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DUAL HYBRID NUMBERS WITH HORADAM NUMBER COEFFICIENTS

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Abstract. In this paper, we present a new class of dual hybrid numbers that incorporate Horadam numbers into their components. We explore some fundamental properties associated with these numbers. In particular, we obtain recurrence relations, generating function, Binet formula of these sequences and, by using Binet formula, we derive Vajda, Cassini, Catalan and d'Ocagne identities. By studying this new class of hybrid numbers, we express many dual hybrid numbers with coefficient special integer sequences such as Fibonacci, Lucas, Pell, etc. in a unified way.

1. Introduction

Dual, complex, and hyperbolic numbers are among the most prominent classes of two-dimensional number systems as extensions of the real numbers. These number systems have many applications on mechanics, robotics, computer graphics, geometry, physics and rigid body motion [38, 57, 63].

In 2018, Ozdemir [28] defined the hybrid numbers as a generalization of complex, dual, and hyperbolic numbers. The set of hybrid numbers denoted by $\mathbb{K}$ is defined as

$$\mathbb{K} := \left\{ p = a + b\mathrm{i} + c\epsilon + d\mathrm{h} : \mathrm{i}^2 = -1, \epsilon^2 = 0, \mathrm{h}^2 = 1 \text{ and } a, b, c, d \in \mathbb{R} \right\}.$$

It is well-known that the hybrid number multiplication is non-commutative.

There have been extensive investigations into various forms of hybrid numbers, which incorporate components from elements sourced from specific integer sequences.

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In particular, in 2018 Szynal-Liana defined Horadam hybrid numbers by taking components of hybrid numbers as Horadam numbers. The Horadam sequence, named after A. F. Horadam [19, 20], is a generalization of many significant and well-known number sequences, including Fibonacci, Lucas, and Jacobsthal sequences etc. In the Appendix, we present a succinct yet comprehensive overview of the relevant literature concerning specific classes of hybrid number sequences, highlighting key developments and contributions in the field (Table 5).

In this paper, we introduce a new class of hybrid number sequence called dual Horadam hybrid numbers. First, we give some basic concepts and notions.

2. Basic Concepts and Notions

The sets of dual, complex and hyperbolic numbers are respectively defined as

$$\begin{aligned}
\mathbb{D} &= \left\{\boldsymbol{\lambda} = \lambda + \lambda^{*}\varepsilon : \lambda, \lambda^{*} \in \mathbb{R}\right\},\\
\mathbb{C} &= \left\{\widehat{\boldsymbol{z}} = z_1 + z_2 i : z_1, z_2 \in \mathbb{R}\right\},\\
\mathbb{P} &= \left\{\mathfrak{a} = a_1 + a_2 h : a_1, a_2 \in \mathbb{R}\right\},
\end{aligned}$$

where ε is the dual unit with $\varepsilon^2 = 0$, $\varepsilon \neq 0$, i is the complex unit with $i^2 = -1$ and h is the hyperbolic unit with $h^2 = 1$, $h \neq 1$. The addition, subtraction, multiplication with scalar and multiplication operations of the dual numbers $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ are respectively defined as

$$\begin{aligned}
\boldsymbol{\lambda} \pm \boldsymbol{\gamma} &= (\lambda + \lambda^{*}\varepsilon) \pm (\gamma + \gamma^{*}\varepsilon) = (\lambda \pm \gamma) + (\lambda^{*} + \gamma^{*})\,\varepsilon,\\
t\boldsymbol{\lambda} &= t\,(\lambda + \lambda^{*}\varepsilon) = t\lambda + t\lambda^{*}\varepsilon, \quad t \in \mathbb{R},\\
\boldsymbol{\lambda}\boldsymbol{\gamma} &= (\lambda + \lambda^{*}\varepsilon)\,(\gamma + \gamma^{*}\varepsilon) = \lambda\gamma + (\lambda\gamma^{*} + \lambda^{*}\gamma)\,\varepsilon.
\end{aligned}$$

For the integers p, q, the Horadam numbers $\mathcal{H}_n = \mathcal{H}_n\left(\mathcal{H}_0, \mathcal{H}_1; p, q\right)$ are defined by Horadam in [19] by the recursive relation

$$\mathcal{H}_n = p\mathcal{H}_{n-1} + q\mathcal{H}_{n-2}, \quad n \geq 2, \tag{2.1}$$

with initial values $\mathcal{H}_0 = a$, $\mathcal{H}_1 = b$. It is well known that for special cases, the Fibonacci and the Lucas numbers are $\mathcal{H}_n\,(0, 1; 1, 1)$ and $\mathcal{H}_n\,(2, 1; 1, 1)$, respectively.

Let α and β be the roots of the characteristic equation $t^2 - pt - q = 0$ associated to (2.1). By solving this equation we get two distinct characteristic roots for $p^2 + 4q > 0$ as

$$\alpha = \frac{p + \Delta}{2} \quad \text{and} \quad \beta = \frac{p - \Delta}{2}, \tag{2.2}$$

where $\Delta = \sqrt{p^2 + 4q}$. Moreover, $\mathcal{H}_n = \mathrm{A}\alpha^n + \mathrm{B}\beta^n$ is the Binet formula of the Horadam numbers, where

$$\mathrm{A} = \frac{\mathcal{H}_1 - \mathcal{H}_0\beta}{\Delta} \quad \text{and} \quad \mathrm{B} = \frac{\mathcal{H}_0\alpha - \mathcal{H}_1}{\Delta}, \tag{2.3}$$

for details see [19].

Let denote n^{th} dual Horadam number as $\mathcal{H}_n = \mathcal{H}_n(\mathcal{H}_0, \mathcal{H}_1; p, q)$. For $n \geq 0$, dual Horadam numbers satisfy the following recursive relation

$$\mathcal{H}_n = p\mathcal{H}_{n-1} + q\mathcal{H}_{n-2}, \quad n \geq 2, \tag{2.4}$$

with initial values $\mathcal{H}_0 = a + b\varepsilon$ and $\mathcal{H}_1 = b + (pb + qa)\varepsilon$. By using the relation (2.4), we have

$$\begin{aligned}
&\mathcal{H}_2 = pb + qa + \Big((p^2+q)b + pqa\Big)\varepsilon, \\
&\mathcal{H}_3 = (p^2+q)b + pqa + \Big((p^2+q)(qa+pb) + pqb\Big)\varepsilon, \\
&\mathcal{H}_4 = (p^2+q)(qa+pb) + pqb + \Big(p(p^2+q)(qa+pb) + pq\left((2p+1)b + qa\right)\Big)\varepsilon.
\end{aligned} \tag{2.5}$$

It is well known that the dual Fibonacci $\mathcal{H}_n(\varepsilon, 1+\varepsilon; 1, 1)$ and dual Lucas numbers $\mathcal{H}_n(2+\varepsilon, 1+3\varepsilon; 1, 1)$ are studied in [16].

The Horadam hybrid numbers are defined by Szynal-Liana [43] as

$$\mathcal{Q}_n = \mathcal{H}_n + \mathcal{H}_{n+1}\mathrm{i} + \mathcal{H}_{n+2}\epsilon + \mathcal{H}_{n+3}\mathrm{h}. \tag{2.6}$$

The author gave the Binet formula, generating function and its characters as in the hybrid numbers. Some important results on Horadam hybrid numbers were also given in [48].

Let n be a non-negative integer. The equation

$$\mathcal{Q}_n = \mathrm{A}\alpha^n\widetilde{\alpha} + \mathrm{B}\beta^n\widetilde{\beta} \tag{2.7}$$

holds, where

$$\widetilde{\alpha} = 1 + \alpha\mathrm{i} + \alpha^2\epsilon + \alpha^3\mathrm{h}, \quad \widetilde{\beta} = 1 + \beta\mathrm{i} + \beta^2\epsilon + \beta^3\mathrm{h}, \tag{2.8}$$

and A, B are defined by (2.3).

The generating function for the Horadam hybrid number sequence $\{\mathcal{Q}_n\}$ is given by

$$\mathrm{GF}_{\mathcal{Q}_n}(t) = \frac{\mathcal{Q}_0 + (\mathcal{Q}_1 - p\mathcal{Q}_0)\,t}{1 - pt - qt^2}. \tag{2.9}$$

The recurrence relation of the Horadam hybrid numbers is

$$\mathcal{Q}_n = p\mathcal{Q}_{n-1} + q\mathcal{Q}_{n-2}, \tag{2.10}$$

with initial conditions

$$\begin{aligned}
&\mathcal{Q}_0 = a + b\mathrm{i} + (pb - qa)\,\epsilon + \Big((p^2+q)b + pqa\Big)\mathrm{h}, \\
&\mathcal{Q}_1 = b + (pb - qa)\,\mathrm{i} + \Big((p^2+q)b + pqa\Big)\epsilon + \Big((p^2+q)(qa+pb) + pqb\Big)\mathrm{h}.
\end{aligned}$$

The set of dual hybrid numbers are defined by Seçgin et al. in [37] as

$$\mathbb{DK} := \Big\{\boldsymbol{a} + \boldsymbol{b}\mathrm{i} + \boldsymbol{c}\epsilon + \boldsymbol{d}\mathrm{h} : \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{D}, \mathrm{i}^2 = -1, \epsilon^2 = 0, \mathrm{h}^2 = 1\Big\},$$

where ε is the dual unit, i, ϵ and h are the hybrid units. It is known that the algebra of dual hybrid numbers is commutative. Moreover, a basis of this algebra is $\{1, \mathrm{i}, \epsilon, \mathrm{h}, \varepsilon, \varepsilon\mathrm{i}, \varepsilon\epsilon, \varepsilon\mathrm{h}\}$. The dual hybridian product is as in the hybridian product. It means that for $Q = \boldsymbol{a} + \boldsymbol{b}\mathrm{i} + \boldsymbol{c}\epsilon + \boldsymbol{d}\mathrm{h}$ and $P = \boldsymbol{x} + \boldsymbol{y}\mathrm{i} + \boldsymbol{z}\epsilon + \boldsymbol{t}\mathrm{h}$, the dual hybridian product is

$$QP = \boldsymbol{ax} + (\boldsymbol{c} - \boldsymbol{b})\,\boldsymbol{y} + \boldsymbol{bz} + \boldsymbol{dt} + \Big(\boldsymbol{bx} + (\boldsymbol{a} - \boldsymbol{d})\,\boldsymbol{y} + \boldsymbol{bt}\Big)\mathrm{i}$$
$$+ \Big(\boldsymbol{cx} - \boldsymbol{dy} + (\boldsymbol{a} + \boldsymbol{d})\,\boldsymbol{z} + (\boldsymbol{b} - \boldsymbol{c})\,\boldsymbol{t}\Big)\epsilon + \Big(\boldsymbol{dx} + \boldsymbol{cy} - \boldsymbol{bz} + \boldsymbol{at}\Big)\mathrm{h}.$$

The products of two dual hybrid units are given by Table 1 below.

TABLE 1. The product of dual hybrid units

$\cdot$	1	i	ϵ	h	ε	$\varepsilon\mathrm{i}$	$\varepsilon\epsilon$	$\varepsilon\mathrm{h}$
1	1	i	ϵ	h	ε	$\mathrm{i}\varepsilon$	$\epsilon\varepsilon$	$\mathrm{h}\varepsilon$
i	i	-1	$1-\mathrm{h}$	$\epsilon+\mathrm{i}$	$\mathrm{i}\varepsilon$	$-\varepsilon$	$(1-\mathrm{h})\,\varepsilon$	$(\epsilon+\mathrm{i})\,\varepsilon$
ϵ	ϵ	$\mathrm{h}+1$	0	$-\epsilon$	$\epsilon\varepsilon$	$(\mathrm{h}+1)\,\varepsilon$	0	$-\epsilon\varepsilon$
h	h	$-\epsilon-\mathrm{i}$	ϵ	1	$\mathrm{h}\varepsilon$	$(-\epsilon-\mathrm{i})\,\varepsilon$	$\epsilon\varepsilon$	ε
ε	ε	$\mathrm{i}\varepsilon$	$\epsilon\varepsilon$	$\mathrm{h}\varepsilon$	0	0	0	0
$\varepsilon\mathrm{i}$	$\mathrm{i}\varepsilon$	$-\varepsilon$	$(1-\mathrm{h})\,\varepsilon$	$(\epsilon+\mathrm{i})\,\varepsilon$	0	0	0	0
$\varepsilon\epsilon$	$\epsilon\varepsilon$	$(\mathrm{h}+1)\,\varepsilon$	0	$-\epsilon\varepsilon$	0	0	0	0
$\varepsilon\mathrm{h}$	$\mathrm{h}\varepsilon$	$(-\epsilon-\mathrm{i})\,\varepsilon$	$\epsilon\varepsilon$	ε	0	0	0	0

The conjugate of a dual hybrid number Q, denoted by $\overline{Q}$, is defined as $\overline{Q} = \mathrm{S}\,(Q) - \mathrm{V}\,(Q)$ or $\overline{Q} = \boldsymbol{a} - \boldsymbol{b}\mathrm{i} - \boldsymbol{c}\epsilon - \boldsymbol{d}\mathrm{h}$. The conjugate of the sum of dual hybrid numbers is equal to the sum of their conjugates. There is also $Q\overline{Q} = \overline{Q}Q$ depending upon the dual hybrid product.

The dual number

$$\mathcal{C}\,(Q) = Q\overline{Q} = \overline{Q}Q = \boldsymbol{a}^2 + (\boldsymbol{b} - \boldsymbol{c}) - \boldsymbol{c}^2 - \boldsymbol{d}^2 \tag{2.11}$$

is called the *character* of the dual hybrid number Q. The vector $\mathcal{V}_Q = (\boldsymbol{a}, \boldsymbol{b} - \boldsymbol{c}, \boldsymbol{c}, \boldsymbol{d})$ is called the vector representation of the dual hybrid number Q. The norm of the hybrid number is the dual number $\sqrt{|Q\overline{Q}|}$ which is denoted by $\boldsymbol{\rho} = \|Q\|$.

For more details related to dual hybrid numbers, we refer to [37].

3. Dual Horadam Hybrid Numbers

In this section, we define dual Horadam hybrid numbers. Through an examination of this novel category of hybrid numbers, we were able to represent numerous dual hybrid numbers utilizing coefficient special integer sequences like Fibonacci, Lucas, Pell, and others.

Definition 3.1. The set of dual Horadam hybrid numbers is defined as

$$\mathbb{DK}_{\mathcal{H}} = \left\{ \boldsymbol{\mathfrak{Q}}_n = \boldsymbol{\mathcal{H}}_n + \boldsymbol{\mathcal{H}}_{n+1}\mathrm{i} + \boldsymbol{\mathcal{H}}_{n+2}\epsilon + \boldsymbol{\mathcal{H}}_{n+3}\mathrm{h} : \mathrm{i}, \epsilon, \mathrm{h} \text{ are hybrid units} \right\}, \tag{3.1}$$

where $\boldsymbol{\mathcal{H}}_n = \mathcal{H}_n + \mathcal{H}_{n+1}\varepsilon$ is the n^{th} dual Horadam number with dual unit ε.

Table 2 presents several noteworthy special cases of dual Horadam hybrid numbers.

TABLE 2. Notable special cases of dual hybrid Horadam numbers obtained by particular choices of $\boldsymbol{\mathcal{H}}_0$, $\boldsymbol{\mathcal{H}}_1$, p, and q

Dual Horadam Hybrid Numbers	$\boldsymbol{\mathcal{Q}}_n = \boldsymbol{\mathcal{H}}_n(\boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{H}}_1; p, q)$
Dual Generalized Fibonacci	$\boldsymbol{\mathcal{H}}_n(\varepsilon, 1+\varepsilon; p, q)$
Dual Generalized Lucas	$\boldsymbol{\mathcal{H}}_n(2+\varepsilon, 1+3\varepsilon; p, q)$
Dual Fibonacci	$\boldsymbol{\mathcal{H}}_n(\varepsilon, 1+\varepsilon; 1, 1)$
Dual Lucas	$\boldsymbol{\mathcal{H}}_n(2+\varepsilon, 1+3\varepsilon; 1, 1)$
Dual Jacobsthal	$\boldsymbol{\mathcal{H}}_n(\varepsilon, 1+\varepsilon; 1, 2)$
Dual Jacobsthal-Lucas	$\boldsymbol{\mathcal{H}}_n(2+\varepsilon, 1+5\varepsilon; 1, 2)$
Dual Pell	$\boldsymbol{\mathcal{H}}_n(\varepsilon, 1+2\varepsilon; 2, 1)$
Dual Pell-Lucas	$\boldsymbol{\mathcal{H}}_n(2+2\varepsilon, 2+6\varepsilon; 2, 1)$
Dual Padovan	$\boldsymbol{\mathcal{H}}_n(1+\varepsilon, 1+\varepsilon; 1, 1)$
Dual Perrin	$\boldsymbol{\mathcal{H}}_n(3, 2\varepsilon; 1, 1)$
Dual Mersenne	$\boldsymbol{\mathcal{H}}_n(\varepsilon, 1+3\varepsilon; 3, 2)$
Dual Gibonacci	$\boldsymbol{\mathcal{H}}_n(a+b\varepsilon, b+(a+b)\varepsilon; 1, 1)$
Dual Gaussian Fibonacci	$\boldsymbol{\mathcal{H}}_n\left(\frac{i}{2}+\varepsilon, 1+(3+i)\varepsilon; 3, 2\right)$
Dual Balancing	$\boldsymbol{\mathcal{H}}_n(\varepsilon, 1+6\varepsilon; 6, 1)$
Dual Lucas-Balancing	$\boldsymbol{\mathcal{H}}_n(1+3\varepsilon, 3+17\varepsilon; 6, 1)$
Dual Oresme	$\boldsymbol{\mathcal{H}}_n\left(\frac{1}{2}\varepsilon, \frac{1}{2}-\frac{1}{6}\varepsilon; 1, \frac{1}{4}\right)$

The n^{th} dual Horadam hybrid number $\boldsymbol{\mathcal{Q}}_n$ consists of two hybrid elements and can be represented as

$$\boldsymbol{\mathcal{Q}}_n = \mathcal{Q}_n + \mathcal{Q}_{n+1}\varepsilon, \tag{3.2}$$

where $\mathcal{Q}_n = \mathcal{H}_n + \mathcal{H}_{n+1}\mathrm{i} + \mathcal{H}_{n+2}\epsilon + \mathcal{H}_{n+3}\mathrm{h}$ is the n^{th} Horadam hybrid number. The scalar and vector part of the dual Horadam hybrid number $\boldsymbol{\mathcal{Q}}_n$ are given, respectively, by

$$\mathrm{S}(\boldsymbol{\mathcal{Q}}_n) = \boldsymbol{\mathcal{H}}_n \quad \text{and} \quad \mathrm{V}(\boldsymbol{\mathcal{Q}}_n) = \boldsymbol{\mathcal{H}}_{n+1}\mathrm{i} + \boldsymbol{\mathcal{H}}_{n+2}\epsilon + \boldsymbol{\mathcal{H}}_{n+3}\mathrm{h}.$$

Let $\boldsymbol{\mathcal{Q}}_n = \mathcal{Q}_n + \mathcal{Q}_{n+1}\varepsilon$ and $\boldsymbol{\mathcal{P}}_n = \mathcal{P}_n + \mathcal{P}_{n+1}\varepsilon$ be two dual Horadam hybrid numbers and $\boldsymbol{\lambda} = \lambda + \lambda^*\varepsilon$ be a dual number. Then, the addition, subtraction, multiplication with scalar and multiplication operations are defined as in the dual hybrid numbers. So, following equations can be given by

$$\boldsymbol{\mathcal{Q}}_n \pm \boldsymbol{\mathcal{P}}_n = (\mathcal{Q}_n \pm \mathcal{P}_n) + (\mathcal{Q}_{n+1} \pm \mathcal{P}_{n+1})\varepsilon,$$
$$\boldsymbol{\lambda}\boldsymbol{\mathcal{P}}_n = \lambda\mathcal{P}_n + (\lambda\mathcal{P}_{n+1} + \lambda^*\mathcal{P}_n)\varepsilon,$$
$$\boldsymbol{\mathcal{Q}}_n\boldsymbol{\mathcal{P}}_n = \mathcal{Q}_n\mathcal{P}_n + (\mathcal{Q}_n\mathcal{P}_{n+1} + \mathcal{Q}_{n+1}\mathcal{P}_n)\varepsilon.$$

The hybrid, dual and total conjugates of the dual Horadam hybrid number $\boldsymbol{\mathcal{Q}}_n$ are, respectively, defined as follows:

$$\boldsymbol{\mathcal{Q}}_n^{\dagger_1} = \boldsymbol{\mathcal{H}}_n - \boldsymbol{\mathcal{H}}_{n+1}\mathrm{i} - \boldsymbol{\mathcal{H}}_{n+2}\epsilon - \boldsymbol{\mathcal{H}}_{n+3}\mathrm{h} = \overline{\mathcal{Q}_n} + \overline{\mathcal{Q}_{n+1}}\varepsilon = \mathrm{S}(\boldsymbol{\mathcal{Q}}_n) - \mathrm{V}(\boldsymbol{\mathcal{Q}}_n),$$
$$\boldsymbol{\mathcal{Q}}_n^{\dagger_2} = \overline{\boldsymbol{\mathcal{H}}_n} + \overline{\boldsymbol{\mathcal{H}}_{n+1}}\mathrm{i} + \overline{\boldsymbol{\mathcal{H}}_{n+2}}\epsilon + \overline{\boldsymbol{\mathcal{H}}_{n+3}}\mathrm{h} = \mathcal{Q}_n - \mathcal{Q}_{n+1}\varepsilon = \overline{\mathrm{S}(\boldsymbol{\mathcal{Q}}_n)} + \overline{\mathrm{V}(\boldsymbol{\mathcal{Q}}_n)},$$

$$\mathfrak{Q}_n^{\dagger_3} = \overline{\boldsymbol{\mathcal{H}}_n} - \overline{\boldsymbol{\mathcal{H}}_{n+1}}\mathrm{i} - \overline{\boldsymbol{\mathcal{H}}_{n+2}}\epsilon - \overline{\boldsymbol{\mathcal{H}}_{n+3}}\mathrm{h} = \overline{\mathcal{Q}_n} - \overline{\mathcal{Q}_{n+1}}\varepsilon = \overline{\mathrm{S}(\mathfrak{Q}_n)} - \overline{\mathrm{V}(\mathfrak{Q}_n)}.$$

Let $\mathfrak{Q}_n$ be a dual Horadam hybrid number. We state the following two corollaries.

Corollary 3.1. *The following identities hold:* $\mathfrak{Q}_n + \mathfrak{Q}_n^{\dagger_1} = 2\boldsymbol{\mathcal{H}}_n$, $\mathfrak{Q}_n + \mathfrak{Q}_n^{\dagger_2} = 2\mathcal{Q}_n$ *and* $\mathfrak{Q}_n + \mathfrak{Q}_n^{\dagger_3} = 2(\mathcal{H}_n + \mathrm{V}(\mathcal{H}_{n+1})\varepsilon)$.

Corollary 3.2. *The following equations are satisfied:*

$$\begin{aligned}
\mathfrak{Q}_n\mathfrak{Q}_n^{\dagger_1} =& \boldsymbol{\mathcal{H}}_n^2 + \boldsymbol{\mathcal{H}}_{n+1}^2 - 2\boldsymbol{\mathcal{H}}_{n+1}\boldsymbol{\mathcal{H}}_{n+2} - \boldsymbol{\mathcal{H}}_{n+3}^2,\\
\mathfrak{Q}_n\mathfrak{Q}_n^{\dagger_2} =& \boldsymbol{\mathcal{H}}_n^2 - \mathcal{H}_{n+1}^2 + \mathcal{H}_{n+3}^2 + \mathcal{H}_{n+1}\mathcal{H}_{n+2} + 2\mathcal{H}_n\mathrm{V}(\mathcal{Q}_n)\\
&+ 2\Big[\Big(\mathcal{H}_{n+2}\mathcal{H}_{n+3} + \mathcal{H}_{n+1}\mathcal{H}_{n+4}\Big)\mathrm{i}\\
&+\Big((\mathcal{H}_{n+2} - \mathcal{H}_{n+3})\mathcal{H}_{n+3} + (\mathcal{H}_{n+2} - \mathcal{H}_{n+1})\mathcal{H}_{n+4}\Big)\epsilon\\
&+ \Big(\mathcal{H}_{n+1}\mathcal{H}_{n+3} - \mathcal{H}_{n+1}^2\Big)\mathrm{h}\Big]\varepsilon,\\
\mathfrak{Q}_n\mathfrak{Q}_n^{\dagger_3} =& \Big[(\mathcal{H}_n - \mathcal{H}_{n+3})^2 + \mathcal{H}_{n+1}(\mathcal{H}_{n+1} - 2\mathcal{H}_{n+2}) + \mathcal{H}_{n+3}(\mathcal{H}_{n+1} + \mathcal{H}_{n+2}) - 2\mathcal{H}_{n+2}^2\\
&+2\Big(-\mathcal{H}_{n+1}^2 + \mathcal{H}_n\mathcal{H}_{n+2}\mathcal{H}_{n+1}\mathcal{H}_{n+4} - \mathcal{H}_{n+2}\mathcal{H}_{n+3}\Big)\mathrm{i}\\
&+ \Big(2\Big(\mathcal{H}_n\mathcal{H}_{n+3} - \mathcal{H}_{n+1}\mathcal{H}_{n+2} + \mathcal{H}_{n+1}\mathcal{H}_{n+4} - \mathcal{H}_{n+2}\mathcal{H}_{n+4}\Big)\\
&- \mathcal{H}_{n+2}\mathcal{H}_{n+3} + \mathcal{H}_{n+3}^2\Big)\epsilon + 2\Big(\mathcal{H}_{n+2}^2 - 2\mathcal{H}_{n+1}\mathcal{H}_{n+3}\mathcal{H}_n\mathcal{H}_{n+4}\Big)\mathrm{h}\Big]\varepsilon.
\end{aligned}$$

Theorem 3.1. *The character of dual Horadam hybrid numbers is*

$$\begin{aligned}
\mathcal{C}(\mathfrak{Q}_n) =& \mathcal{H}_n^2\Big(1 - p^2q^2\Big) + 2\mathcal{H}_n\mathcal{H}_{n+1}\Big(p + p^3q - pq^2\Big)\\
&+ \mathcal{H}_{n+1}^2\Big(1 - 2p - p^4 + 2p^2q - q^2\Big)\\
&+ \Big[-\mathcal{H}_n^2q + 2\mathcal{H}_n\mathcal{H}_{n+1}\Big(1 - 4p^2q^2 + p^2 + p^3q - q + 2pq + p^4 + q^3\Big)\\
&+\mathcal{H}_{n+1}^2\Big(3p + 5p^3q + pq^2 - 4p^2 - 2p^5\Big)\Big]\varepsilon.
\end{aligned}$$

Proof. Let $\boldsymbol{\mathcal{H}}_{n+2} = p\boldsymbol{\mathcal{H}}_{n+1} - q\boldsymbol{\mathcal{H}}_n$, $\boldsymbol{\mathcal{H}}_{n+3} = (p^2 - q)\boldsymbol{\mathcal{H}}_{n+1} - pq\boldsymbol{\mathcal{H}}_n$, $\boldsymbol{\mathcal{H}}_{n+1}\boldsymbol{\mathcal{H}}_{n+2} = p\boldsymbol{\mathcal{H}}_{n+1}^2 - q\boldsymbol{\mathcal{H}}_n\boldsymbol{\mathcal{H}}_{n+1}$ and $\mathcal{C}(\mathfrak{Q}_n) = \boldsymbol{\mathcal{H}}_n^2 + \boldsymbol{\mathcal{H}}_{n+1}^2 - 2\boldsymbol{\mathcal{H}}_{n+1}\boldsymbol{\mathcal{H}}_{n+2} - \boldsymbol{\mathcal{H}}_{n+3}^2$. Then,

$$\begin{aligned}
\mathcal{C}(\mathfrak{Q}_n) =& \boldsymbol{\mathcal{H}}_n^2 + \boldsymbol{\mathcal{H}}_{n+1}^2 - 2\boldsymbol{\mathcal{H}}_{n+1}(p\boldsymbol{\mathcal{H}}_{n+1} - q\boldsymbol{\mathcal{H}}_n) - \Big(\Big(p^2 - q\Big)\boldsymbol{\mathcal{H}}_{n+1} - pq\boldsymbol{\mathcal{H}}_n\Big)^2\\
=& (\mathcal{H}_n + \mathcal{H}_{n+1}\varepsilon)^2\Big(1 - p^2q^2\Big)\\
&+ 2(\mathcal{H}_n + \mathcal{H}_{n+1}\varepsilon)(\mathcal{H}_{n+1} + \mathcal{H}_{n+2}\varepsilon)\Big(q + p^3q - pq^2\Big)\\
&+ (\mathcal{H}_{n+1} + \mathcal{H}_{n+2}\varepsilon)^2\Big(1 - 2p - p^4 + 2p^2q - q^2\Big)\\
=& \Big(\mathcal{H}_n^2 + 2\mathcal{H}_n\mathcal{H}_{n+1}\varepsilon\Big)\Big(1 - p^2q^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&+2\left(\mathcal{H}_n\mathcal{H}_{n+1}+\mathcal{H}_n\mathcal{H}_{n+2}\varepsilon+\mathcal{H}_{n+1}^2\varepsilon\right)\left(p+p^3q-pq^2\right)\\
&+\left(\mathcal{H}_{n+1}^2+2\mathcal{H}_{n+1}\mathcal{H}_{n+2}\varepsilon\right)\left(p+p^3q-pq^2\right)\\
=&\left(\mathcal{H}_n^2+2\mathcal{H}_n\mathcal{H}_{n+1}\varepsilon\right)\left(1-p^2q^2\right)\\
&+2\left(\mathcal{H}_n\mathcal{H}_{n+1}+\left(p\mathcal{H}_n\mathcal{H}_{n+1}-q\mathcal{H}_n^2+\mathcal{H}_{n+1}^2\right)\varepsilon\right)\left(p+p^3q-pq^2\right)\\
&+\left(\mathcal{H}_{n+1}^2+2\left(p\mathcal{H}_{n+1}^2-q\mathcal{H}_n\mathcal{H}_{n+1}\right)\varepsilon\right)\left(1-2p-p^4+2p^2q-q^2\right)\\
=&\mathcal{H}_n^2\left(1-p^2q^2\right)+2\mathcal{H}_n\mathcal{H}_{n+1}\left(p+p^3q-pq^2\right)\\
&+\mathcal{H}_{n+1}^2\left(1-2p-p^4+2p^2q-q^2\right)\\
&+\Big[-\mathcal{H}_n^2q+2\mathcal{H}_n\mathcal{H}_{n+1}\left(1-4p^2q^2+p^2+p^3q-q+2pq+p^4+q^3\right)\\
&+\mathcal{H}_{n+1}^2\left(3p+5p^3q+pq^2-4p^2-2p^5\right)\Big]\varepsilon.
\end{aligned}
$$

□

Remark 3.1. Let $n \geq 0$ be integer. Then,

$$
\begin{aligned}
\mathcal{C}\left(\mathbf{\mathfrak{Q}}_n\right)=&\mathrm{A}\alpha^{2n}\left(1+\alpha^2-2\alpha^3-\alpha^6\right)+\mathrm{B}^2\beta^{2n}\left(1+\beta^2-2\beta^3-\alpha^6\right)\\
&+2\mathrm{AB}\alpha^n\beta^n\left(1+\alpha\beta-\alpha^2\beta^2-\alpha^2\beta-\alpha^3\beta^3\right)\\
&+2\Big[\mathrm{A}^2\alpha^{2n+1}\left(1+\alpha^2-2\alpha^3-\alpha^6\right)+\mathrm{B}^2\beta^{2n+1}\left(1+\beta^2-2\beta^3-\alpha^6\right)\\
&+\left(\mathrm{AB}\alpha^n\beta^n\right)(\alpha+\beta)\left(1+\alpha\beta-\alpha\beta^2-\alpha^2\beta-\alpha^3\beta^3\right)\Big]\varepsilon.
\end{aligned}
$$

Theorem 3.2. *The dual Horadam hybrid numbers satisfy the recurrence relation*

$$\mathbf{\mathfrak{Q}}_n = p\mathbf{\mathfrak{Q}}_{n-1} + q\mathbf{\mathfrak{Q}}_{n-2}, \quad n \geq 2, \tag{3.3}$$

with initial conditions $\mathbf{\mathfrak{Q}}_0 = \boldsymbol{\mathcal{H}}_0+\boldsymbol{\mathcal{H}}_1\mathrm{i}+\boldsymbol{\mathcal{H}}_2\epsilon+\boldsymbol{\mathcal{H}}_3\mathrm{h}$ *and* $\mathbf{\mathfrak{Q}}_1 = \boldsymbol{\mathcal{H}}_1+\boldsymbol{\mathcal{H}}_2\mathrm{i}+\boldsymbol{\mathcal{H}}_3\epsilon+\boldsymbol{\mathcal{H}}_4\mathrm{h}$, *where the conditions of* $\boldsymbol{\mathcal{H}}_n$ *for* $n \in \{0,1,2,3,4\}$ *are given in* (2.5).

Proof. It can be easily proven by using the equations (2.1) and (2.5). □

Theorem 3.3. *The Binet formula for dual Horadam hybrid numbers is*

$$\mathbf{\mathfrak{Q}}_n = \mathrm{A}\alpha^*\alpha^n + \mathrm{B}\beta^*\beta^n, \tag{3.4}$$

where $\alpha^* = \widetilde{\alpha}(1+\alpha\varepsilon)$, $\beta^* = \widetilde{\beta}(1+\beta\varepsilon)$, *and* A, B, *and* $\widetilde{\alpha}$, $\widetilde{\beta}$ *are defined by* (2.3) *and* (2.8), *respectively.*

Proof. By using the Binet formula of Horadam hybrid numbers given in (2.7), we obtain

$$
\begin{aligned}
\mathbf{\mathfrak{Q}}_n &= \mathcal{Q}_n + \mathcal{Q}_{n+1}\varepsilon\\
&= \mathrm{A}\alpha^n\widetilde{\alpha} + \mathrm{B}\beta^n\widetilde{\beta} + \left(\mathrm{A}\alpha^{n+1}\widetilde{\alpha} + \mathrm{B}\beta^{n+1}\widetilde{\beta}\right)\varepsilon
\end{aligned}
$$

$$= \mathrm{A}\widetilde{\alpha}\,(1+\alpha\varepsilon)\,\alpha^n + \mathrm{B}\widetilde{\beta}\,(1+\beta\varepsilon)\,\beta^n$$
$$= \mathrm{A}\alpha^*\alpha^n + \mathrm{B}\beta^*\beta^n.$$

□

From the definitions of α^* and β^*, we give the following multiplicative relationship in Table 3.

TABLE 3. Multiplicative relationship between α^* and β^*

	α^*	β^*
α^*	$\left(2\widetilde{\alpha} - \mathcal{C}(\widetilde{\alpha})\right)(1+2\alpha\varepsilon)$	$\left(2\widetilde{\beta} - \vartheta + \Delta(\mathcal{V}_0 + q\xi)\right)(1+p\varepsilon)$
β^*	$\left(2\widetilde{\alpha} - \vartheta - \Delta(\mathcal{V}_0 + q\xi)\right)(1+p\varepsilon)$	$\left(2\widetilde{\beta} - \mathcal{C}(\widetilde{\beta})\right)(1+2\beta\varepsilon)$

Here, $\mathcal{F}_n = \frac{\alpha^n-\beta^n}{\Delta}$, $\vartheta = 1+q-pq-q^3$, $\xi = -\mathcal{F}_2 + (q\mathcal{F}_1 - \mathcal{F}_2)\,\epsilon + \mathcal{F}_1\mathrm{h}$ and $\mathcal{V}_0 = \mathcal{F}_1\mathrm{i} + \mathcal{F}_2\epsilon + \mathcal{F}_3\mathrm{h}$, where $\mathcal{F}_n = \mathcal{H}_n(0,1;p,q)$ is the generalized Fibonacci number (for details, see [48]).

Theorem 3.4. *The generating function for the dual Horadam hybrid numbers is*

$$\mathrm{GF}_{\mathbf{\mathfrak{Q}}_n}(t) = \frac{\mathbf{\mathfrak{Q}}_0 + (\mathbf{\mathfrak{Q}}_1 - p\mathbf{\mathfrak{Q}}_0)\,t}{1-pt-qt^2}. \tag{3.5}$$

Proof. Let $\mathrm{GF}_{\mathcal{H}_n}(t)$ be the generating function for dual Horadam hybrid numbers such that

$$\mathrm{GF}_{\mathcal{H}_n}(t) = \mathbf{\mathfrak{Q}}_0 + \mathbf{\mathfrak{Q}}_1 t + \mathbf{\mathfrak{Q}}_2 t^2 + \cdots + \mathbf{\mathfrak{Q}}_n t^n + \cdots. \tag{3.6}$$

Multiplying both sides of (3.6) by $-pt$ and $-qt^2$, we have

$$-pt\mathrm{GF}_{\mathcal{H}_n}(t) = -p\left(\mathbf{\mathfrak{Q}}_0 t + \mathbf{\mathfrak{Q}}_1 t^2 + \mathbf{\mathfrak{Q}}_2 t^3 + \cdots + \mathbf{\mathfrak{Q}}_n t^{n+1} + \cdots\right),$$
$$-qt^2\mathrm{GF}_{\mathcal{H}_n}(t) = -q\left(\mathbf{\mathfrak{Q}}_0 t^2 + \mathbf{\mathfrak{Q}}_1 t^3 + \mathbf{\mathfrak{Q}}_2 t^4 + \cdots + \mathbf{\mathfrak{Q}}_n t^{n+2} + \cdots\right).$$

We have anticipated result (3.5) by using the relation (3.2). □

Theorem 3.5. *For non-negative integers n, r and s, the Vajda's identity for the dual Horadam hybrid numbers is*

$$\mathbf{\mathfrak{Q}}_{n+r}\mathbf{\mathfrak{Q}}_{n+s} - \mathbf{\mathfrak{Q}}_n\mathbf{\mathfrak{Q}}_{n+r+s} = \mathrm{AB}q^n\Delta^2\Big[-2\mathbb{H}_{\mathcal{F},n} + \vartheta\mathcal{F}_s + (\mathcal{V}_0 + q\xi)(\alpha^s+\beta^s)\Big](1+p\varepsilon)\,\mathcal{F}_r,$$

where $\mathbb{H}_{\mathcal{F},n}$ is the n^{th} Fibonacci hybrid number.

Proof. By using the Binet formula for dual Horadam hybrid numbers, we obtain

$$\begin{aligned}
&\mathbf{\mathfrak{Q}}_{n+r}\mathbf{\mathfrak{Q}}_{n+s} - \mathbf{\mathfrak{Q}}_n\mathbf{\mathfrak{Q}}_{n+r+s}\\
&= \left(\mathrm{A}\alpha^*\alpha^{n+r} + \mathrm{B}\beta^*\beta^{n+r}\right)\left(\mathrm{A}\alpha^*\alpha^{n+s} + \mathrm{B}\beta^*\beta^{n+s}\right)\\
&\quad - \left(\mathrm{A}\alpha^*\alpha^n + \mathrm{B}\beta^*\beta^n\right)\left(A\alpha^*\alpha^{n+r+s} + \mathrm{B}\beta^*\beta^{n+r+s}\right)\\
&= (\mathrm{A}\alpha^*)^2\,\alpha^{2n+r+s} + \mathrm{AB}\alpha^*\beta^*\alpha^{n+r}\beta^{n+s} + \mathrm{AB}\beta^*\alpha^*\alpha^{n+s}\beta^{n+r} + (\mathrm{B}\beta^*)\,\beta^{2n+r+s}
\end{aligned}$$

$$
\begin{aligned}
&-\left(\mathrm{A}\alpha^*\right)^2\alpha^{2n+r+s}-\mathrm{AB}\alpha^*\beta^*\alpha^n\beta^{n+r+s}-\mathrm{AB}\beta^*\alpha^*\alpha^{n+r+s}\beta^n-B\beta^*\beta^{n+r+s}\\
=&\mathrm{AB}\alpha^*\beta^*\left(\alpha\beta\right)^n\Big(\alpha^r\beta^s-\beta^{r+s}\Big)+\mathrm{AB}\beta^*\alpha^*\left(\alpha\beta\right)^n\Big(\alpha^s\beta^r-\alpha^{r+s}\Big)\\
=&\mathrm{AB}\left(\alpha\beta\right)^n\Big[\alpha^*\beta^*\beta^s\left(\alpha^r-\beta^r\right)+\beta^*\alpha^*\alpha^s\left(\beta^r-\alpha^r\right)\Big]\\
=&\mathrm{AB}\left(\alpha\beta\right)^n\left[\alpha^*\beta^*\beta^s-\beta^*\alpha^*\alpha^s\right]\left(\alpha^r-\beta^r\right)\\
=&\mathrm{AB}q^n\Big[\big(2\widetilde{\beta}-\vartheta+\Delta\left(\mathcal{V}_0+q\xi\right)\big)\left(1+p\varepsilon\right)\beta^s-\big(2\widetilde{\alpha}-\vartheta-\Delta\left(\mathcal{V}_0+q\xi\right)\big)\left(1+p\varepsilon\right)\alpha^s\Big]\Delta\mathcal{F}_r\\
=&\mathrm{AB}q^n\Delta\Big[\big(2\widetilde{\beta}-\vartheta+\Delta\left(\mathcal{V}_0+q\xi\right)\big)\beta^s-\big(2\widetilde{\alpha}-\vartheta-\Delta\left(\mathcal{V}_0+q\xi\right)\big)\alpha^s\Big]\left(1+p\varepsilon\right)\mathcal{F}_r\\
=&\mathrm{AB}q^n\Delta\Big[2\widetilde{\beta}\beta^s-\vartheta\beta^s+\Delta\left(\mathcal{V}_0+q\xi\right)\beta^s-2\widetilde{\alpha}\alpha^s+\vartheta\alpha^s+\Delta\left(\mathcal{V}_0+q\xi\right)\alpha^s\Big]\left(1+p\varepsilon\right)\mathcal{F}_r\\
=&\mathrm{AB}q^n\Delta\Big[2(\widetilde{\beta}\beta^s-\widetilde{\alpha}\alpha^s)+\vartheta\left(\alpha^s-\beta^s\right)+\Delta\left(\mathcal{V}_0+q\xi\right)\left(\alpha^s+\beta^s\right)\Big]\left(1+p\varepsilon\right)\mathcal{F}_r\\
=&\mathrm{AB}q^n\Delta\Big[-2\Delta\mathbb{H}_{\mathcal{F},n}+\Delta\big(\vartheta\mathcal{F}_s+\left(\mathcal{V}_0+q\xi\right)\left(\alpha^s+\beta^s\right)\big)\Big]\left(1+p\varepsilon\right)\mathcal{F}_r\\
=&\mathrm{AB}q^n\Delta^2\Big[-2\mathbb{H}_{\mathcal{F},n}+\vartheta\mathcal{F}_s+\left(\mathcal{V}_0+q\xi\right)\left(\alpha^s+\beta^s\right)\Big]\left(1+p\varepsilon\right)\mathcal{F}_r,
\end{aligned}
$$

where $\mathbb{H}_{\mathcal{F},n}$ is the n^{th} Fibonacci hybrid number. □

We have the particular cases from Vajda's identity in the following corollaries.

Corollary 3.3 (Catalan's identity). *For non-negative integers n and s, such that $n \geq s$, we have*

$$\mathfrak{Q}_{n-s}\mathfrak{Q}_{n+s}-\mathfrak{Q}_n^2=(-1)^{s+1}\,\mathrm{AB}q^n\Delta^2\Big(-2\mathbb{H}_{\mathcal{F},n}+\vartheta\mathcal{F}_s+\left(\mathcal{V}_0+q\xi\right)\left(\alpha^s+\beta^s\right)\Big)\left(1+p\varepsilon\right)\mathcal{F}_s.$$

It is clear that if we take $m = 1$ in the Catalan identity, then we obtain the following result.

Corollary 3.4 (Cassini's identity). *For positive integer n, we have*

$$\mathfrak{Q}_{n-1}\mathfrak{Q}_{n+1}-\mathfrak{Q}_n^2=\mathrm{AB}q^n\Delta^2\Big(-2\mathbb{H}_{\mathcal{F},n}+\vartheta+\left(\mathcal{V}_0+q\xi\right)\left(\alpha^s+\beta^s\right)\Big)\left(1+p\varepsilon\right).$$

It is clear that if we take $s = m - n$ and $r = 1$ in Theorem 3.5, then we obtain the following result.

Corollary 3.5 (d'Ocagne's identity). *For non-negative integers n and m, such that $m \geq n$, we have*

$$\mathfrak{Q}_{n+1}\mathfrak{Q}_m-\mathfrak{Q}_n\mathfrak{Q}_{m+1}=\mathrm{AB}q^n\Delta^2\Big(-2\mathbb{H}_{\mathcal{F},n}+\vartheta+\left(\mathcal{V}_0+q\xi\right)\left(\alpha^{m-n}+\beta^{m-n}\right)\Big)\left(1+p\varepsilon\right).$$

4. Conclusions

Dual hybrid numbers, as defined by [3, 37], have been given for a special case by replacing real number coefficients by Horadam number coefficients. We first define elementary operations on the set of dual Horadam hybrid numbers. Then we give the definitions of the character and the norm of a dual hybrid number. We then investigate recurrence relations and the Binet formula of dual Horadam hybrid numbers. We also provide the generating function. We finally investigate Vajda's identity and give some corollaries for special cases such as Catalan, Cassini and d'Ocagne's identities. Note that if we let the dual part of each coefficient of a dual Horadam hybrid number approach zero, then the dual Horadam hybrid numbers reduce to the Horadam hybrid numbers and our results reduce to the results of [43].

In Table 4, we present the numbers mentioned in this paper.

TABLE 4. An overview of the number systems considered in this paper

Set	**Number**	**Definition**
$\mathbb{K}$	Hybrid Numbers [28]	$Q = a + b\mathrm{i} + c\epsilon + d\mathrm{h}$ where a, b, c and d are real numbers
$\mathbb{DK}$	Dual Hybrid Numbers [37]	$\boldsymbol{Q} = \boldsymbol{a} + \boldsymbol{b}\mathrm{i} + \boldsymbol{c}\epsilon + \boldsymbol{d}\mathrm{h}$ where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are dual numbers
$\mathcal{H}_n$	Horadam Numbers [20]	$\mathcal{H}_n = p\mathcal{H}_{n-1} - q\mathcal{H}_{n-2}$ where $\mathcal{H}_n\left(\mathcal{H}_0, \mathcal{H}_1; p, q\right)$
$\boldsymbol{\mathcal{H}}_n$	Dual Horadam Numbers	$\boldsymbol{\mathcal{H}}_n = p\boldsymbol{\mathcal{H}}_{n-1} - q\boldsymbol{\mathcal{H}}_{n-2}$ where $\boldsymbol{\mathcal{H}}_n\left(\boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{H}}_1; p, q\right)$
$\mathbb{K}_{\mathcal{H},n}$	Horadam Hybrid Numbers [43]	$\mathcal{Q}_n = \mathcal{H}_n + \mathcal{H}_{n+1}\mathrm{i} + \mathcal{H}_{n+2}\epsilon + \mathcal{H}_{n+3}\mathrm{h}$
$\mathbb{DK}_{\boldsymbol{\mathcal{H}},n}$	Dual Horadam Hybrid Numbers	$\boldsymbol{\mathcal{Q}}_n = \boldsymbol{\mathcal{H}}_n + \boldsymbol{\mathcal{H}}_{n+1}\mathrm{i} + \boldsymbol{\mathcal{H}}_{n+2}\epsilon + \boldsymbol{\mathcal{H}}_{n+3}\mathrm{h}$

5. Appendix

In Table 5, we give a brief literature review related to special type of hybrid number sequences:

Table 5: Timeline of published articles related to the theory and applications of special types of hybrid number sequences

Year	**Articles**
2018	The Horadam hybrid numbers [43] On Pell and Pell-Lucas hybrid numbers [44]
2019	On Jacobsthal and Jacobsthal-lucas hybrid numbers [45] On k-Pell hybrid numbers [10] Tribonacci and Tribonacci-Lucas hybrid numbers [50]
2020	Hybrid numbers with Fibonacci and Lucas hybrid number coefficients [33] A new generalization of Fibonacci hybrid and Lucas hybrid numbers [25] A note on generalized hybrid tribonacci numbers [59] On $\mathcal{J}\left(r,n\right)$-Jacobsthal hybrid numbers [8] On generalized Mersenne hybrid numbers [46] The hybrid numbers of Padovan and some identities [36]

	On the Horadam hybrid quaternions [Dağdeviren et al., arXiv:2012.08277] A study on Horadam hybrid numbers [48]
2021	A note on ratios of Fibonacci hybrid and Lucas hybrid numbers [34] More identities on Fibonacci and Lucas hybrid numbers [61] Investigation of generalized Fibonacci hybrid numbers and their properties [11] On k-Fibonacci hybrid numbers and their matrix representations [6] Generalized hybrid Fibonacci and Lucas p-numbers [27] Generalized k-order Fibonacci and Lucas hybrid numbers [4] Generalized tetranacci hybrid number [39] Pentanacci and pentanacci-Lucas hybrid numbers [21] On k-Kacobsthal and k-Jacobsthal-Lucas hybrid numbers [24] Some properties between Mersenne, Jacobsthal and Jacobthal-Lucas hybrid numbers [51] Mersenne-Lucas hybrid numbers [29] Hybrid leonardo numbers [2] A combined approach to Perrin and Padovan hybrid sequences [32] Padovan and Perrin hybrid number identities [56] Unrestricted Gibonacci hybrid numbers [7]
2022	On the generalized Gaussian Fibonacci numbers and Horadam hybrid numbers: A unified approach [62] An introduction to harmonic complex numbers and harmonic hybrid Fibonacci numbers: A unified approach [22] Introduction to k-Horadam hybrid numbers [23]
2023	On a new generalization of Fibonacci hybrid numbers [52] A new class of Leonardo hybrid numbers and some remarks on Leonardo quaternions over finite fields [53] Hybrid hyper-Fibonacci and hyper-Lucas numbers [1] On hybrid hyper k-Pell, k-Pell-Lucas, and modified k-Pell numbers [40] On hybrid numbers with Gaussian Mersenne coefficients [60] Balancing and Lucas-balancing hybrid numbers and some identities [55] On some k-Oresme hybrid numbers [17] On Horadam finite operator hybrid numbers [58]
2024	Oresme hybrid number [47] Cobalancing hybrid numbers [35] On a new generalization of Pell hybrid numbers [9] On Vietoris' hybrid number sequence [15] Introduction to generalized Leonardo-Alwyn hybrid numbers [Cerda-Morales, arXiv:2405.13074] On the linear recurrence of (generalized) hybrid numbers sequences and moment problems [49] A note on hybrid hyper-Leonardo numbers [41]

	Fibonacci-Lucas spinors obtained from hybrid numbers [Özçevik et al., arXiv:2406.15393] On the Mersenne and Mersenne-Lucas hybrid quaternions [30] Hybrid numbers with hybrid Leonardo number coefficients [42] The (s,t)-Jacobsthal hybrid numbers and (s,t)-Jacobsthal-Lucas hybrid numbers [31] On higher order Lucas hybrid quaternions [5]
2025	On Higher-Order Generalized Fibonacci Hybrid Numbers with q-Integer Components: New Properties, Recurrence Relations, and Matrix Representations [26] On the Lichtenberg hybrid quaternions [14] On some k-Oresme hybrid numbers including negative indices [18]

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