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NOTE ON HAMILTONIAN GRAPHS IN ABELIAN 2-GROUPS

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Abstract. We analyze a graph G whose vertices are subgroups of $\mathbb{Z}_2^k$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Two vertices are joined if their respective subgroups have nontrivial intersection. We prove that such a graph is $6(2^{k-2}-1)$-regular. If a graph is regular, a classical theorem by Ore claims that a graph is Hamiltonian if the degree of any vertex is at least one half of the number of vertices. Using Ore's theorem, we show that G is Hamiltonian for $k \in \{3,4\}$. Ore's theorem cannot be applied when $k \geq 5$. Nevertheless, we manage to construct a Hamiltonian cycle for $k=5$. Our construction uses orbits of one $\mathbb{Z}_2^4$ group under an action of an automorphism of order 31. It is highly likely that this approach could be generalized for $k > 5$.

1. Introduction and notation

Many algebraic structures, including groups, have nice interpretations in graph theory (see for example [1,3] and [4]). Readers can find more on groups and graphs in [5]. If there is a cycle in a graph that visits every vertex, then the graph is Hamiltonian. In this paper we are interested in Hamiltonian graphs defined on Abelian groups of exponent 2. For some classical results on Hamiltonian graphs see [5]. The main tool in our analysis will be the application of various group rings, for example see [2]. An elementary Abelian group of order 2^k is denoted by E_{2^k}. If $x_1, x_2, \ldots, x_k$ are generators, then we can write $E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle$. Additionally, $x_i^2 = 1$ for all $i \in [k] = \{1, 2, \ldots, k\}$. With $E_{2^l}[H]$ we denote a collection of all subgroups of order 2^l that are contained in $H \leq E_{2^k}$.

We introduce a set $E_{2^s}[T, H]^{-1} = \{S \mid T \leq S \leq H,\ S \cong E_{2^s}\}$ of all E_{2^s}-subgroups that contain T and that are also contained in H. One can see that if $t \leq s \leq m$,

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$H \cong E_{2^m}$, and $T \cong E_{2^t}$, then $|E_{2^s}[T,H]^{-1}| = |E_{2^{s-t}}[H/T]| = |E_{2^{s-t}}[E_{2^{m-t}}]| = \begin{bmatrix} m-t \\ s-t \end{bmatrix}_2$, where H/T is a quotient group isomorphic to $E_{2^{m-t}}$ and $\begin{bmatrix} a \\ b \end{bmatrix}_2$ is a Gaussian coefficient.

Let $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ be a graph with vertices $T \leq E_{2^k}$, where $T \cong E_{2^2} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Edges $\mathcal{E}_k$ are defined as follows:

$$\{T_1, T_2\} \in \mathcal{E}_k \Leftrightarrow T_1 \cap T_2 \cong \mathbb{Z}_2.$$

This means that two E_{2^2} groups are joined if and only if they have a common involution (nontrivial intersection). Our main goal is to see when such graphs are Hamiltonian. We will show that Ore's Theorem immediately yields that $(E_{2^2}[E_{2^3}], \mathcal{E}_3)$ and $(E_{2^2}[E_{2^4}], \mathcal{E}_4)$ are Hamiltonian.

We will use $\deg(u)$ to denote the degree of a vertex.

Theorem 1.1 (Ore). *Let G be a connected graph with $n > 3$ vertices. If $\deg(x) + \deg(y) > n$ for all non-adjacent vertices x and y, then G is Hamiltonian.*

A graph $G = (V, E)$ is a r-regular graph if $\deg(x) = r$ for all vertices $x \in V$. As an immediate consequence of Theorem 1.1 we have the following.

Corollary 1.1. *If $G = (V, E)$ is r-regular graph and if $\deg(x) > \frac{1}{2}|V|$, then G is Hamiltonian.*

2. Regularity

In this section we will prove that $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is a regular graph. This means that we need to show that for any $T \in E_{2^2}[E_{2^k}]$ there is a constant number of $S \in E_{2^2}[E_{2^k}]$ such that $|T \cap S| = 2$.

From this point on, we will assume that $k > 2$. Furthermore, we will show that if $k \in \{3, 4\}$, then a graph $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is Hamiltonian.

Theorem 2.1. *A graph $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is $6(2^{k-2} - 1)$-regular. The inequality*

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - \deg(V) < 0$$

holds for all $V \in E_{2^2}[E_{2^k}]$ if any only if $k < 5$.

Proof. Let V be a vertex of $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$. Put $V^* = V \setminus \{1\}$. Let us denote with $n(V)$ the collection of all vertices adjacent to V. If $P \in n(V)$, then $P \cong E_{2^2}$ and $P \cap V = \langle g \rangle$ for some $g \in E^*_{2^k}$. Also, $P \in E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}$. Hence,

$$n(V) = \left[\bigcup_{g \in V^*} E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \right] \setminus \{V\}.$$

On the other hand, we have

$$|E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| = |E_2[E_{2^k}/\langle g \rangle]| = |E_2[E_{2^{k-1}}]| = 2^{k-1} - 1.$$

If $g, h \in V^*$ and $g \neq h$, then

$$|E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1}| = |E_{2^2}[E_{2^k}] \cap \{V\}| = 1.$$

Also, for three mutually different $g, h, k \in T^*$ we get

$$|E_{2^2}[\langle g\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k\rangle, E_{2^k}]^{-1}| = 1.$$

Using the inclusion-exclusion formula, the following holds

$$\begin{aligned}\deg(V) =& \sum_{g\in V^*} |E_{2^2}[\langle g\rangle, E_{2^k}]^{-1}| - \sum_{g\neq h,\ g,h\in V^*} \left|E_{2^2}[\langle g\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h\rangle, E_{2^k}]^{-1}\right| \\ &+ \sum_{g\neq h\neq k\neq g,\ g,h,k\in V^*} \left|E_{2^2}[\langle g\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k\rangle, E_{2^k}]^{-1}\right| - 1 \\ =&\binom{3}{1}(2^{k-1}-1) - \binom{3}{2}\cdot 1 + 1 - 1 \\ =&6(2^{k-2}-1).\end{aligned}$$

Notice that $|E_{2^2}[E_{2^k}]| = \begin{bmatrix}k\\2\end{bmatrix}_2 = \frac{1}{3}(2^k-1)(2^{k-1}-1)$. Put $t = 2^{k-2}$. Therefore,

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - \deg(V) = \frac{1}{6}(4t-1)(2t-1) - 6(t-1) = \frac{1}{6}(8t^2 - 42t + 37).$$

For $k = 3$ and $k = 4$ we get $8t^2 - 42t + 37 < 0$. For $k \geq 5$ we have $8t^2 - 42t + 37 > 0$. This proves our claim. □

Now, using Corollary 1.1, we see that the following holds.

Corollary 2.1. *Graphs* $(E_{2^2}[E_{2^3}], \mathcal{E}_3)$ *and* $(E_{2^2}[E_{2^4}], \mathcal{E}_4)$ *are Hamiltonian. Furthermore, necessary conditions for application of Ore's theorem are not satisfied for* $k \geq 5$.

3. Hamiltonian Cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$

Let $E_{2^5} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle = \langle a, b, c, d, e\rangle$, where a, b, c, d, e are generators of E_{2^5}. Any automorphism $\alpha \in \mathrm{Aut}(E_{2^5})$ is represented by its action on generators. We can denote any $\alpha \in \mathrm{Aut}(E_{2^5})$ by

$$\alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix},$$

for some $g_i \in E_{2^5}^*$. This means $\alpha(a) = g_1,\ \alpha(b) = g_2$ and so on. The order of an automorphism $o(\alpha)$ is the smallest nonnegative integer n such that α^n is an identity map. If $X \subseteq E_{2^5}$ and $\alpha \in \mathrm{Aut}(E_{2^5})$, then with $X^{\langle\alpha\rangle}$ we will denote one α-orbit of X. If α is of order n, then an orbit $X^{\langle\alpha\rangle}$ can be represented in a group ring $\mathbb{Z}[E_{2^5}]$ like this:

$$X^{\langle\alpha\rangle} = X + X^{\alpha} + \cdots + X^{\alpha^{n-1}}.$$

The following lemma will be crucial for a construction of a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$.

Lemma 3.1. *Let* $E_{2^5} = \langle a, b, c, d, e\rangle$ *and let* $\alpha \in Aut(E_{2^5})$ *be given by*

$$\alpha = \begin{pmatrix} a & b & c & d & e \\ bc & cd & bcd & de & a \end{pmatrix},$$

then $o(\alpha) = 31$ *and* $H^{\langle\alpha\rangle} = E_{2^4}[E_{2^5}]$, *where* $H = \langle a,b,c,d\rangle$. *If* $T = \langle a,b,c\rangle$ *and* $\Delta_i = T \cap T^{\alpha^i}$ *for* $i \in \mathbb{Z}_{31}$, *then*

$$\Delta_i = \begin{cases} \langle b,c\rangle, & \text{if } i = 1, 14, \\ \langle a,bc\rangle, & \text{if } i = 13, 30, \\ \langle ab,c\rangle, & \text{if } i = 17, 18, \\ \cong \mathbb{Z}_2, & \text{otherwise.} \end{cases}$$

Proof. We can rewrite an automorphism α in a simplified form like this: $\alpha = (bc, cd, bcd, de, a)$. For the purpose of finding α^i we represent α in a matrix form over $\mathbb{Z}_2$

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Rows and columns are indexed by a,b,c,d,e. After calculating powers of α over $\mathbb{Z}_2$, we get that α^{31} is an identity matrix. Furthermore, α^i is not an identity matrix for all $i < 31$. Therefore, $o(\alpha) = 31$. For example, using the same approach, we get $\alpha^{13} = (de, abcde, bc, abde, d)$ and $\alpha^{14} = (ade, acde, b, abe, de)$. Hence, $T^{\alpha^{13}} = \langle de, abcde, bc\rangle = \langle de, abc, bc\rangle = \langle de, a, bc\rangle$ and $\Delta_{13} = T \cap T^{\alpha^{13}} = \langle a, bc\rangle$. Furthermore, $T^{\alpha^{14}} = \langle ade, acde, b\rangle = \langle ade, c, b\rangle$ and $\Delta_{14} = \langle b,c\rangle$. Also, $\alpha^{17} = (ae, c, ab, acd, acde)$, $\alpha^{18} = (abc, bcd, bd, e, ae)$ and $\alpha^{30} = (e, bc, abc, ac, acd)$. For all other cases Δ_i is a group of order 2. In the Appendix, one can find all powers α^i together with the images T^{α^i}.

Assume that $H^{\alpha^i} = H$ for some power $i < 31$. Then $\Delta_i = T \cong E_{2^3}$. This is a contradiction with $|\Delta_i| \leq 4$, hence $H^{\alpha^i} \neq H$. Since the number of all E_{2^4} subgroups of E_{2^5} is $|E_{2^4}[E_{2^5}]| = \begin{bmatrix} 5 \\ 4 \end{bmatrix}_2 = 2^5 - 1 = 31$, this means that an α-orbit of H contains all E_{2^4} subgroups of E_{2^5}. Therefore, $H^{\langle\alpha\rangle} = E_{2^4}[E_{2^5}]$. □

Throughout the rest of the paper the subgroup $\langle a,b,c\rangle \leq E_{2^5} = \langle a,b,c,d,e\rangle$ shall be denoted by T and α shall be the automorphism defined in the Lemma 3.1. We are now ready to sketch the main idea for a construction of a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$. A main building block will be an α-orbit of T. There are 7 vertices or subgroups of order 4 in T^{α^i}, $i \in \mathbb{Z}_{31}$. We will show, in Theorem 3.4, that a collection of all vertices from $\bigcup_{i=0}^{31} E_{2^2}[T^{\alpha^i}]$ is in fact the set of all vertices $E_{2^2}[E_{2^5}]$. Also $T \cap T^{\alpha} \cong E_{2^2}$ is a vertex. The same holds for all other $T^{\alpha^i} \cap T^{\alpha^{i+1}}$. As we will see from Theorem 3.5, vertices $T^{\alpha^i} \cap T^{\alpha^{i+1}}$ are all mutually different. As a final step, we will introduce a recursive procedure that will enable us to choose vertices from each $E_{2^2}[T^{\alpha^i}]$ so that they all together form a Hamiltonian cycle.

Motivated by the previous lemma we introduce slightly different notation:

$$\Delta_{\Omega_1} = \langle b,c\rangle, \quad \Omega_1 = \{1, 14\},$$

$$\begin{aligned}\Delta_{\Omega_2} =& \langle a, bc\rangle, \quad \Omega_2 = \{13, 30\},\\ \Delta_{\Omega_3} =& \langle ab, c\rangle, \quad \Omega_3 = \{17, 18\}.\end{aligned}$$

Lemma 3.2. *Groups $\Delta_{\Omega_i}^{\alpha^k}$ and Δ_{Ω_i} are distinct for all $i \in [3]$ and $k \in [30]$.*

Proof. Assume the opposite. Let $i \in [3]$ and $k \in [30]$ such that $\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_i}$. Since $o(\alpha) = 31$ is a prime, then α^k generate entire $\langle\alpha\rangle$. Hence $\langle\alpha\rangle = \langle\alpha^k\rangle$. Let $H = \langle a, b, c, d\rangle$. Lemma 3.1 implies that $H^{\langle\alpha^k\rangle} = E_{2^4}[E_{2^5}]$. There is $s \in \mathbb{Z}_{31}$ such that $\Delta_{\Omega_i} \leq H^{(\alpha^k)^s}$. Since $\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_i}$, then $\Delta_{\Omega_i} = \Delta_{\Omega_i}^{(\alpha^k)^t} \leq (H^{(\alpha^k)^s})^{(\alpha^k)^t} = H^{(\alpha^k)^{s+t}}$ for all $t \in \mathbb{Z}_{31}$. A mapping $t \mapsto s + t$ is one-to-one map on $\mathbb{Z}_{31}$. Hence, we can write in a group ring $\mathbb{Z}[E_{2^4}[E_{2^5}]]$ the following:

$$\sum_{t=0}^{30} H^{(\alpha^k)^{s+t}} = \sum_{t\in\mathbb{Z}_{31}} ((H)^{\alpha^k})^t = E_{2^4}[E_{2^5}].$$

From $\Delta_{\Omega_i} \leq H^{(\alpha^k)^{s+t}}$ for all $t \in \mathbb{Z}_{31}$ it follows $|E_{2^4}[\Delta_{\Omega_i}, E_{2^5}]^{-1}| \geq 31$. This is a contradiction with

$$|E_{2^4}[\Delta_{\Omega_i}, E_{2^5}]^{-1}| = |E_{2^2}[E_{2^5}/\Delta_{\Omega_i}]| = |E_{2^2}[E_{2^3}]| = \begin{bmatrix}3\\2\end{bmatrix}_2 = 2^3 - 1 = 7. \qquad \square$$

Corollary 3.1. *If $\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_j}$, then α^k is a unique element from $\langle\alpha\rangle$.*

Proof. Suppose that k_1 and k_2 are integers such that $\Delta_{\Omega_i}^{\alpha^{k_1}} = \Delta_{\Omega_i}^{\alpha^{k_2}} = \Delta_{\Omega_j}$. It follows that $\Delta_{\Omega_i}^{\alpha^{k_1-k_2}} = \Delta_{\Omega_i}$. By Lemma 3.2, a map $\alpha^{k_1-k_2}$ is an identity map. Thus $k_1 = k_2$. □

Lemma 3.3. *Subgroups Δ_{Ω_i}, $i \in [3]$, satisfy the following: $\Delta_{\Omega_1}^{\alpha^{30}} = \Delta_{\Omega_2}$, $\Delta_{\Omega_2}^{\alpha^{18}} = \Delta_{\Omega_3}$, $\Delta_{\Omega_3}^{\alpha^{14}} = \Delta_{\Omega_1}$.*

Proof. From Lemma 3.1 we have $\Delta_{\Omega_1}^{\alpha^{30}} = (T \cap T^{\alpha})^{\alpha^{30}} = T^{\alpha^{30}} \cap T = \Delta_{\Omega_2}$. Hence $\Delta_{\Omega_1}^{\alpha^{30}} = \Delta_{\Omega_2}$. Furthermore, $\Delta_{\Omega_1}^{\alpha^{17}} = (T \cap T^{\alpha^{14}})^{\alpha^{17}} = T^{\alpha^{17}} \cap T = \Delta_{\Omega_3}$. Now we have $\Delta_{\Omega_3}^{\alpha^{14}} = \Delta_{\Omega_1}$. Moreover $\Delta_{\Omega_3}^{\alpha^{13}} = \Delta_{\Omega_2}$ and $\Delta_{\Omega_2}^{\alpha^{18}} = \Delta_{\Omega_3}$. This proves our claim. □

Theorem 3.1. *For T and α the following holds*

$$\sum_{0\leq i<j\leq 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}]| = 31 \cdot 3.$$

Proof. Take some i and j such that $T^{\alpha^i} \cap T^{\alpha^j} \cong E_{2^2}$. Then

$$T^{\alpha^i} \cap T^{\alpha^j} = (T \cap T^{\alpha^{j-i}})^{\alpha^i} = (\Delta_{j-i})^{\alpha^i} = (\Delta_{i-j})^{\alpha^j} \cong E_{2^2}.$$

This means that $\Delta_{j-i} = \Delta_{i-j} \cong E_{2^2}$. Thus, by Lemma 3.1, we get $\{i-j, j-i\} \in \{\{1, 30\}, \{13, 18\}, \{14, 17\}\}$. Since $i \in \mathbb{Z}_{31}$, each $\{i, j\}$ contributes 31 to the sum $\sum_{0\leq i<j\leq 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}]|$. Therefore, the final number is $31 \cdot 3$. This proves our assertion. □

Theorem 3.2. *For T and α the following holds*

$$\sum_{0\le i<j<k\le 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}] \cap E_{2^2}[T^{\alpha^k}]| = 31.$$

Proof. Let $A = T^{\alpha^i} \cap T^{\alpha^j} \cap T^{\alpha^k} \cong E_{2^2}$ for some $0 \le i < j < k \le 31$. Then $A = (T^{\alpha^i} \cap T^{\alpha^j}) \cap (T^{\alpha^i} \cap T^{\alpha^k})$. This means $A = (T \cap T^{\alpha^{j-i}})^{\alpha^i} \cap (T \cap T^{\alpha^{k-i}})^{\alpha^i} = (\Delta_{j-i} \cap \Delta_{k-i})^{\alpha^i}$. Hence $\Delta_{j-i} \cap \Delta_{k-i} \cong E_{2^2}$. Since $|\Delta_t| \le 4$ we get $\Delta_{j-i} = \Delta_{k-i} \cong E_{2^2}$. Since $j-i \ne k-i$, we get $\{j-i, k-i\} = \Omega_s$ for some $s \in [3]$.

If $s = 1$, then $\{j-i, k-i\} = \{1, 14\}$. This implies that $\{i, j, k\}$ can be represented as $\{i, i+1, i+14\}$ where $i \in \mathbb{Z}_{31}$.

The case $s = 2$ gives us $\{j-i, k-i\} = \{13, 30\}$. Hence, $\{i, j, k\}$ can be represented as $\{i, i+13, i+30\}$ where $i \in \mathbb{Z}_{31}$. However, we get

$$\{\{i, i+13, i+30\} \mid i \in \mathbb{Z}_{31}\} = \{\{(i-1)+1, (i-1)+1+13, (i-1)+1+30\} \mid i \in \mathbb{Z}_{31}\},$$

and this set is equal to $= \{\{j, j+1, j+14\} \mid j \in \mathbb{Z}_{31}\}$ where $j = i-1$ in $\mathbb{Z}_{31}$. Therefore, the previous two cases are in fact the same.

If $s = 3$, then $\{j-i, k-i\} = \{17, 18\}$. Now we get $\{i, j, k\}$ is of the form $\{i, i+17, i+18\}$ where $i \in \mathbb{Z}_{31}$. Notice that

$$\{\{i, i+17, i+18\} \mid i \in \mathbb{Z}_{31}\} = \{\{(i+17)-17, i+17, (i+17)+1\} \mid i \in \mathbb{Z}_{31}\}.$$

It follows

$$\{\{j-17, j, j+1\} \mid j \in \mathbb{Z}_{31}\} = \{\{j+14, j, j+1\} \mid j \in \mathbb{Z}_{31}\},$$

where $j = i+17$ in $\mathbb{Z}_{31}$. Thus, all the three cases are the same and so we have one representative.

This means that we have one representative of a triple $\{i, j, k\}$ such that $T^{\alpha^i} \cap T^{\alpha^j} \cap T^{\alpha^k} \cong E_{2^2}$ where $i \in \mathbb{Z}_{31}$. This proves the claim of the theorem. ☐

Theorem 3.3. *For T and α the following holds*

$$\sum_{0\le i<j<k<s\le 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}] \cap E_{2^2}[T^{\alpha^k}] \cap E_{2^2}[T^{\alpha^s}]| = 0.$$

Proof. Assume that $A = T^{\alpha^i} \cap T^{\alpha^j} \cap T^{\alpha^k} \cap T^{\alpha^s} \cong E_{2^2}$ for some $0 \le i < j < k < s \le 30$. It implies that

$$A = (T \cap T^{\alpha^{j-i}})^{\alpha^i} \cap (T \cap T^{\alpha^{k-i}})^{\alpha^i} \cap (T \cap T^{\alpha^{s-i}})^{\alpha^i} = (\Delta_{j-i} \cap \Delta_{k-i} \cap \Delta_{s-i})^{\alpha^i}.$$

This means that $\Delta_{j-i} = \Delta_{k-i} = \Delta_{s-i} \cong E_{2^2}$. Since T^{α^i}, T^{α^j}, T^{α^k}, T^{α^s} are mutually different, we get $|\{j-i, k-i, s-i\}| = 3$. Also, $\Delta_{j-i} = \Delta_{k-i} = \Delta_{s-i} \cong E_{2^2}$ implies $\{j-i, k-i, s-i\} \subseteq \Omega_i$ for some i. That is a contradiction since $|\Omega_i| = 2$. ☐

The next result finally shows that orbit $T^{\langle\alpha\rangle}$ contains all E_{2^2} subgroups of E_{2^5}.

Theorem 3.4. *For T and α the following holds*

$$\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}] = E_{2^2}[E_{2^5}].$$

Proof. The total number of all E_{2^2} subgroups of E_{2^5} is $|E_{2^2}[E_{2^5}]| = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_2 = 31 \cdot 5$. Using the inclusion-exclusion formula and Theorems 3.1, 3.2 and 3.3 we get

$$\begin{aligned}\left|\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}]\right| =& \sum_{i=0}^{30} \left|E_{2^2}[T^{\alpha^i}]\right| - \sum_{0\le i<j\le 30} \left|E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}]\right| \\ &+ \sum_{0\le i<j<k\le 30} \left|E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}] \cap E_{2^2}[T^{\alpha^k}]\right| + \cdots + \\ =& 31 \cdot 7 - 31 \cdot 3 + 31 - 0 + 0 - \cdots \\ =& 31 \cdot 5.\end{aligned}$$

Therefore, every group from $E_{2^2}[E_{2^5}]$ is contained in $\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}]$. □

Theorem 3.5. *A graph* $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$ *is Hamiltonian.*

Proof. Since $T \cong E_{2^3}$ and $AB = T$, where $A, B \in E_{2^2}[T^{\alpha^i}]$, it follows that $|A \cap B| = \frac{|A|\cdot|B|}{|E_{2^3}|} = 2$. Hence, A and B are adjacent. Therefore, the vertices in $E_{2^2}[T^{\alpha^i}] \cong K_7$ induce a complete graph on 7 vertices denoted by K_7. Thus, if we delete some vertices together with the edges incident to them from $E_{2^2}[T^{\alpha^i}]$, there will be a path in a remaining graph that visits each remaining vertex.

The subgraphs $E_{2^2}[T^{\alpha^{i-1}}]$, $E_{2^2}[T^{\alpha^i}]$ and $E_{2^2}[T^{\alpha^{i+1}}]$ have common vertices $T^{\alpha^i} \cap T^{\alpha^{i-1}}$ and $T^{\alpha^i} \cap T^{\alpha^{i+1}}$. Let $L(T^{\alpha^i}) = \{T^{\alpha^i} \cap T^{\alpha^{i-1}}, T^{\alpha^i} \cap T^{\alpha^{i+1}}\}$. Notice that $L(T^{\alpha^i}) = \{\Delta_1^{\alpha^{i-1}}, \Delta_1^{\alpha^i}\}$ (since $T \cap T^\alpha = \Delta_1$). We may look at vertices $L(T^{\alpha^i})$ as links between neighboring graphs $E_{2^2}[T^{\alpha^{i-1}}]$, $E_{2^2}[T^{\alpha^i}]$ and $E_{2^2}[T^{\alpha^{i+1}}]$.

Suppose that there are at least two equal vertices in $\bigcup_{i=0}^{30} L(T^{\alpha^i})$. Let $T^{\alpha^i} \cap T^{\alpha^{i+1}} = T^{\alpha^s} \cap T^{\alpha^{s+1}}$ for some $i \neq s$. Thus, $(T \cap T^\alpha)^{\alpha^i} = (T \cap T^\alpha)^{\alpha^s}$. Hence, $\Delta_1^{\alpha^i} = \Delta_1^{\alpha^s}$ and $\Delta_1^{\alpha^{i-s}} = \Delta_1$ for $\alpha^{i-s} \neq id$. This is a contradiction with Lemma 3.2. Therefore, all vertices in $\bigcup_{i=0}^{30} L(T^{\alpha^i})$ are mutually different.

As the initial step of a recursive construction of a Hamiltonian cycle, we define $E_{2^2}[T^{\alpha^i}]_0 = E_{2^2}[T^{\alpha^i}]$ for all $i \in \mathbb{Z}_{31}$. Assume that we have formed a sequence $\left(E_{2^2}[T^{\alpha^i}]_{m_i}\right)_{i\in\mathbb{Z}_{31}}$, where m_i is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within $E_{2^2}[T^{\alpha^i}]$.

If there is a vertex A and $j \neq i$ such that $A \in \left(E_{2^2}[T^{\alpha^i}]_{m_i} \setminus L(T^{\alpha^i})\right) \cap E_{2^2}[T^{\alpha^j}]_{m_j}$, then A is not a link, but it is a vertex in graphs $E_{2^2}[T^{\alpha^i}]_{m_i}$ and $E_{2^2}[T^{\alpha^j}]_{m_j}$. Then, we delete a vertex A and the edges incident to it. In this case let $E_{2^2}[T^{\alpha^i}]_{m_i+1} = E_{2^2}[T^{\alpha^i}]_{m_i} \setminus \{A\}$.

If such a vertex A does not exist, we leave $E_{2^2}[T^{\alpha^i}]_{m_i}$ unchanged and denote that by $\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$. Now, continue the same procedure with $E_{2^2}[T^{\alpha^{i+1}}]_{m_{i+1}}$. Following this process, after finite number of steps, we will construct a sequence $\left(\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}\right)_{i\in\mathbb{Z}_{31}}$. Using a notation in a group ring $\mathbb{Z}[E_{2^2}[E_{2^5}]]$, we have the following:

$$\bigcup_{i\in\mathbb{Z}_{31}} \bigcup_{A\in\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}} A = E_{2^2}[E_{2^5}].$$

Note that by Theorem 3.4, $\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}]$ contains all edges in E_{2^5}. From $|E_{2^2}[T^{\alpha^i}]| = 7$ and the fact that we do not delete links in this procedure, we get $m_i \leq 5$ and $\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i} \cong K_{7-m_i}$.

Therefore, there is always a path through each vertex of $\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$, where endvertices belong to $L(T^{\alpha^i})$. Since all links are preserved, the mentioned paths, after being joined together, make a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$. $\square$

4. Appendix

We list here all the powers α^i together with the images T^{α^i}:

$$\begin{aligned}
\alpha =&(bc, cd, bcd, de, a), \quad T^{\alpha} = \langle bc, cd, bcd\rangle,\\
\alpha^2 =&(b, bce, bde, ade, bc), \quad T^{\alpha^2} = \langle b, bce, bde\rangle,\\
\alpha^3 =&(bc, ab, ace, abcde, b), \quad T^{\alpha^3} = \langle bc, ab, ace\rangle,\\
\alpha^4 =&(bce, bd, ad, acde, cd), \quad T^{\alpha^4} = \langle bce, bd, ad\rangle,\\
\alpha^5 =&(ab, ce, bcde, ae, bce), \quad T^{\alpha^5} = \langle ab, ce, bcde, ae, bce\rangle,\\
\alpha^6 =&(bd, abcd, abde, abc, ab), \quad T^{\alpha^6} = \langle bd, abcd, abde\rangle,\\
\alpha^7 =&(ce, cde, abe, c, bd), \quad T^{\alpha^7} = \langle ce, cde, abe\rangle,\\
\alpha^8 =&(abcd, abce, abd, abc, ce), \quad T^{\alpha^8} = \langle abcd, abce, abd\rangle,\\
\alpha^9 =&(cde, ac, be, bde, abcd), \quad T^{\alpha^9} = \langle cde, ac, be\rangle,\\
\alpha^{10} =&(abce, d, acd, ace, cde), \quad T^{\alpha^{10}} = \langle abce, d, acd\rangle,\\
\alpha^{11} =&(ac, de, e, ad, abce), \quad T^{\alpha^{11}} = \langle ac, de, e\rangle,\\
\alpha^{12} =&(d, ade, a, bcde, ad), \quad T^{\alpha^{12}} = \langle d, ade, a\rangle,\\
\alpha^{13} =&(de, abcde, bc, abde, d), \quad T^{\alpha^{13}} = \langle de, abcde, bc\rangle,\\
\alpha^{14} =&(ade, acde, b, abe, de), \quad T^{\alpha^{14}} = \langle ade, acde, b\rangle,\\
\alpha^{15} =&(abcde, ae, cd, abd, ade), \quad T^{\alpha^{15}} = \langle abcde, ae, cd\rangle,\\
\alpha^{16} =&(acde, abc, bce, be, abcde), \quad T^{\alpha^{16}} = \langle acde, abc, bce\rangle,\\
\alpha^{17} =&(ae, c, ab, acd, acde), \quad T^{\alpha^{17}} = \langle ae, c, ab\rangle,\\
\alpha^{18} =&(abc, bcd, bd, e, ae), \quad T^{\alpha^{18}} = \langle abc, bcd, bd\rangle,\\
\alpha^{19} =&(c, bde, ce, a, abc), \quad T^{\alpha^{19}} = \langle c, bde, ce\rangle,\\
\alpha^{20} =&(bcd, ace, abcd, bc, c), \quad T^{\alpha^{20}} = \langle bcd, ace, abcd\rangle,\\
\alpha^{21} =&(bde, ad, cde, b, bcd), \quad T^{\alpha^{21}} = \langle bde, ad, cde\rangle,\\
\alpha^{22} =&(ace, bcde, abce, cd, bde), \quad T^{\alpha^{22}} = \langle ace, bcde, abce\rangle,
\end{aligned}$$

$$\alpha^{23} =(ad, abde, ac, bc, ace), \quad T^{\alpha^{23}} = \langle ad, abde, ac\rangle,$$
$$\alpha^{24} =(bcde, abe, d, ab, ad), \quad T^{\alpha^{24}} = \langle bcde, abe, d\rangle,$$
$$\alpha^{25} =(abde, abd, de, bd, bcde), \quad T^{\alpha^{25}} = \langle abde, abd, de\rangle,$$
$$\alpha^{26} =(abe, be, ade, ce, abde), \quad T^{\alpha^{26}} = \langle abe, be, ade\rangle,$$
$$\alpha^{27} =(abd, acd, abcde, abcd, abe), \quad T^{\alpha^{27}} = \langle abd, acd, abcde\rangle,$$
$$\alpha^{28} =(be, e, acde, cde, abd), \quad T^{\alpha^{28}} = \langle be, e, acde\rangle,$$
$$\alpha^{29} =(ace, a, ae, abce, be), \quad T^{\alpha^{29}} = \langle ace, a, ae\rangle,$$
$$\alpha^{30} =(e, bc, abc, ac, acd), \quad T^{\alpha^{30}} = \langle e, bc, abc\rangle,$$
$$\alpha^{31} =(bc, cd, bcd, de, a), \quad T^{\alpha^{31}} = \langle bc, cd, bcd\rangle.$$

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