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DIRECT LIMIT OF (m,n)-ARY HYPERMODULES

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Abstract. The purpose of this paper is the study of direct limit in the category of (m,n)-ary hypermodules over (m,n)-hyperring R. In this regards, we introduce and study $R_{(m,n)} - Hmod$, the category of $R_{(m,n)} - Hmod$, and direct limit in this category. In particular, we study a direct limit of morphisms, direct systems of kernels, and cokernels. Finally, we investigate the relationship between the functor home and direct limit and prove that the functor *hom* preserves direct limit in category $R_{(m,n)} - Hmod$.

1. Introduction

The notion of n-ary groups (also called n-group or multi-ary group) is a generalization of groups. An n-ary group (G,f) is a pair of a set G and a map $f : G\times\cdots\times G \to G$, which is called an n-ary operation. The earliest work on these structures was done in 1904 by Krasner [23] and in 1928 by Dörnet [20]. Such n-ary groups have many applications in computer science, coding theory, topology, combinatorics, and quantum physic (for more details see [16–19, 30, 31]). One of the applications in algebraic hyperstructures theory was defined by Marty [28]. Many researchers developed this theory of view point of theory and application(for more see [5, 11, 12, 14, 15, 36]).

Ameri et al. [3] introduced and studied the notion of hyperalgebraic, a framework to formulate algebraic hyperstructures in a general manner, also R. Ameri and I. G. Rosenberg [2]. Davvaz and Vougiouklis [15] studied n-ary hypergroups. After that, a generalization of it, such as (m,n)-hyperrings and (m,n)-hypermodules were introduced and studied in different contexts(some of the studies can be found in [4, 6, 7, 9, 24–26, 29]). On the other hand, fundamental relations, as the smallest equivalence

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relation on an algebraic hyperstructure such as a hypergroup, hyperring, hypermodules or in general a hyperalgebra such that its quotient is a group, ring, module, or algebra respectively, play an important role to study the theory of algebraic hyperstructure. In fact, the fundamental relation on an algebraic hyperstructure induces a functor from a category of algebraic hyperstructures such as a category of hypergroups and hypermodules to its related classical algebraic structure such as the category of group and modules. R. Ameri in [1] introduced and studied the category of hypergroups and hypermodules. Recently, various kinds of categories of hyperstructures have been studied in numerous papers(for instance see [1, 21, 22, 27, 32–35]). In this paper, we follow [21] and introduce and study direct limit in the category of (m, n)-hypermodules. This work is a generalization of the paper A. Asadi, R. Ameri, Direct Limit of Krasner (m, n)-Hyperrings [8], with more details of categorical properties related to direct limit. In Section 2, we give some basic preliminaries about (m, n)-rings and (m, n)-hypermodules. In Section 3, we introduce a direct system of (m, n)-ary hypermodules and use it to introduce direct limit in category (m, n)-hypermodules. In Section 4, the properties of direct limit of a direct system of (m, n)-ary hypermodules are investigated. In Section 5, the direct limit of morphisms is studied and some basic properties of the are obtained. In section 6, direct systems of kernels and cokernels of a direct system of (m, n)-ary hypermodules are studied. Finally, in section 7, the behavior of direct limits under home representable functors is studied, and it is shown that these functors preserve limits.

2. Preliminaries

In this section, we give some definitions and results of n-array hyperstructures which we need in what follows.

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \to P^*(H)$ is called an n-ary *hyperoperation,* where $P^*(H)$ is the set of all nonempty subsets of H. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid.*

We shall use the following abbreviated notation.

The sequence $x_i, x_{i+1}, \dots, x_j$ will be denoted by x_i^j. For $j < i, x_i^j$ is the empty set. Using this notation, $f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_n)$ will be written as $f\left(x_1^i, y_{i+1}^j, z_{j+1}^n\right)$. In the case when $y_{i+1} = \cdots = y_j = y$ the last expression will be written $f\left(x_1^i, y_{(j-i)}, z_{j+1}^n\right)$.

If f is an n-array hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then t-array hyperoperation f_l is given by

$$f_l\left(x_1^{l(n-1)+1}\right) = \underbrace{f\Big(f\Big(\dots, f(f}_{l}(x_1^n), x_{n+1}^{2n-1}\Big), \dots, \Big), x_{(l-1)(n-1)+1}^{l(n-1)+1}\Big).$$

For nonempty subsets $A_1, A_2, \dots, A_n$ of H, define

$$f(A_1^n) = f(A_1, A_2, \dots, A_n) = \bigcup\{f\left(x_1^n\right) \mid x_i \in A_i, i = 1, 2, \dots, n\}.$$

An n-array hyperoperation f is called *associative* if

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right),$$

hold for every $1 \leq i < j \leq n$ and all $x_1, \ldots, x_{n-1} \in H$. An n-array hypergroupoid with the associative n-array hyperoperation is called an *n-ary semihypergroup.*

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution, $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an n-ary quasihypergroup. If (H, f) is an n-ary semihypergroup and n-array quasihypergroup, then (H, f) is called an n-ary hypergroup. An n-ary hypergroupoid (H, f) is commutative if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$, then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$.

Definition 2.1 ([15])**.** Let (H, f) be an n-array hypergroup and B be a non-empty subset of H. B is called an n-ary subhypergroup of (H, f), if $f(x_1^n) \subseteq B$ for all $x_1^n \in B$, and the equation $b \in f\left(b_1^{i-1}, x_i, b_{i+1}^n\right)$ has a solution, $x_i \in B$ for every $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

Definition 2.2 ([15])**.** Let (H, f) be a commutative n-ary hypergroup. (H, f) is called canonical n-ary hypergroup if the following statements are satisfied:

(1) there exists unique $e \in H$, such that for every $x \in H$, $f(x, \underbrace{e, \ldots, e}_{(n-1)}) = x$;

(2) for all $x \in H$ there exists unique $x^{-1} \in H$, such that $e \in f(x, x^{-1} \underbrace{e, \ldots, e}_{(n-2)})$;

(3) if $x \in f(x_1^n)$, then for all i, we have $x_i \in f\left(x, x^{-1}, \ldots, x_{i-1}^{-1}, x_{i+1}^{-1}, \ldots, x_n^{-1}\right)$.

We say that e is the scaler identity of (H, f) and x^{-1} is the inverse of x. Notice the inverse of e is e.

Definition 2.3 ([29])**.** A (Krasner) (m, n)-hyperring is algebraic hyperstructure (R, h, k) which satisfies the following axioms:

(1) (R, h) is a canonical m-ary hypergroup;

(2) (R, k) is an n-ary semigroup;

(3) the n-ary operation k is distributive to the m-array hyperoperation h, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$k\left(a_1^{i-1}, h(x_1^m), a_{i+1}^n\right) = h\left(k(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, k(a_1^{i-1}, x_m, a_{i+1}^n)\right);$$

(4) 0 is a zero element (absorbing element), of the n-ary operation k, i.e., for $x_2^n \in R$ we have $k(0, x_2^n) = k(x_2, 0, x_3^n) = \cdots = k(x_2^n, 0)$.

A nonempty subset S of R is called a subhyperring of R if (R, h, k) is a Krasner (m, n)-hyperring. Let I be a non-empty subset of R. We say that I is a hyperideal of (R, h, k) if (I, h) is a canonical m-ary hypergroup of (R, h) and $k(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for every $x_1^n \in R$ and $1 \leq i \leq n$.

Definition 2.4. Let M be a nonempty set. Then (M, f, g) is an (m, n)-hypermodule over an (m, n)-hyperring (R, h, k), if (M, f) is an m-ary hypergroup and the map $g : \underbrace{R \times \cdots \times R}_{n-1} \times M \to P^*(M)$ satisfies the following conditions:

(i) $g\left(r_1^{n-1}, f(x_1^m)\right) = f\left(g(r_1^{n-1}, x_1), \ldots, g(r_1^{n-1}, x_m)\right)$;
(ii) $g\left(r_1^{i-1}, h(s_1^m), r_{i+1}^{n-1}, x\right) = f\left(g(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, g(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x\right)$;
(iii) $g\left(r_1^{i-1}, k(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x\right) = g\left(r_1^{n-1}, g(r_m^{n+m-2}, x)\right)$;
(iv) $0 \in g\left(r_1^{i-1}, 0, r_{i+1}^{n-1}, x\right)$.

If g is an n-ary hyperoperation, $S_1, \ldots, S_{n-1}$ are subsets of R and $M_1 \subseteq M$, we set

$$g(S_1^{n-1}, M_1) = \bigcup\{g(r_1^{n-1}, x) \mid r_i \in S_i, i = 1, \ldots, n-1, x \in M_1\}.$$

If $n = m = 2$ then an (m, n)-ary hypermodule M is hypermodule.

Let (M, f, g) be an (m, n)-hypermodule over an (m, n)-hyperring (R, h, k). A nonempty subset N of M is called an (m, n)-ary sub-hypermodule of M if (N, f) is m-array subhypergroup of (M, f) and $g(R^{(n-1)}, N) \in P^*(N)$.

Definition 2.5. A canonical (m, n)-hypermodule (M, f, g) is an (m, n)-hypermodule with a canonical m-array hypergroup (M, f) over a Krasner (m, n)-hyperring (R, h, k).

A Krasner (m, n)-hyperring (R, h, k) is *commutative* if (R, k) is a commutative n-ary semigroup. Also, we say that (R, h, k) is a *scaler identity* if there exists an element 1_R, such that $x = k(x, 1_R^{(n-1)})$ for all $x \in R$. Later on, let (R, h, k) be a commutative Krasner (m, n)-hyperring with a scaler identity 1_R. For all $r_1^{n-1} \in R$ and $x \in M$ we have

$$g(r_1^{n-1}, 0_M) = \{0_M\}, \quad g(0_R^{n-1}, x) = \{0_M\} \quad \text{and} \quad g(1_R^{n-1}, x) = \{x\}.$$

Moreover, let $g(r_1^{i-1}, -r_i, r_{i+1}^{n-1}, x) = -g(r_1, \ldots, r_{n-1}, x) = g(r_1^{n-1}, -x)$.

Definition 2.6 ([29])**.** Let (M_1, f_1, g_1) and (M_2, f_2, g_2) be two (m, n)-hypermodules over an (m, n)-hyperring (R, h, k). We say that $\phi : M_1 \to M_2$ is a homomorphism of (m, n)-hypermodules if for all x_1^m, x of M_1 and $r_1^{n-1} \in R$: $\phi\left(f_1(x_1, \ldots, x_m)\right) = f_2\left(\phi(x_1), \ldots, \phi(x_m)\right)$, $\phi\left(g_1(r_1^{n-1}, x)\right) = g_2\left(r_1^{n-1}, \phi(x)\right)$.

If in the above definition we consider a map $\phi : M_1 \to P^*(M_2)$, then we obtain a *multivalued homomorphism,* shortly we write m-homomorphism.

Example 2.1. We shall provide an example of an m-homomorphism. Let A and B be two canonical hypergroup as Tables 1 and 2.

Define $0 * x = 0$ and $1 * x = x$ for all $x \in A, B$. Then, it is easy to check that $(A, +, *)$ is a Krasner hyperring, and A and B are also A-hypermodule with the external multiplication $*$. Let $\varphi : B \to A$ with $\varphi(1) = \varphi(-1) = 1$ and $\varphi(0) = 0$. Clearly, φ is an m-homomorphism.

+	0	1
0	0	1
1	1	$\{o, 1\}$

TABLE 1. $(A, +)$

$+'$	0	1	-1
0	0	1	-1
1	1	1	$\{o, 1,\text{-}1\}$
-1	-1	$\{o, 1,\text{-}1\}$	-1

TABLE 2. $(B, +')$

Definition 2.7 ([9])**.** A linear combination of family $A = \{x_i \mid i \in I\}$ of elements of M is a sum of the form $f\left(g(r_{11}^{1(n-1)}, x_1), \ldots, g(r_{11}^{l(n-1)}, x_l), o^{(m-l)}\right)$ with $l \leq m$ and if $l > m, l = t(m-1)+1$, a linear combination of A is the form of

$$\underbrace{f\Big(f\Big(\ldots, f\Big(f}_{t}\left(g(r_{11}^{1(n-1)}, x_1\Big), \ldots, g\Big(r_{m1}^{m(n-1)}, x_m\Big)\Big), g\Big(r_{(m+1)1}^{(m+1)(n-1)}, x_{m+1}\Big), \ldots,\right.$$

$$g\Big(r_{(2m-1)1}^{(2m-1)(n-1)}, x_{2m-1}\Big), \ldots\Big), g\Big(r_{((l-1)(m-1)+2)1}^{((l-1)(m-1)+2))(n-1)}, \ldots, g\Big(r_{(l(m-1)+1)1}^{(l(m-1)+1)(n-1)}\Big)\Big)\Big),$$

where $r_{ij} \in R$ and set $\{r_{ij}, r_{ij} \neq 0\}$ is finite.

A linear combination of family $\{x_i \mid i \in I\}$ of elements of M is a sum of the form

$$\Big\{f\Big(g\Big(r_{11}^{1(n-1)}, x_1\Big), \ldots, g\Big(r_{l1}^{l(n-1)}, x_l\Big)\Big) \mid x_i, i \in I\Big\}$$

is linear dependent if there exists a linear combination

$$f\Big(g\Big(r_{11}^{1(n-1)}, x_1\Big), \ldots, g\Big(r_{l1}^{l(n-1)}, x_l\Big)\Big)$$

containing 0, without being all r_{ij} equal to 0. Otherwise, $\{x_i \mid i \in I\}$ is called linear independent.

Definition 2.8 ([9])**.** A subset X of M generates M if every element of M belongs to linear combination of elements from X.

Definition 2.9 ([21])**.** The category $R_{(m,n)} - Hmod$ of (m,n)-ary hypermodules defined as follows:

(i) the objects of $R_{(m,n)} - Hmod$ are (m,n)-hypermodules,
(ii) for the objects M and K, the set of all morphisms from M to K is defined as follows:

$$Hom_R(M, K) = \{f \mid f : M \to P^*(K) \text{ is an m-homomorphism}\};$$

(iii) the composition gf of morphisms $f: M \to P^*(K)$ and $g: K \to P^*(L)$ defined as follows:

$$gf: H \to P^*(K), \quad gf(x) = \bigcup_{t \in f(x)} g(t);$$

(iv) for any object H, the morphism $1_H: H \to P^*(H)$, defined by $1_H(x) = \{x\}$, is the identity morphism.

Remark 2.1. Consider a category whose objects are all (m,n)-hypermodules and whose morphisms are all R-homomorphisms denoted by $R_{(m,n)} - hmod$. The class of all R-homomorphisms from A into B is denoted by $hom_R(A,B)$. In addition, $R_{s(m,n)} - hmod$ is the category of all (m,n)-hypermodules whose morphisms are all strong R-homomorphisms. The class of all strong R-homomorphisms from A into B is denoted by $hom_{R_S}(A,B)$. It is easy to observe that $R_{s(m,n)} - hmod$ is a subcategory of $R_{(m,n)} - hmod$.

Definition 2.10 ([21])**.** Let $\{M_i \mid i \in I\}$ be a family of (m,n)-hypermodules. We define a hyperoperation on $\prod_{i\in I} M_i$ as follows:

$$F\{a_{i1}^{im}\} = \left(\{t_i\} \mid t_i \in f_i(a_{i1}^{im}), \{a_{i1}^{im}\} \in \prod_{i\in I} M_i\right).$$

For $r \in R$ and $a_i \in \prod_{i\in I} M_i$, define

$$G\left(r_1^{(n-1)}\{a_i\}_{(i\in I)}\right) = \left\{g_i\left(r_1^{(n-1)}, a_i\right)\right\}_{i\in I}.$$

then $\prod_{i\in I} M_i$, together with m-array hyperoperation F and n-array operation G is called *direct hyper product* $\{M_i \mid i \in I\}$.

Theorem 2.1 ([21])**.** *Let $\{M_i \mid i \in I\}$ be a family of (m,n)-hypermodules, and $\{\phi_i : M \to p^*(M_i) \mid i \in I\}$ be a family of m-homomorphisms. Then there exists a unique m-homomorphism*

$$\left(\left\{\phi : M \to p^*\left(\prod_{i\in I} M_i\right)\right)\right\}$$

such that, $\Pi_i\phi = \phi_i$ for all $i \in I$, and this property determines $\prod_{i\in I} M_i$ uniquely up to isomorphism. In other words, $\prod_{i\in I} M_i$ is a product in the category of $R_{(m,n)} - Hmod$.

Definition 2.11 ([21])**.** The direct hypersum of a family $\{M_i \mid i \in I\}$ of (m,n)-hypermodules, denoted by $\coprod_{i\in I} M_i$ is the set of all $\{a_i\}_{i\in I}$, where a_i can be non-zero only for a finite number of indices.

Proposition 2.1 ([21])**.** *If $\{M_i \mid i \in I\}$ is a family of (m,n)-hypermodules, then*

(i) *$\coprod_{i\in I} M_i$ is an (m,n)-hypermodule.*

(ii) *for each* $k \in I$, *the map* $\ell_k : M_k \to \coprod_{i\in I} M_i$, *given by* $l_k(a) = \{a_i\}_{i\in I}$, *where* $a_i = 0$, *for* $i \neq k$, *and* $a_k = a$, *is m-homomorphism.*

(iii) *for each* $i \in I$, $\ell_i(M_i)$ *is a subhypermodule of* $\coprod_{i\in I} M_i$. *The map* ℓ_k *is called the canonical injection.*

Theorem 2.2 ([21])**.** *Let* $\{M_i \mid i \in I\}$ *be a family of* (m,n)*-hypermodules and* $\{\phi_i : M_i \to M \mid i \in I\}$ *be a family of m-homomorphisms of* (m,n)*-hypermodules. Then, there is a unique m-homomorphism* $\phi : \coprod_{i\in I} M_i \to M$ *such that* $\phi\ell_i = \phi_i$, *for all* $i \in I$ *and this property determines* $\coprod_{i\in I} M_i$ *uniquely up to isomorphism. In the other words* $\coprod_{i\in I} M_i$ *is a coproduct in the category of* $R_{(m,n)} - Hmod$.

Remark 2.2. In the following sections of this paper, we consider the category of all (m,n)-hypermodules over a (m,n)-hyperring R, in the sense of Canonical (m,n)-hypermodules over Krasner (m,n)-hyperring R with a scaler identity. We denote this category by $R_{(m,n)} - Khmod$. Hence the objects of $R_{(m,n)} - Khmod$ are Canonical (m,n)-hypermodules over Krasner (m,n)-herringbone.

3. The Direct Limit

Definition 3.1 ([37])**.** Let (A, Λ) be a quasi-ordered directed(to the right) set, i.e. for the two elements $i, j \in \Lambda$ there exists (at least one) $k \in \Lambda$ with $i \leq K$ and $j \leq K$.

A direct system of (m,n)-ary hypermodules $(M_i, \phi_{ij})_\Lambda$ consists of

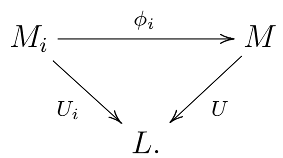
(1) a family of (m,n)-ary hypermodules $(M_i)_\Lambda)$ and
(2) a family of morphisms $\phi_{ij} : M_i \to M_j$ for all pairs (i,j) with $i \leq j$, satisfying

$$\phi_{ii} = id_{M_i} \quad \text{and} \quad \phi_{jk}\phi_{ij} = \phi_{ik}, \quad \text{for } i \leq j \leq k.$$

A direct system of morphisms from $(M_i, \phi_{ij})_\Lambda$ into an $R - (m,n)$-hypermodules L is a a family of morphisms $\{U_i : M_i \to L\}$ with $U_j\phi_{ij} = U_i$ whenever $i \leq j$.

Definition 3.2. Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R - (m,n)$-hypermodules and M an $R - (m,n)$-hypermodule.

A direct system of morphisms $\{\phi_i : M_i \to M\}_\Lambda$ is said to be a direct limit of $(M_i, \phi_{ij})_\Lambda$ if, for every direct system of morphisms $\{U_i : M_i \to L\}_\Lambda, L \in R_{(m,n)} - hmod$, there is a unique morphism $U : M \to L$ which makes the following diagram commutative for every $i \in \Lambda$



If $\{\phi'_i : M_i \to M'\}_\Lambda$ is another direct limit of $(M_i, \phi_{ij})_\Lambda$, then by definition there is an isomorphism $H : M_i \to M'$ with $H\phi_i = \phi'_i$ for $i \in \Lambda$. Hence M is uniquely determined up to isomorphism.

We write $M = \varinjlim M_i$ and $(\phi_i, \varinjlim M_i)$ for the direct limit.

Example 3.1. A collection of subsets M_i of a set M can be partially ordered by inclusion. If the collection is directed, its direct limit is the union $\cup M_i$. The same is true for a directed collection of subgroups of a given group.

Theorem 3.1. *Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R-(m,n)$-hypermodules. For every pair $i \leq j$ we put $M_{i,j} = M_i$ and obtain with canonical embedding ℓ_i the following mappings:*

$$M_{i,j} \xrightarrow{\phi_{ij}} M_j \xrightarrow{\ell_j} \coprod_\Lambda M_k,$$

$$M_{i,j} \xrightarrow{id_{M_i}} M_i \xrightarrow{\ell_i} \coprod_\Lambda M_k.$$

The difference yields morphisms $F\{-\ell_i, \ell_j\phi_{ij}, o^{(m-2)}\} : M_{i,j} \longrightarrow \coprod_\Lambda M_k$ and with the coproduct we obtain a morphism $\phi : \coprod_{i\leq j} M_{i,j} \longrightarrow \coprod_\Lambda M_k$.

$Cok\phi$ together with the morphisms

$$\phi_i = Cok\phi\ell_i : M_i \longrightarrow \coprod_\Lambda M_k \longrightarrow Cok\phi$$

form a direct limit of $(M_i, \phi_{ij})_\Lambda$.

Proof. Let $\{U_i : M_i \to L\}_\Lambda$ be a direct limit of morphisms and $\overline{U} : \coprod_\Lambda M_k \longrightarrow L$ with $\overline{U}\ell_k = U_k$. We have $0 \in \overline{U}(F(-\ell_i, \ell_j\phi_{ij}, o^{(m-2)})) = f_l(-U_i, U_j\phi_{ij}, o^{(m-2)})$ for $i \leq j$. Hence, $\overline{U}\phi = 0$ and the diagram

$$\begin{array}{ccccc} \coprod_{i\leq j} M_{ij} & \xrightarrow{\phi} & \coprod_\Lambda M_k & \longrightarrow & Coke\phi \\ & & \downarrow \overline{U} & & \\ & & L & & \end{array}$$

can be extended to a commutative diagram by a unique $U : Coke\phi \to L$ (definition of cokernel). □

Remark 3.1 ([37]). Regarding the quasi-ordered set Λ as a (directed) category, a directed system of (m,n)-hypermodules corresponds to a functor $\phi : \Lambda \to R_{(m,n)} - hmod$. Then direct system of morphisms is functorial morphisms between ϕ and constant functor $\Lambda \to R_{(m,n)} - hmod$. Then the direct limit is called the colimit of the functor ϕ. Instead of Λ, more general categories can serve as source and Instead of $R_{(m,n)} - hmod$, other categories may be used as target.

4. Properties of the Direct Limit

Theorem 4.1. *Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R-(m,n)$-hypermodules with direct limit $(\phi_i, \varinjlim M_i)$.*

(1) *For* $m_j \in M_j, j \in \Lambda$, *we have* $0 \in \phi_j(m_j)$ *if and only if, for some* $k \geq j$, $0 \in \phi_{jk}(m_j)$.
(2) *For* $m, n \in \varinjlim M_i$, *there exist* $k \in \Lambda$ *and elements* $m_k, n_k \in M_k$ *with* $m \in \phi_k(m_k)$ *and* $n \in \phi_k(n_k)$.
(3) *If* N *is a finitely generated submodules of* $\varinjlim M_i$, *then there exist* $k \in \Lambda$ *with* $N \subset \phi_k(m_k)(= \operatorname{Im}\phi_k)$.
(4) $\varinjlim M_i = \bigcup_{\Lambda} \operatorname{Im}\phi_i$.

Proof. (1) If $0 \in \phi_{jk}(m_j)$, then also $0 \in \phi_j(m_j) = \phi_k\phi_{jk}m_j$.

Assume on the other hand $0 \in \phi_j(m_j)$, i.e., within Theorem 3.1

$$\ell_j m_j \in \operatorname{Im} F, \quad \ell_j m_j = \sum_{(i,l)\in E} f(-\ell_i, \ell_i\phi_{il}, 0^{(m-2)})m_{il}, \quad m_{il} \in M_{i,l},$$

where E is a finite set of pairs $i \leq l$.

Choose any $k \in \Lambda$ bigger than all the indices occurring in E and $j \leq k$.

For $i \leq k$ the $\phi_{ik} : M_i \to M_k$ yield a morphism $\psi_k : \coprod_{i\leq k} M_i \to M_k$ with $\psi_k\ell_i = \phi_{ik}$ and

$$\begin{aligned}\phi_{jk}m_j =&\psi_k\ell_j m_j = \sum_E f\left(\psi_k\ell_l\phi_{il}, -\ell_i\psi_k, 0^{(m-2)}\right) m_{il}\\ =&\sum_E f\left(\phi_{lk}\phi_{il}, -\phi_{ik}, 0^{(m-2)}\right) m_{il} \ni 0.\end{aligned}$$

(2) For $m \in \varinjlim M_i$, let $(m_{i_1}, \ldots, m_{i_r})$ be a preimage of m in $\coprod_\lambda M_k$ (under Coke F). For $k \geq i_1, \ldots, i_r$ we get

$$m \in f_i\left(\phi_{i_1}(m_{i_1}), \ldots, \phi_{i_r}(m_{i_r})\right) = \phi_k\left(f_i(\phi_{i_1k}(m_{i_1}), \ldots, \phi_{i_rk}(m_{i-r}))\right).$$

For $m, n \in \varinjlim M_i$, and $k, l \in \Lambda, m_k \in M_k, n_l \in M_l$ with $m \in \phi_k(m_k), n \in \phi_l(n_l)$, we choose $s \geq k, s \geq l$ to obtain $m \in \phi_s(\phi_{ks}(m_k)), n \in \phi_s(\phi_{ls}(n_l))$.

(3), (4) are consequences of (2). □

5. Direct Limit of Morphisms

Theorem 5.1. *Let* $(M_i, \phi_{ij})_\Lambda$ *and* $(N_i, \psi_{ij})_\Lambda$ *be two direct systems of* $R-(m,n)$*-hypermodules over the same set* Λ *and* $(\phi_i, \varinjlim M_(i)$, *resp.* $(\psi_i, \varinjlim N_i)$ *their direct limits.*

For any family of morphisms $\{u_i : M_i \to N_i\}_\Lambda$, *with* $\phi_{ij}u_j = \psi_{ij}u_i$ *for all indices* $i \leq j$, *there is unique morphism*

$$u : \varinjlim M_i \to \varinjlim N_i,$$

such that, for every $i \in \Lambda$, the following diagram is commutative

$$\begin{array}{ccc} M_i & \xrightarrow{u_i} & N_i \\ \phi_i \downarrow & & \downarrow \psi_i \\ \varinjlim M_i & \xrightarrow{u} & \varinjlim N_i \end{array}.$$

If all the u_i are monic (epic), then u is monic (epic).

Notation: $u = \varinjlim u_i$.

Proof. The mappings $\{\psi_i u_i : M_i \to \varinjlim N_i\}_\Lambda$ form a direct system of morphisms since for $i \leq j$ we get $\psi_j u_j = \psi_j \psi_{ij} u_i = \psi_i u_i$. Hence the existence of u follows from the defining property of the direct limit.

Consider $m \in \varinjlim M_i$ with $0 \in u(m)$. By (4.1), there exist $k \in \Lambda$ and $m_k \in M_k$ with $m \in \phi_k(m_k)$ and hence $0 \in u(\phi_k(m_k)) = \psi_k(u_k(m_k))$. Now there exists $l \geq K$ whith $0 \in \psi_{lk}(u_k(m_k)) = u_l(\phi_{kl}(m_k))$. If u_l is monic, then $\phi_{kl}(m_k) = 0$ and also $m \in \phi_k(m_k) = 0$. Consequently, if all $\{u_i\}_\Lambda$ are monic, then u is monic.

For $n \in \varinjlim N_i$ By (4.1), there exist $k \in \Lambda$ and $n_k \in N_k$ with $n \in \psi_k(n_k)$. If u_k is surjective, then $n_k \in u_k(m_k)$ for some $m_k \in M_k$ and $n \in \psi_k(u_k(m_k)) = u(\phi_k(m_k))$. If all the $\{u_i\}_\Lambda$ are surjective, then u is surjective. □

6. Direct Systems of Kernels and Cokernels

Using Theorem 5.1, we obtain, for $i \leq j$, commutative diagrams

$$\begin{array}{ccccccc} Ke\ u_i & \longrightarrow & M_i & \xrightarrow{u_i} & N_i & \longrightarrow & Coke\ u_i \\ & & \downarrow & & \downarrow & & \\ Ke\ u_j & \longrightarrow & M_j & \xrightarrow{u_j} & N_j & \longrightarrow & Coke\ u_j \end{array}$$

which can be extended by $k_{ij} : Ke\ u_i \to Ke\ u_j$ and $h_{ij} : coke\ u_i \to coke\ u_j$ to commutative diagrams.

$(Ke\ u_i, k_{ij})_\Lambda$ and $(coke\ u_i, h_{ij})_\Lambda$ also form direct system of (m, n)-hypermodules.

Theorem 6.1. *Consider direct systems of $R-(m,n)$-hypermodules*

$$(L_i, \phi_{ij})_\Lambda, (M_i, \psi_{ij})_\Lambda, (N_i, \mu_{ij})_\Lambda,$$

with direct limits $(\phi_i, \varinjlim L_i)$, $(\psi_i, \varinjlim M_i)$, $(\mu_i, \varinjlim N_i)$ and families of morphism $\{u_i\}_\Lambda, \{v_i\}_\Lambda$, which make the following diagrams commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_i & \xrightarrow{u_i} & M_i & \xrightarrow{v_i} & N_i & \longrightarrow & 0 \\ & & \downarrow \varphi_{ij} & & \downarrow \psi_{ij} & & \downarrow \mu_{ij} & & \\ 0 & \longrightarrow & L_j & \xrightarrow{u_j} & M_j & \xrightarrow{v_j} & N_j & \longrightarrow & 0. \end{array}$$

Then, $U = \varinjlim u_i$ *and* $V = \varinjlim v_i$*, the following sequence is also exact:*

$$0 \longrightarrow \varinjlim L_i \xrightarrow{U} \varinjlim M_i \xrightarrow{V} \varinjlim N_i \longrightarrow 0 \ .$$

Proof. It has already been shown in (5.1) that U is monic and V is epic. $\operatorname{Im} U \subseteq Ke\, V$ is obvious. Consider $m \in Ke\ V$. There exist $k \in \Lambda$ and $m_k \in M_k$ with $m \in \psi_k(m_k)$ and $0 \in V(m) \in V(\psi_k(m_k)) = \mu_k(V_k(m_k))$.

Now by (4.1), we can find an $s \in \Lambda$ with $0 \in V_s(\psi_{ks}(m_k)) = \psi_{ks}(V_{ks}(m_k))$.

This implies $\psi_{ks}(m_k) = u_s l_s$ for some $l_s \in L_s$ and

$$U(\varphi_s(l_s)) = \psi_s(u_s(l_s)) = \psi_s(\psi_{ks}(m_s)) = \psi_k(m_k) \ni m.$$

Consequently, $m \in \operatorname{Im} U$ and $\operatorname{Im} U = Ke\, V$. □

Theorem 6.2. *Let* M *be an* $R-(m,n)$*-hypermodule,* Λ *a set, and* $\{M_i\}_\Lambda$ *a family of subhypermodules of* M *directed with respect to inclusion and with* $\bigcup_\Lambda M_i = M$*, then* $\varinjlim M_i = M$*.*

Proof. Defining $i \le j$ if $M_i \subset M_j$ for $i, j \in \Lambda$, the set Λ becomes quasi-ordered and directed. With the inclusion $\varphi_{ij} : M_i \to M_j$ for $i \le j$, the family $\{M_i, \varphi_{ij}\}_\Lambda$ is a direct system of (m,n)-hypermodules and $\varinjlim M_i = M$.

In particular, every (m,n)-hypermodule is a direct limit of its finitely generated subhypermodules. □

7. Home-Functor and Direct Limit

Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R-(m,n)$-hypermodules with direct limit $(\phi_i, \varinjlim M_i)$ and K an $R-(m,n)$-hypermodule. with the assignments, for $i \le j$,

$$h_{ij} := hom(k, \phi_{ij}) : hom(k, M_i) \to hom(k, M_j), \quad \alpha_i \mapsto \phi_{ij}\alpha_i,$$

we obtain a direct system of $\mathbb{Z}-(m,n)$-hypermodules $(hom(k, M_i), h_{ij})_\Lambda$ with direct limit $(h_i, \varinjlim hom(k, M_i))$ and the assignment

$$u_i := hom(k, \phi_i) : hom(k, M_i) \to hom(k, \varinjlim M_i), \quad \alpha_i \mapsto \phi_i\alpha_i,$$

defines a direct system of $\mathbb{Z}$-morphisms ($\mathbb{Z}$ is as an (m,n)-hyperring) and hence a $\mathbb{Z}$-morphism

$$\Phi_K := \varinjlim u_i : \varinjlim hom(k, M_i) \to hom(k, \varinjlim M_i).$$

These $\mathbb{Z}$-morphisms may be regarded as $End(K)$-morphisms.

Theorem 7.1. *If* K *is a finitely generated* $R-(m,n)$*-hypermodule, then* Φ_K *is monic.*

Proof. Consider $\alpha \in Ke\, \Phi_K$. There exist $i \in \Lambda$ and $\alpha_i \in hom(K, M_i)$ with $\alpha \in h_i(\alpha_i)$ and $0 \in \varphi_i(\alpha_i)$, Since $\alpha_i(K) \subset Ke\varphi_i$ is a finitely generated (m,n)-subhypermodule of M_i, There exists $i \le j \in \Lambda$ with $0 \in \varphi_{ij}(\alpha_i(K))$ (by (4.1)). This implies $h_{ij}(\alpha_i) = \varphi_{ij}(\alpha_i) = 0$ and $h_i(\alpha_i) = 0$ in $hom(k, \varinjlim M_i)$. □

Theorem 7.2. *An $R-(m,n)$-hypermodule K is finitely generated if and only if*

$$\Phi_K : \varinjlim hom(k, M_i) \to hom(k, \varinjlim M_i)$$

is an isomorphism for every direct system $(M_i, \psi_{ij})_\Lambda$ of (m,n)-hypermodules with ψ_{ij} monomorphisms.

Proof. Let K be finitely generated. By (7.1), Φ_K is monic. With the φ_{ij} monic, the φ_i are monic. For every $\alpha \in hom(k, \varinjlim M_i)$, the image $\alpha(K)$ is finitely generated. By (4.1), $\alpha(K) \subset \varphi_k(M_k)$ for some $k \in \Lambda$., with $\varphi_k^{-1} : \psi_k(M_k)$ we get $\varphi_k^{-1}\alpha \in hom(k, M_k)$ and $\Phi_k h_k(\varphi_k^{-1}\alpha) = \varphi_k(\varphi_k^{-1}\alpha) \ni \alpha$, i.e., Φ_k is surjective.

On the other hand. Assume Φ_k is an isomorphisms for the direct system $(K_i, \varphi_{ij})_\Lambda$ of the finitely generated (m,n)-subhypermodules $K_i \subset K$, i.e.,

$$\varinjlim hom(K, K_i) \simeq hom(K, \varinjlim K_i) \simeq hom(K, K).$$

By (4.1), there exist $j \in \Lambda$ and $\alpha_j \in hom(K, K_j)$ with $\alpha_j \varphi_j = id_K$, i.e., $K = \alpha_j \varphi_j \alpha_j K = \varphi_j K_j$. Hence, K is finitely generated. □

8. Conclusions and Future Works

In this paper, the category of (m,n)-hypermodules introduced and studied, especially the subclass of canonical (m,n)-hypermodules was investigated. Also, direct limit in category (m,n)-hypermodules was introduced and its basic properties has been discussed. In this regards, the relationship between direct limit and functor home in this category was investigated. The paper provided a good introduction to study the category of (m,n)- hypermodules as a generalization of category of (m,n)-modules as well as hypermodules. At the end, the paper provide a good introduction to study the homology of (m,n)-hypermodules, as well as hyperstructures in general.

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