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# ON ZERO FREE REGIONS FOR DERIVATIVES OF A POLYNOMIAL

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Abstract. Let $P_n$ denote the set of polynomials of the form

$$p(z)=(z-a)^m\prod_{k=1}^{n-m}(z-z_k),$$

with $|a|\leq 1$ and $|z_k|\geq 1$ for $1\leq k\leq n-m$. For the polynomials of the form $p(z)=z\prod_{k=1}^{n-1}(z-z_k)$, with $|z_k|\geq 1$, where $1\leq k\leq n-1$, Brown [2] stated the problem "Find the best constant $C_n$ such that $p'(z)$ does not vanish in $|z|<C_n$". He also conjectured in the same paper that $C_n=\frac{1}{n}$. This problem was solved by Aziz and Zarger [1]. In this paper, we obtain the results which generalizes the results of Aziz and Zarger.

## 1. Introduction and statement of results

Let $p(z)=\prod_{k=1}^{n}(z-z_k)$ be a complex polynomial of degree $n$. The classical Gauss-Lucas theorem states that every critical point of a complex polynomial $p$ of degree at least 2 lies in the convex hull of its zeros. This theorem has been further investigated and developed. About the location of critical point relative to each individual zero, a possible answer is given by the famous conjecture known in literature as Sendov's conjecture.

*Conjecture* 1 (Sendov's Conjecture). If all the zeros of a polynomial $p(z)$ lie in $|z|\leq 1$, then for any zero $z_0$ of $p$, the disc $|z-z_0|\leq 1$ contains at least one critical point of $p$.

This conjecture has attracted much attention. About 100 papers have been published related to this conjecture. This conjecture has so far been verified for general

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polynomials of degree less than or equal to 8. However the problem is still unproved in general.

In connection with this conjecture, Brown [2] observed that, if $p(z) = z(z-1)^{n-1}$, then $p'(\frac{1}{n}) = 0$ and posed the following problem.

"Let $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$. Find the best constant $C_n$ such that $p'(z)$ does not vanish in $|z| < C_n$".

However, Brown himself conjectured that $C_n = \frac{1}{n}$. This problem has been settled by Aziz and Zarger [1], in fact they proved the following.

**Theorem 1.1.** *If $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$ is a polynomial of degree $n$, with $|z_k| \geq 1$, where $1 \leq k \leq n-1$, then $p'(z)$ does not vanish in $|z| < \frac{1}{n}$.*

As a generalization of Theorem 1.1, N. A. Rather and F. Ahmad [3] have proved the following result.

**Theorem 1.2.** *Let $p(z) = (z-a)\prod_{k=1}^{n-1}(z - z_k)$ with $|a| \leq 1$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-1$, then $p'(z)$ does not vanish in the region*

$$\left|z - \left(\frac{n-1}{n}\right)a\right| < \frac{1}{n}.$$

*The result is best possible as is shown by the polynomial*

$$p(z) = (z-a)(z - e^{i\alpha})^{n-1}, \quad 0 \leq \alpha < 2\pi.$$

N. A. Rather and F. Ahmad also proved the following result in the same paper.

**Theorem 1.3.** *Let $p(z) = (z-a)^m\prod_{k=1}^{n-m}(z - z_k)$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then $p'(z)$ has $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros of $p'(z)$ lie in the region*

$$\left|z - \left(\frac{n-m}{n}\right)a\right| \geq \frac{m}{n}.$$

*The result is best possible as is shown by the polynomial*

$$p(z) = (z-a)^m(z - e^{i\alpha})^{n-m}, \quad 0 \leq \alpha < 2\pi.$$

Zarger and Manzoor [4] have extended Theorem 1.1 to the second derivative $p''(z)$ of a polynomial of the form $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$, with $|z_k| \geq 1$ for $1 \leq k \leq n-m$. In fact they proved the following.

**Theorem 1.4.** *If $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$ with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p''(z)$ does not vanish in $0 < |z| < \frac{m(m-1)}{n(n-1)}$.*

Zarger and Manzoor [4] also obtained the following result for the polynomial $p^{(m)}(z)$, $m \geq 1$.

**Theorem 1.5.** *If $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$ is a polynomial of degree $n$ with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$, does not vanish in $|z| < \frac{m!}{n(n-1)\cdots(n-m+1)}$.*

In this paper, we first prove the following theorem which generalize the result of Theorem 1.4.

**Theorem 1.6.** *Let* $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ *be a polynomial of degree* $n$ *with* $|a| \le 1$, *and* $|z_k| \ge 1$ *for* $1 \le k \le n-m$, *then* $p''(z)$ *has* $(m-2)$ *fold zero at* $z=a$ *and remaining* $(n-m)$ *zeros lie in the region*

$$\left| z - \left(1 - \frac{m(m-1)}{n(n-1)}\right) a \right| \ge \frac{m(m-1)}{n(n-1)}.$$

*Proof.* We can write

$$p(z) = (z-a)^m Q(z),$$

where $Q(z) = \prod_{k=1}^{n-m}(z-z_k)$, then by Theorem 1.3, the polynomial

$$p'(z) = (z-a)^{m-1} R(z),$$

where $R(z) = (z-a)Q'(z) + mQ(z)$ has $(m-1)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(\frac{n-m}{n}\right) a \right| \ge \frac{m}{n}.$$

Now, consider the polynomial

$$S(z) = p'\left(\frac{m}{n}z + \frac{n-m}{n}a\right) \tag{1.1}$$

or

$$S(z) = \left(\frac{m}{n}\right)^{m-1} (z-a)^{m-1} R\left(\frac{m}{n}z + \frac{n-m}{n}a\right),$$

then $S(z)$ is a polynomial of degree $n-1$ with $(m-1)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in $|z| \ge 1$.

Now, applying Theorem 1.3 to the polynomial $S(z)$, the derivative $S'(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(\frac{(n-1)-(m-1)}{n-1}\right) a \right| \ge \frac{m-1}{n-1},$$

which is equivalent to

$$\left| z - \left(\frac{n-m}{n-1}\right) a \right| \ge \frac{m-1}{n-1}.$$

Replacing $z$ by $\frac{n}{m}z + \left(\frac{m-n}{m}\right)a$, in equation (1.1) and differentiating, we obtain

$$p''(z) = (z-a)^{m-2} T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$.

Applying above, we see $p''(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in the region

$$\left| z - \left(1 - \frac{m(m-1)}{n(n-1)}\right) a \right| \ge \frac{m(m-1)}{n(n-1)}.$$

This completes the proof. ☐

*Remark* 1.1. For $a=0$, it reduces to Theorem 1.4.

Our next result generalizes Theorem 1.5 to the polynomial of the form $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$.

**Theorem 1.7.** *If $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$, has all its zeros in the region*

$$\left|z - \left(1 - \frac{m!}{n(n-1)\cdots(n-m+1)}\right)a\right| \geq \frac{m!}{n(n-1)\cdots(n-m+1)}.$$

*Proof.* We can write

$$p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$$

or

$$p(z) = (z-a)^m Q(z),$$

where $Q(z) = \prod_{k=1}^{n-m}(z-z_k)$, $|z_k| \geq 1$, $1 \leq k \leq n-m$.

From the proof of Theorem 1.6, we can write

$$p''(z) = (z-a)^{m-2}T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$. Also, $p''(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \frac{n(n-1)-m(m-1)}{n(n-1)}a\right| \geq \frac{m(m-1)}{n(n-1)}.$$

Now, consider the polynomial

$$U(z) = p''\left(\frac{m(m-1)}{n(n-1)}z + \frac{n(n-1)-m(m-1)}{n(n-1)}a\right) \tag{1.2}$$

or

$$U(z) = \left(\frac{m(m-1)}{n(n-1)}\right)^{m-2}(z-a)^{m-2}T\left(\frac{m(m-1)}{n(n-1)}z + \frac{n(n-1)-m(m-1)}{n(n-1)}a\right).$$

Then $U(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in $|z| \geq 1$.

Again, applying Theorem 1.3 to $U(z)$, which is a polynomial of degree $n-2$, the derivative $U'(z)$ has $(m-3)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \left(\frac{n-2-(m-2)}{n-2}\right)a\right| \geq \frac{m-2}{n-2},$$

which is equivalent to

$$\left|z - \left(\frac{n-m}{n-2}\right)a\right| \geq \frac{m-2}{n-2}.$$

Replacing $z$ by $\frac{n(n-1)}{m(m-1)}z + \frac{m(m-1)-n(n-1)}{m(m-1)}a$, in (1.2) and differentiating, we obtain

$$p'''(z) = (z-a)^{m-3}V(z),$$

where $V(z) = (z-a)T'(z) + (m-2)T(z)$ has $(m-3)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie

$$\left|z - \left(1 - \frac{m(m-1)(m-2)}{n(n-1)(n-2)}\right) a\right| \geq \frac{m(m-1)(m-2)}{n(n-1)(n-2)}.$$

Proceeding similarly, for any positive integer $m = 1, 2, \ldots, n-1$, we see that the polynomial $p^{(m)}(z)$ has all its zeros in the region

$$\left|z - \left(1 - \frac{m!}{n(n-1)\cdots(n-m+1)}\right) a\right| \geq \frac{m!}{n(n-1)\cdots(n-m+1)}.$$

This completes the proof. □

*Remark* 1.2. For $a = 0$, it reduces to Theorem 1.5.

*Remark* 1.3. For $m = 1$, it reduces to Theorem 1.2.

*Remark* 1.4. For $a = 0$ and $m = 1$, it reduces to the result of Aziz and Zarger.

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