigma_d(\mathcal{L}) = \left\{\lambda = 2\cos z : z \in D, \det \widetilde{J}(z) = 0\right\},$$

$$\sigma_{ss}(\mathcal{L}) = \left\{\lambda = 2\cos z : z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}, \det \widetilde{J}(z) = 0\right\},$$

respectively.

Theorem 4.2. *Assume* (1.4). *Then the Jost function* $\widetilde{J}$ *satisfies the following asymptotic equation*

$$\widetilde{J}(z) = \nu_1\left(\widetilde{K}\widetilde{M}\right)^{-1}\left(\widetilde{K} - \widetilde{M}\right)[I + o(1)]\left(e^{5iz} + e^{3iz}\right), \quad z \in B_0, |z| \to +\infty.$$

Proof. Since the polynomial function $P_n(z)$ is of $(n-1)$. degree and polynomial function $Q_n(z)$ is of $(n-2)$. degree with respect to λ, we get

$$(\nu_0 + \nu_1\lambda) P_n^T(z) e^{i(n-1)z} = \nu_1 [I + o(1)], \quad |z| \to +\infty, \, z \in B_0. \tag{4.2}$$

It is clear that

$$\begin{aligned}\widetilde{J}(z) =& \widetilde{K}^{-1}\widetilde{M}^{-1}(\nu_0+\nu_1\lambda)\Big[\widetilde{K}P_{m_0-1}^T(z)e^{i(m_0-2)z}e^{-i(m_0-2)z}E_{m_0+2}(z)e^{-i(m_0+2)z}e^{i(m_0+2)z} \\ & -\widetilde{M}P_{m_0-2}^T(z)e^{i(m_0-3)z}e^{-i(m_0-3)z}E_{m_0+1}(z)e^{-i(m_0+1)z}e^{i(m_0+1)z}\Big].\end{aligned}$$

By using (2.2) and (4.2), we write the following asymptotic equation

$$\widetilde{J}(z) = \nu_1 \left(\widetilde{K}\widetilde{M}\right)^{-1}\left(\widetilde{K} - \widetilde{M}\right)[I + o(1)]\left(e^{5iz} + e^{3iz}\right), \quad z \in B_0, \, |z| \to +\infty.$$

Theorem 4.3. *If the condition* (1.4) *satisfies, then* $\sigma_c(\mathcal{L}) = [-2, 2]$, *where* $\sigma_c(\mathcal{L})$ *denotes the continuous spectrum of* $\mathcal{L}$.

Proof. Let us introduce the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ generated by the following difference expression in $l_2(\mathbb{N}, \mathbb{C}^\mu)$ with (1.2) and (1.3)

$$\begin{aligned}(\mathcal{L}_0 y)_n =& Y_{n-1} + Y_{n+1}, \quad n \in \mathbb{N}\backslash\{m_0-1, m_0+1\}, \\ (\mathcal{L}_1 Y)_n =& D_n Y_n, \quad n \in \mathbb{N}\backslash\{m_0\},\end{aligned}$$

respectively. Under the condition (1.4), it is clear to see the compactness of $\mathcal{L}_1$ [24]. On the other hand, we write $\mathcal{L} = \mathcal{L}_0^1 + \mathcal{L}_0^2 + \mathcal{L}_1$, where L_0^1 is a selfadjoint operator with $\sigma_c(\mathcal{L}_0^1) = [-2, 2]$ and L_0^2 is a finite dimensional operator in $l_2(\mathbb{N}, \mathbb{C}^\mu)$. Then, by the help of Weyl theorem of a compact perturbation [19], we find the continuous spectrum of $\mathcal{L}$. ☐

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