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ON THE LIMITS OF PROXIMATE SEQUENCES

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Abstract. We investigate the continuity of the pointwise limits of proximate sequences. Both general proximate sequences and a subclass are considered. We obtain some results related to the fixed points of the limit functions and fixed point like properties of the proximate sequences.

1. Introduction

Theory of shape is shown to be a good alternative to homotopy theory for locally complicated spaces.

The first definitions of shape theory are mainly using external spaces, first in Borsuk's approach the spaces are embedded in the Hilbert cube and after in the categorical approach pioneered by Mardesic and Segal we see the use of inverse systems of polyhedra-again external spaces.

Recently, in the last decades some intrinsic descriptions of shape emerged. In the latter approach, functions between original spaces are investigated. From [2], in compact metric spaces there exists a cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of finite coverings.

The morphisms in the intrinsic approach, as described in [3], are characterized by sequences of functions (f_n) that map objects in the category to one another. In the case of compact metric spaces, these functions are continuous over the members of a cofinal sequence of coverings for the space. As the index of the function in the sequence increases, its level of continuity improves, moving closer and closer to a state of being completely continuous. This idea is intuitive and straightforward to understand.

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Usually while investigating natural processes we can observe them only approximately. If the resolution of the observation becomes higher we get higher approximation - by the same analogy in the proximate approach increasing the index of covering means increasing the precision. The advantage here is that we can compute those processes because there are only finite many members of each covering.

Considering the intrinsic approach allows us to look at the limit of these functions. In this paper we investigate a special class of proximate sequences and we obtain some relations between proximate sequences and their limit functions.

2. Definition of Proximate Sequences

Along this paper by a covering we mean an open covering of the space.

For arbitrary space W by $Cov(W)$ we denote the set of all open coverings of W.

In this chapter we will introduce the intrinsic definition for shape. For more detailed explanations about intrinsic approach of shape we suggest [3].

If $\mathcal{U}, \mathcal{V}$ are two coverings of the space X, then $\mathcal{V}$ is refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{V} \prec \mathcal{U}$.

If $U \in \mathcal{U}$, then the star of U is the set $St(U, \mathcal{U}) = \cup\{W \in \mathcal{U} \mid W \cap U \neq \emptyset\}$ and by $St\mathcal{U}$ will be denoted the collection of all $St(U, \mathcal{U}), U \in \mathcal{U}$.

Let $f : X \to Y$ be a function and let $\mathcal{V}$ be a covering of Y. We say that $g : X \to Y$ is $\mathcal{V}$ - near to f if for every $x \in X$, $f(x)$ and $g(x)$ lie in the same member of $\mathcal{V}$. It is denoted by $f =_{\mathcal{V}} g$.

Let $f : X \to X$ be a function and let $\mathcal{U}$ be a covering of X. We say that the point $x \in X$ is $\mathcal{U}$ - invariant for f if there exists $U \in \mathcal{U}$ such that $x, f(x) \in U$.

Definition 2.1. Let $\mathcal{V}$ be a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$ - continuous at the point $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$. A function $f : X \to Y$ is $\mathcal{V}$-continuous on X if it is $\mathcal{V}$ - continuous at every point $x \in X$.

(The family of all such U_x forms a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}$ - continuous, if there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.)

Definition 2.2. For arbitrary covering $\mathcal{V}$ of the space Y, we say that two functions $f, g : X \to Y$ are $\mathcal{V}$ - homotopic, if there exists a function $F : X \times I \to Y$ such that:

1) $F : X \times I \to Y$ is $st\mathcal{V}$ - continuous;
2) $F : X \times I \to Y$ is $\mathcal{V}$ - continuous at all points of $X \times \partial I$;
3) $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

We denote this by $f \overset{\mathcal{V}}{\sim} g$.

Proposition 2.1. *The relation "* $\overset{\mathcal{V}}{\sim}$ *" is an equivalence relation.*

Proof. See [2, 4]. □

Further on we will work only with compact metric spaces. In this case it is enough to work with finite coverings.

Definition 2.3 ([3])**.** The sequence (f_n) of functions $f_n : X \to Y$ is a proximate sequence from X to Y, if there exists a cofinal sequence of finite coverings of Y, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ and for all indices $m \geq n$, f_n and f_m are $\mathcal{V}_n$ - homotopic. In this case we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$.

Definition 2.4 ([3])**.** If (f_n) and (f'_n) are proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Two proximate sequences (f_n) and $(f'_n) : X \to Y$ are homotopic if for some cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$, (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$, and for all integers f_n and f'_n are $\mathcal{V}_n$ - homotopic.

Now we will define the composition of proximate sequences.

Definition 2.5. If $(f_n) : X \to Y$ is a proximate sequence over $(\mathcal{V}_n)$ and $(g_k) : Y \to Z$ is a proximate sequence over $(\mathcal{W}_k)$, for a covering $\mathcal{W}_k$ of Z, there exists a covering $\mathcal{V}_{n_k}$ of Y such that $g_k(\mathcal{V}_{n_k}) \prec \mathcal{W}_k$. Then, the composition is the proximate sequence $(h_k) = (g_k \circ f_{n_k}) : X \to Z$.

Compact metric spaces and homotopy classes of proximate sequences form the shape category, i.e., isomorphic spaces in this category have the same shape. See [4].

Now we will define regular coverings from [4].

Definition 2.6. Let X be a set and $V = \{V_i \mid i = 1, 2, \ldots, n\}$ be a finite set of subsets of X. If $V \in \mathcal{V}$, we define depth of V in $\mathcal{V}$, to be the biggest number $k \in \mathbb{N}$ such that there exists sequence of elements of $\mathcal{V}$ such that $V \subset V_2 \subset V_3 \subset \cdots \subset V_k$. (If V is not a proper subset of any element in $\mathcal{V}$, then depth of V is 1). It will be denoted by depth(V).

Definition 2.7. A covering $\mathcal{V}$ of the topological space Y is regular if it satisfies the following conditions.

1) If $V \in \mathcal{V}$, then $V \cap Y \neq \emptyset$.

2) If $U, V \in \mathcal{V}$ and $U \cap V \neq \emptyset$, then $U \cap V \in \mathcal{V}$.

Since for every finite covering of the space there are finitely many nonempty intersections of the elements we have the following property. If Y is compact metric space, then there exists a cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of regular coverings.

Definition 2.8. Let $(f_n) : X \to Y$ be a proximate sequence over the coverings $(\mathcal{V}_n)$. We will call (f_n) super proximate sequence if for $m \geq n$, it follows $f_m =_{\mathcal{V}_n} f_n$.

It is clear that if a sequence of functions $(f_n) : X \to Y$ fulfills the property for $m \geq n$, $f_m =_{\mathcal{V}_n} f_n$ than it will be proximate sequence. This follows from the fact that every $\mathcal{V}$ - near functions are $\mathcal{V}$ - homotopic [4].

3. Limits of Proximate Sequences

Since, in the intrinsic shape - proximate sequences from X to Y consist of functions with codomain Y, we can investigate the limit

$$\lim_{n\to+\infty} f_n(x).$$

For the general situation the limit function is not even Darboux even if the component functions are continuous surjections.

Example 3.1. The pointwise limit of proximate sequences is not continuous in general. Take $X = Y = I$ with Euclidean topology and let $f_n(x) = x^n$. It is clear that the sequence (f_n) is proximate sequence, but the limit function is:

$$f(x) = \begin{cases} 0, & \text{if } x < 1, \\ 1, & \text{if } x = 1, \end{cases}$$

which, clearly is not Darboux hence not almost continuous. (Every almost continuous function on the closed unit interval is Darboux, see [5]).

Remark. Let $(\mathcal{V}_n)$ be a cofinal sequence of (regular) coverings for the space Y. Lets define e sequence $(\mathcal{W}_n)$ of coverings of the space Y by $\mathcal{W}_n = st(\mathcal{V}_n)$. From $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$, it follows that $st(\mathcal{V}_1) \succ st(\mathcal{V}_2) \succ \cdots$. At the other side, if $\mathcal{W}$ is an open covering of Y, from the fact that in compact (paracompact) space for every cover has an open star refinement there exists an open covering $\mathcal{K}$ of Y such that $st(\mathcal{K}) \prec \mathcal{W}$. From cofinality of $(\mathcal{V}_n)$ there exists a member $\mathcal{V}_{n_0}$ such that $\mathcal{V}_{n_0} \prec \mathcal{K}$. Hence, we have $st(\mathcal{V}_{n_0}) \prec st(\mathcal{K}) \prec \mathcal{W}$, i.e., $(st(\mathcal{V}_n))$ is also cofinal sequence.

Theorem 3.1. *The pointwise limit of every super proximate sequence is a continuous function.*

Proof. Let $(f_n) : X \to Y$ be a super proximate sequence and let $\lim_{n\to+\infty} f_n(x) = f(x)$. Let $\mathcal{V}$ be arbitrary finite covering of Y. From the fact that Y is compact and from the Remark we can choose a covering $\mathcal{V}_{n'}$ such that $st(\mathcal{V}_{n'}) \prec \mathcal{V}$. Now, let's take arbitrary $x \in X$ and let V be an element from $\mathcal{V}_{n'}$ such that $f(x) \in V$. Now, there exists $n'' > n'$ such that $f_{n''}(x) \in V$. On the other hand, from $f_{n'} =_{\mathcal{V}_{n'}} f_{n''}$, there exists $V' \in \mathcal{V}$ with property $f_{n'}(x), f_{n''}(x) \in V'$. Finally we have that $f(x), f_{n'}(x) \in st(V, \mathcal{V}_{n'})$ and we can write $f =_{st(\mathcal{V}_{n'})} f_{n'}$. Using the fact $st(\mathcal{V}_{n'}) \prec \mathcal{V}$ we can say the function f is $\mathcal{V}$-near to the $\mathcal{V}$-continuous function $f_{n'}$. From [6, Lemma 4.3.] it follows that f is continuous. □

The following example ensures us that in general super proximate sequences need not to have continuous component functions.

Example 3.2. Take the space I and the proximate sequence $(f_n(x)) : I \to I$ defined by:

$$f_n(x) = \begin{cases} x, & \text{if } x > 0, \\ 1/n, & \text{if } x = 0. \end{cases}$$

We define a sequence $(\mathcal{V}_n)$ of coverings of Y in the following way.

Let $\mathcal{V}_1 = \{[0, 1/2)\} \cup \{B_i^1 \mid i \in 1, 2, \ldots, m_1\}$ where B_i^1 are balls with radius smaller than $1/2$ such that $0 \notin B_i^1$.

Now, by Lebesgue lemma choose $\mathcal{V}_2 = \{[0, 1/3)\} \cup \{B_i^2 \mid i \in 1, 2, \ldots, m_2\}$ such that $\mathcal{V}_2 \prec \mathcal{V}_1$, B_i^2 have radius smaller than $1/3$ and the only element of $\mathcal{V}_2$ that contains zero is $[0, 1/3)$.

Inductively, define $\mathcal{V}_n = \{[0, 1/(n+1))\} \cup \{B_i^n | i \in 1, 2, \ldots, m_n\}$ where $\mathcal{V}_n \prec \mathcal{V}_{n-1} \prec \cdots \prec \mathcal{V}_2 \prec \mathcal{V}_1$ and B_i^n are balls with radius smaller than $1/(n+1)$ such that $0 \notin B_i^n$.

We can see that $(f_n(x))$ is a super proximate sequence over $(\mathcal{V}_n)$ with noncontinuous components.

We will show now that for every continuous function can be expressed as limit of a nontrivial super proximate sequence.

Theorem 3.2. *Let $f : X \to Y$ be a continuous function where X, Y are compact, Haussdorf spaces. There exists a cofinal sequence $(\mathcal{W}_n)$ of Y and a super proximate sequence over $(\mathcal{W}_n)$ such that $f_n \to f$, $n \to +\infty$.*

Proof. Let's define the super proximate sequence $(f_n) : X \to Y$ in the following way.

Lets fix $n \in \mathbb{N}$ and take $x \in X$. Choose W_x a member of $\mathcal{V}_n$ with the maximal depth that is contained in $st(f(x), \mathcal{V}_n)$. We define $f_n(x)$ to be one fixed selected element from W_x.

1) f_n is $st(\mathcal{V}_n)$ - continuous.

Let $x \in X$, take $V \in \mathcal{V}_n$ be the element of $\mathcal{V}_n$ such that $f(x) \in V$, from the continuity of f we can choose an open set $U \subset X$ such that $f(U) \subset V$. We have $f_n(U) \subset st(V, \mathcal{V}_n)$, so f_n is $st(\mathcal{V}_n)$ - continuous.

2) If $m > n$, then $f_m =_{st(\mathcal{V}_n)} f_n$. This follows from the fact that $st(f(x), \mathcal{V}_m) \subset st(f(x), \mathcal{V}_n)$.

3) $\lim_{n\to+\infty} f_n(x) = f(x)$. For this part let O be open neighborhood of $f(x)$ in Y. Take the covering $\mathcal{O} = \{O, Y \setminus f(x)\}$ of Y. There exists n_0 such that $st^2(\mathcal{V}_{n_0}) \prec \mathcal{O}$ and there exists $V \in \mathcal{V}_{n_0}$ such that $f(x) \in V$. Now, let $m > n_0$. From $f_m =_{st(\mathcal{V}_{n_0})} f_{n_0}$ and from the fact that $st(V, \mathcal{V}_{n_0})$ is an element of $st(\mathcal{V}_{n_0})$ that contains $f(x)$ we have $f_m(x) \in st^2(V, \mathcal{V}_{n_0}) \subset O$. □

In the proof we can ommit the requirement the covering to be regular, but in this way if a neighborhood U_x of some point x has the property $st(u, \mathcal{V}_n) = st(x, \mathcal{V}_n)$ for all $u \in U_x$ then function f_n will be constant at that neighborhood.

4. Fixed Point Property of Limits

In this section we will establish a connection between fixed points of the limit function and some properties of the corresponding super proximate sequence.

Theorem 4.1. *Let $(f_n) : X \to X$ be a super proximate sequence over the cofinal sequence $(\mathcal{V}_n)$, where X is compact metric space and let $\lim_{n\to+\infty} f_n(x) = f(x)$. The following statements are equivalent.*

1) *f has fixed point.*

2) *For every $n \in \mathbb{N}$ there exists $V_n \in \mathcal{V}_n$ and $x_n \in X$ such that $f_n(x_n)$, x_n lie in the same set of $st(\mathcal{V}_n)$.*

Proof. 1) $\Rightarrow$ 2) Let $f : X \to X$ be the limit of super proximate sequence $(f_n) : X \to X$ and $f(x') = x'$ for a point $x' \in X$. Lets assume the opposite, that there exists $n_0 \in N$ such that $x', f_{n_0}(x') \notin st(V, \mathcal{V}_{n_0})$ for all $V \in \mathcal{V}_{n_0}$. There exists $n_1 > n_0$ such that $f(x') = x', f_{n_1}(x') \in V_{n_0}$ for some $V_{n_0} \in \mathcal{V}_{n_0}$. But, $f_{n_1} =_{\mathcal{V}_{n_0}} f_{n_0}$, so there exists element V'_{n_0} in $\mathcal{V}_{n_0}$ such that $f_{n_1}(x'), f_{n_0}(x') \in V'_{n_0}$ so $f_{n_0}(x'), x' \in st(V_{n_0}, \mathcal{V}_{n_0})$, which is contradiction.

2) $\Rightarrow$ 1) From compactness of X the sequence (x_n) has a convergent sub-sequence (x_{n_k}) in X. Let's assume that $\lim_{k\to+\infty} x_{n_k} = x'$. We claim that $f(x') = x'$. If we suppose the contrary, i.e., $f(x') \neq x'$, then from the fact that X is Hausdorff there exist open sets U' and U'' such that $f(x') \in U'$, $x' \in U''$, $f(U'') \subset U'$ and $U' \cap U'' = \emptyset$. Take the covering $\mathcal{O} = \{X \setminus \{f(x')\}, U'\}$ of X. There exists $n_0 \in \mathbb{N}$ such that:

$$st(\mathcal{V}_{n_0}) \prec \mathcal{O}, x_{n_0} \in U'' \quad \text{and} \quad f(x_{n_0}) \in U'.$$

From the fact that f_{n_0} is $\mathcal{V}_{n_0}$ - continuous it follows that there exist a neighborhood $U_{x'}$ of x' and an element V_{n_0} from $\mathcal{V}_{n_0}$ that contains $f_{n_0}(x')$ such that $f_{n_0}(U_{x'}) \subset V_{n_0}$. Now, choose $n_1 > n_0$ to be large enough such that $f_{n_1}(x'), f(x')$ lie in same element of $\mathcal{V}_{n_0}$ and $x_{n_1} \in U_{x'}$ it follows that $f_{n_0}(x_{n_1}) \in V_{n_0}$. From $f_{n_1} =_{\mathcal{V}_{n_0}} f_{n_0}$ we have that $f_{n_1}(x_{n_1}), f_{n_0}(x_{n_1})$ lie in the same element of $\mathcal{V}_{n_0}$, i.e., $f_{n_1}(x_{n_1}) \in st(V_{n_0}, \mathcal{V}_{n_0})$.

Considering the fact that $f(x') \in st(V_{n_0}, \mathcal{V}_{n_0})$, we have $f_{n_1}(x_{n_1}), x_{n_1}$ must lie in different elements of $st(\mathcal{V}_{n_0})$, which is contradiction. ☐

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