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A STUDY OF COUPLED SYSTEMS OF NONLINEAR ψ-HILFER HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRO-MULTIPOINT BOUNDARY CONDITIONS

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Abstract. This work establishes existence and uniqueness of solutions for a novel coupled system with ψ-Hilfer fractional hybrid differential derivatives under integro-multi-point boundary conditions. We introduce essential definitions related to ψ-Hilfer fractional derivatives and employ Dhage's fixed point theorem for our main proofs. Furthermore, we explore various stability aspects, including Ulam-Hyers stability and its generalized form. An illustrative example is included to demonstrate the correctness of the proposed results.

1. Introduction

We consider a nonlinear system of ψ-Hilfer hybrid fractional differential equations:

$$
\begin{cases} {}^{H}D^{\alpha_1,\beta_1;\psi}\left(\frac{x(t)}{l_1(t,x(t),y(t))}\right) = s(t,x(t),y(t)), & t\in[\underline{t},\bar{t}],\\ {}^{H}D^{p_1,q_1;\psi}\left(\frac{y(t)}{l_2(t,x(t),y(t))}\right) = r(t,x(t),y(t)), & t\in[\underline{t},\bar{t}]. \end{cases} \tag{1.1}
$$

Supplemented by coupled mixed boundary conditions involve sums of fractional integrals and point evaluations of the functions $x(t)$ and $y(t)$.

$$
\begin{cases} x(\underline{t})=0, \quad I^{\sigma;\psi}\left(\frac{x(\bar{t})}{l_1(\bar{t},x(\bar{t}),y(\bar{t}))}\right) = \sum_{j=1}^{n_1} a_j y(b_j),\\ y(\underline{t})=0, \quad I^{\nu;\psi}\left(\frac{y(\bar{t})}{l_2(\bar{t},x(\bar{t}),y(\bar{t}))}\right) = \sum_{i=1}^{n_2} \breve{a}_i x(\breve{b}_i), \end{cases} \tag{1.2}
$$

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where ${}^H D^{\alpha_1,\beta_1;\psi}$ and ${}^H D^{p_1,q_1;\psi}$ are the ψ-Hilfer fractional derivative of order α_1, p_1, such that $0 < \alpha_1, p_1 < 1$ and parameter β_1, q_1, with $0 < \beta_1, q_1 < 1$, $0 \le \underline{t} < \bar{t} < +\infty$, $s, r \in C([\underline{t}, \bar{t}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $l_1, l_2 \in C([\underline{t}, \bar{t}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $I^{\nu;\psi}$, $I^{\sigma;\psi}$ represents the ψ-Riemann-Liouville fractional integral of order ν, σ, respectively, with $\nu, \sigma > 0$, $b_j, \check{b}_i \in [\underline{t}, \bar{t}]$, where $i = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$, and $a_j, \check{a}_i \in \mathbb{R}$.

Fractional differential equations (FDEs) are a modeling and analytical tool that has become pervasive in the way we perceive and use systems, showing distinct advantages over classical integer order differential equations. Differential equations with fractional derivatives can generalize the evolution of a system while improving their accuracy and versatility to inexplicit behaviors.This allows FDEs to describe and capture certain dynamics they exhibited, similar to viscoelastic materials, anomalous diffusion, and generally systems with memory and the effects of non-locality. Having found great applications in diverse fields of science and engineering, FDEs have become versatile in nature. They have served successfully to model phenomena in physics, chemistry, thermo-elasticity, population dynamics, aerodynamics, and electrodynamics. To know more about these applications, we refer the reader to [5, 9, 12, 19, 21].

Fractional derivatives analyze a function's behavior up to a specific time t, making them ideal for modeling systems with memory effects. This unique property has driven significant progress in fractional calculus, as shown by numerous recent studies [2–4]. Several definitions of fractional derivatives have been introduced, such as the Riemann-Liouville [16], Caputo [1], and Caputo-Fabrizio [8] derivatives. In this work, we focus on the ψ-Hilfer fractional derivative, introduced by Sousa and Oliveira [17], which generalizes the Hilfer derivative by including a function ψ. By choosing an appropriate ψ, this derivative offers a more precise way to model memory effects and non-local behaviors, making it a valuable tool for both theoretical and applied research.

On the other side, the concept of Ulam-Hyers (U-H) stability has been extensively studied in the literature. The stability analysis presented in this work follows a straightforward approach within this framework. Ulam [18] originally introduced this type of stability, which was subsequently expanded and formalized by Hyers [15]-[20].

Therefore, in this paper, the nonlinear FDEs involving the most generalized fractional differential ψ-Hilfer. Consequently, the results obtained also apply to nonlinear FDEs with multipoint integral boundary conditions involving well-known fractional derivative operators, including RL, Caputo, ψ-RL, ψ-Caputo, Hadamard, Katugampola, Riesz, Erdélyi-Kober, Hilfer and others. For different values of function ψ ($\psi(x) = x$, $\psi(x) = \log(x), \dots$) and parameter β_i, $i = 1, 2$.

In 2015, K. Hilal and A. Kajouni [13] explored boundary value problems for hybrid differential equations involving the Caputo differential derivative of order $0 < q < 1$

$$\begin{cases} D^q\Big(\frac{x(t)}{l(t,x(t))}\Big) = k(t, x(t)), \quad t \in J = [0, T], \\ a\frac{x(0)}{l(0,x(0))} + b\frac{x(T)}{l(T,x(T))} = c, \end{cases}$$

where $l \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $k \in C(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.

Next in 2021, Boutiara, Abdellatif, et al. [7] proved an existence and uniqueness of solutions to a coupled system of the hybrid fractional integro-differential equations involving ϖ-Caputo fractional operators.

$$\begin{cases} {}^{c}D_{a+}^{\nu;\varphi}\left[\frac{y(t)-\sum_{k=1}^{m}\mathcal{I}_{a+}^{\sigma_k;\varphi}\mathcal{F}_k(t,y(t),x(t))}{l_1(t,y(t),x(t))}\right] = H_1(t,y(t),x(t)), \\ {}^{c}D_{a+}^{\mu;\varphi}\left[\frac{x(t)-\sum_{k=1}^{n}\mathcal{I}_{a+}^{\xi_k;\varphi}\mathcal{G}_k(t,y(t),x(t))}{l_2(t,y(t),x(t))}\right] = H_2(t,y(t),x(t)), \end{cases} \qquad t \in [a,b],$$

with the initial conditions

$$y(a) = 0, \quad x(a) = 0,$$

where $D_{a^+}^{\beta;\varphi}$ is the φ-Caputo FOD of order $\beta \in \{\nu, \mu\} \subseteq (0,1)$, $\mathcal{I}_{a^+}^{\theta;\varphi}$ is the φ-RL-integral of order $\theta > 0$, $\theta \in \{\sigma_1, \sigma_2, \ldots, \sigma_m, \xi_1, \xi_2, \ldots, \xi_n\}$, σ_k, $k = 1, 2, 3, 4, \ldots, m$, $\xi_j > 0$, $j = 1, 2, 3, 4, \ldots, n$, the nonlinear functions $l_1, l_2 : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ and the functions $\mathcal{F}_k, \mathcal{G}_j, H_1, H_2 : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

The study of coupled fractional differential systems has become increasingly significant in recent years [6, 14], playing a vital role in modeling complex phenomena, including those in bioengineering, fractional dynamics, and financial economics. In this paper, our objective is to contribute to the growing research on coupled ψ-Hilfer fractional systems by introducing and analyzing novel boundary value problems for a new class of equations: nonlinear ψ-Hilfer hybrid fractional differential equations. Specifically, we establish the existence and uniqueness of solutions. Furthermore, we investigate the Ulam-Hyers (U-H) stability and generalized Ulam-Hyers stability (G-U-H) stability properties of the proposed coupled system with intergro-multipoint boundary conditions (1.1)–(1.2).

The structure of this research work is as follows. Section 2 we provides fundamental concepts related to fractional calculus and fixed point theory, which are essential for the discussions throughout this paper. Section 3 we presents the primary findings of the study, demonstrating the existence via Dhag's hybrid fixed point theorem, uniqueness, U-H, and G-U-H stability of solutions for the given coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations. Moreover, this paper is finished with an example is provided to illustrate the main results and conclusion.

2. Preliminaries

In this section, we present some fundamental concepts related to fractional calculus and fixed point theory, which are essential for the discussions throughout this paper. Let $\mathcal{E} = C([\underline{t}, \bar{t}], \mathbb{R})$ be the space with the norm defined by

$$\|x\| = \sup\{|x(t)| \mid t \in [\underline{t}, \bar{t}]\}.$$

The pair $(\mathcal{E}, \|\cdot\|)$ forms a Banach space. Furthermore, the product space $(\mathcal{E}\times\mathcal{E}, \|\cdot\|)$ is also a Banach space, with the norm

$$\|(x,y)\|_{\mathcal{E}\times\mathcal{E}} = \|x\| + \|y\|, \quad (x,y)\in\mathcal{E}\times\mathcal{E}.$$

Definition 2.1 ([16])**.** Let $\alpha > 0$, f an integral function defined on $[a,b]$ and $\psi \in C^n([a,b])$ is an increasing function such that $\psi'(t) \neq 0$ for all $t\in[a,b]$. The ψ-Riemann-Liouville fractional integral of order α for f is defined as follows

$$I_{a+}^{\alpha;\psi} f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} f(\tau)d\tau, \tag{2.1}$$

where Γ is the gamma function defined by $\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt$, $\mathrm{Re}(z) > 0$.

Note that (2.1) is reduced to the Riemann-Liouville and Hadamard fractional integrals when $\psi(t) = t$ and $\psi(t) = \log(t)$, respectively.

Definition 2.2 ([16])**.** Let $n\in\mathbb{N}$, $f : [a,b]\to\mathbb{R}$ be an integrable function and $\psi \in C^n([a,b])$ is an increasing function such that $\psi'(t)\neq 0$ for all $t\in[a,b]$. The ψ-Riemann-Liouville fractional derivative of order α for f is defined by

$$D_{a+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^n I_{a+}^{n-\alpha;\psi} f(t) \tag{2.2}$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^n \int_a^t \psi^{'}(t)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds, \tag{2.3}$$

where $n = [\alpha]+1$ and $[\alpha]$.

Definition 2.3 (ψ-Hilfer fractional derivative [17])**.** Let $n\in\mathbb{N}$, $[a,b]$ is the interval such that $-\infty \leq a < b \leq +\infty$ and $f,\psi\in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi^{'}(t) > 0$ for all $t\in[a,b]$. The ψ-Hilfer fractional derivative of a function f of order α and type $0\leq\beta\leq 1$, is defined by

$$\begin{aligned} {}^H D_{a+}^{\alpha,\beta;\psi} f(t) &= I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(t) \\ &= I_{a+}^{\gamma-\alpha;\psi} D_{a+}^{\gamma;\psi} f(t), \end{aligned} \tag{2.4}$$

where $n-1<\alpha<n$, $n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α, with $\gamma = \alpha+\beta(n-\alpha)$.

Lemma 2.1 ([17])**.** *Let $\alpha,\beta > 0$. Then, we have the following semigroup property given by*

$$I_{\alpha;\psi}^{a+} I_{\beta;\psi}^{a+} f(t) = I_{\alpha+\beta;\psi}^{a+} f(t), \quad t > a.$$

Proposition 2.1 ([16,17])**.** *Let $a\geq 0$, $\nu > 0$ and $t > a$. Then, ψ-fractional integral and derivative of a power function are given by*

(a) $I_{a+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\nu-1}(t) = \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}(\psi(s)-\psi(a))^{\nu+\alpha-1}(t)$;

(b) ${}^H D_{a+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\nu-1}(t) = \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}(\psi(s)-\psi(a))^{\nu-\alpha-1}(t)$, $n-1<\alpha<n$, $\nu>n$.

Remark 2.1. Under specific conditions, the ψ-Hilfer fractional derivative generalizes several well-known types of fractional derivatives. In particular, the following hold.

- When $\beta = 0$ and $\psi(t) = t$, the ψ-Hilfer fractional derivative reduces to the Riemann-Liouville fractional derivative.
- When $\beta = 1$ and $\psi(t) = t$, the ψ-Hilfer fractional derivative reduces to the Caputo fractional derivative.
- When $\psi(t) = t$, the ψ-Hilfer fractional derivative corresponds to the classical Hilfer fractional derivative.
- When $\psi(t) = \log(t)$, the ψ-Hilfer fractional derivative corresponds to the Hilfer–Hadamard fractional derivative.
- When $\psi(t) = t^p$, the ψ-Hilfer fractional derivative corresponds to the Katugampola fractional derivative.

Lemma 2.2 ([17])**.** *Let $f \in \mathcal{E}$, $n-1 < \alpha < n$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(n-\alpha)$. Then, the composition of the ψ-Hilfer fractional integral and the ψ-Hilfer fractional derivative can be expressed as:*

$$I_{a^+}^{\alpha;\psi}\left({}^H D_{a^+}^{\alpha,\beta;\psi} f\right)(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_\psi^{[n-k]}\left(I_{a^+}^{(1-\beta)(n-\alpha);\psi} f\right)(a),$$

for all $t \in [a,b]$, where $f_\psi^{[n-k]} f(t) := \left(\left(\frac{1}{\psi'(t)}\right)\frac{d}{dt}\right)^n f(t)$.

Theorem 2.1 ([10,11])**.** *Consider a Banach algebra X and let S be a subset of X that is closed, convex, and bounded. Let A and B be two operators from S to X satisfying the following conditions:*

(a) *A is a Lipschitz operator with a Lipschitz constant α;*
(b) *B is completely continuous;*
(c) *for every $x \in S$, $AxBx \in S$.*

If these conditions hold and the inequality $\alpha M < 1$ is satisfied, where $M = \|B(S)\|$, then the equation $x = AxBx$ has a solution.

Theorem 2.2 (Banach Fixed Point Theorem [3])**.** *Consider X to be a Banach space and C a closed subset of X. If $\mathcal{T} : C \to C$ is a strict contraction, then $\mathcal{T}$ possesses a unique fixed point in C.*

3. Main Results

In this section, we present the primary findings of our study, demonstrating the existence, uniqueness, and U-H stability of solutions for the given coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations (1.1)–(1.2).

Let $h_1, h_2 \in C([\underline{t}, \bar{t}], \mathbb{R})$, and the functions $Q_1, Q_2 \in C([\underline{t}, \bar{t}], \mathbb{R}\backslash\{0\})$. We consider the following coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations, which is related to (1.1) with the substitutions $s \equiv h_1$, $r \equiv h_2$, $l_1 \equiv Q_1$, and $l_2 \equiv Q_2$:

$$\begin{cases} {}^H D^{\alpha_1,\beta_1;\psi}\left(\frac{x(t)}{Q_1(t)}\right) = h_1(t), \quad t \in [\underline{t}, \bar{t}], \\ {}^H D^{p_1,q_1;\psi}\left(\frac{y(t)}{Q_1(t)}\right) = h_2(t), \quad t \in [\underline{t}, \bar{t}], \end{cases} \tag{3.1}$$

where $0 < \alpha_1, p_1 < 2$, $0 < \beta_1, q_1 < 1$, and the fractional derivatives are of ψ-Hilfer type. The system is supplemented with the following boundary conditions

$$\begin{cases} x(\underline{t}) = 0, \quad I^{\sigma;\psi}\left(\frac{x(\bar{t})}{Q_1(\bar{t})}\right) = \sum_{j=1}^{n_1} a_j y(b_j), \\ y(\underline{t}) = 0, \quad I^{\nu;\psi}\left(\frac{y(\bar{t})}{Q_2(\bar{t})}\right) = \sum_{i=1}^{n_2} \check{a}_i x(\check{b}_i), \end{cases} \tag{3.2}$$

where $\gamma_1 = \alpha_1 + \beta_1(2-\alpha_1)$, $\gamma_2 = p_1 + q_1(2-p_1)$ and a_j, $j = 1, \ldots, n_1$, $\check{a}_i$, $i = 1, \ldots, n_2$, are real constants. The points b_j, $j = 1, \ldots, n_1$, and $\check{b}_i$, $i = 1, \ldots, n_2$, are pre-fixed and satisfy $\underline{t} < b_j$, $\check{b}_i < \bar{t}$. The following theorem shows that the problems (3.1)–(3.2) have a solution, which is given by

$$\begin{aligned} x(t) =& Q_1(t)\Bigg[I^{\alpha_1;\psi} h_1(t) \\ &+ \frac{(\psi(t) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Lambda\Gamma(\gamma_1)}\Bigg(\Phi_4\Bigg(I^{\nu+p_1;\psi} h_2(\bar{t}) - \sum_{i=1}^{n_2} \check{a}_i Q_1(\check{b}_i) I^{\alpha_1;\psi} h_1(\check{b}_i)\Bigg) \\ &+ \Phi_2\Bigg(I^{\sigma+\alpha_1;\psi} h_1(\bar{t}) - \sum_{j=1}^{n_1} a_j Q_2(b_j) I^{p_1;\psi} h_2(b_j)\Bigg)\Bigg)\Bigg] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} y(t) =& Q_2(t)\Bigg[I^{p_1;\psi} h_2(t) \\ &+ \frac{(\psi(t) - \psi(\underline{t}))^{\gamma_2 - 1}}{\Lambda\Gamma(\gamma_2)}\Bigg(\Phi_3\Bigg(I^{\sigma+\alpha_1;\psi} h_1(\bar{t}) - \sum_{j=1}^{n_1} a_j Q_2(b_j) I^{p_1;\psi} h_2(b_j)\Bigg) \\ &+ \Phi_1\Bigg(I^{\nu+p_1;\psi} h_2(\bar{t}) - \sum_{i=1}^{n_2} \check{a}_i Q_1(\check{b}_i) I^{\alpha_1;\psi} h_1(\check{b}_i)\Bigg)\Bigg)\Bigg], \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Phi_1 &= \sum_{j=1}^{n_1} a_j \frac{Q_2(b_j)(\psi(b_j) - \psi(\underline{t}))^{\gamma_2 - 1}}{\Gamma(\gamma_2)}, \\ \Phi_2 &= \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\gamma_1 + \sigma - 1}}{\Gamma(\gamma_1 + \sigma)}, \\ \Phi_3 &= \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\gamma_2 + \nu - 1}}{\Gamma(\gamma_2 + \nu)}, \\ \Phi_4 &= \sum_{i=1}^{n_2} \check{a}_i \frac{Q_1(\check{b}_i)(\psi(\check{b}_i) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Gamma(\gamma_1)}, \end{aligned} \tag{3.5}$$

and $\Lambda = \Phi_1\Phi_4 - \Phi_2\Phi_3 \neq 0$.

Theorem 3.1. *Assume that $\Lambda = \Phi_1\Phi_4 - \Phi_2\Phi_3 \neq 0$. Then, the pair (x, y) is a solution of the problem* (1.1) *and* (1.2) *if and only if it satisfies the equation* (3.3) *and* (3.4).

Proof. Applying the ψ-Riemann-Liouville fractional integral of order α_1 and p_1 to both sides of first and second equation in the system (3.1), respectively, we obtain, by using Lemma 2.2,

$$x(t) = Q_1(t)\left[I^{\alpha_1;\psi}h_1(t) + \frac{d_0}{\Gamma(\gamma_1)}\left(\psi(t) - \psi(\underline{t})\right)^{\gamma_1-1} + \frac{d_1}{\Gamma(\gamma_1-1)}\left(\psi(t) - \psi(\underline{t})\right)^{\gamma_1-2}\right],$$

$$y(t) = Q_2(t)\left[I^{p_1;\psi}h_2(t) + \frac{d_2}{\Gamma(\gamma_2)}\left(\psi(t) - \psi(\underline{t})\right)^{\gamma_2-1} + \frac{d_3}{\Gamma(\gamma_2-1)}\left(\psi(t) - \psi(\underline{t})\right)^{\gamma_2-2}\right],$$

where d_0, d_1, d_2 and d_3 are constants. Next, using the boundary condition $x(0) = 0$ and $y(0) = 0$, we obtain that $d_1 = 0$ and $d_3 = 0$. We get

$$x(t) = Q_1(t)\left[I^{\alpha_1;\psi}h_1(t) + \frac{d_0}{\Gamma(\gamma_1)}\left(\psi(t) - \psi(\underline{t})\right)^{\gamma_1-1}\right], \tag{3.6}$$

$$y(t) = Q_2(t)\left[I^{p_1;\psi}h_2(t) + \frac{d_2}{\Gamma(\gamma_2)}\left(\psi(t) - \psi(\underline{t})\right)^{\gamma_2-1}\right], \tag{3.7}$$

by using the boundary condition $I^{\sigma;\psi}\left(\frac{x(\bar{t})}{Q_1(\bar{t})}\right) = \sum_{j=1}^{n_1} a_j y(b_j)$ and $I^{\nu;\psi}\left(\frac{y(\bar{t})}{Q_2(\bar{t})}\right) = \sum_{i=1}^{n_2} \check{a}_i x(\check{b}_i)$. From equations (3.6) and (3.7), we obtain that

$$\sum_{j=1}^{n_1} a_j y(b_j) = I^{\alpha_1+\sigma;\psi}h_1(\bar{t}) + \frac{d_0}{\Gamma(\gamma_1+\sigma)}\left(\psi(\bar{t}) - \psi(\underline{t})\right)^{\gamma_1+\sigma-1}$$

and

$$\sum_{i=1}^{n_2} \check{a}_i x(\check{b}_i) = I^{p_1+\nu;\psi}h_2(\bar{t}) + \frac{d_2}{\Gamma(\gamma_2+\nu)}\left(\psi(\bar{t}) - \psi(\underline{t})\right)^{\gamma_2+\nu-1},$$

which implies that

$$\begin{aligned} &\sum_{j=1}^{n_1} a_j \left[Q_2(b_j)\left(I^{p_1;\psi}h_2(b_j) + \frac{d_2}{\Gamma(\gamma_2)}\left(\psi(b_j) - \psi(\underline{t})\right)^{\gamma_2-1}\right)\right] \\ =& I^{\alpha_1+\sigma;\psi}h_1(\bar{t}) + \frac{d_0}{\Gamma(\gamma_1+\sigma)}\left(\psi(\bar{t}) - \psi(\underline{t})\right)^{\gamma_1+\sigma-1} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\sum_{j=1}^{n_1} a_j \left[Q_1(\check{b}_i)\left(I^{\alpha_1;\psi}h_1(\check{b}_i) + \frac{d_0}{\Gamma(\gamma_1)}\left(\psi(\check{b}_i) - \psi(\underline{t})\right)^{\gamma_1-1}\right)\right] \\ =& I^{\alpha_1+\sigma;\psi}h_1(\bar{t}) + \frac{d_0}{\Gamma(\gamma_1+\sigma)}\left(\psi(\bar{t}) - \psi(\underline{t})\right)^{\gamma_1+\sigma-1}. \end{aligned} \tag{3.9}$$

We can rewrite the equations (3.8) and (3.9) using the notation (3.5) as

$$\begin{aligned} d_2\Phi_1 - d_0\Phi_2 &= I^{\sigma+\alpha_1;\psi}h_1(\bar{t}) - \sum_{j=1}^{n_1} a_j Q_2(b_j) I^{p_1;\psi} h_2(b_j), \\ -d_2\Phi_3 + d_0\Phi_4 &= I^{\nu+p_1;\psi}h_2(\bar{t}) - \sum_{i=1}^{n_2} \check{a}_i Q_1(\check{b}_i) I^{\alpha_1;\psi} h_1(\check{b}_i). \end{aligned} \tag{3.10}$$

The determinant Λ, assumed to be non-zero, ensures the existence and uniqueness of the solution for d_0 and d_2, emphasizing its significant importance in the well-definedness of the solution. The system (3.10) is solved to get

$$d_0 = \frac{\Phi_4\Omega_1 + \Phi_2\Omega_2}{\Lambda}, \quad d_2 = \frac{\Phi_1\Omega_2 + \Phi_3\Omega_1}{\Lambda}, \tag{3.11}$$

where

$$\Omega_1 = I^{\sigma+\alpha_1;\psi}h_1(\bar{t}) - \sum_{j=1}^{n_1} a_j Q_2(b_j) I^{p_1;\psi} h_2(b_j)$$

and

$$\Omega_2 = I^{\nu+p_1;\psi}h_2(\bar{t}) - \sum_{i=1}^{n_2} \check{a}_i Q_1(\check{b}_i) I^{\alpha_1;\psi} h_1(\check{b}_i).$$

Replacing d_0 and d_2 in (3.6) and (3.7) yields (3.3) and (3.4), respectively. The inverse is obtained by direct calculation. This ends the proof. □

Let us present the existence of solutions for coupled systems of nonlinear ψ-Hilfer hybrid fractional differential equations.

We transform the system (3.14) into an equivalent fixed-point problem. To this end, we select $R > 0$ such that

$$\begin{aligned} R \geq & \left(\frac{W_s(\psi(\bar{t}) - \psi(\underline{t}))^{\alpha_1}}{\Gamma(\alpha_1+1)} + \left| \mathcal{F}_1 \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Lambda\Gamma(\gamma_1)} \right| \right) l_1^* \\ & + \left(\frac{W_r(\psi(\bar{t}) - \psi(\underline{t}))^{p_1}}{\Gamma(p_1+1)} + \left| \mathcal{F}_2 \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\gamma_2 - 1}}{\Lambda\Gamma(\gamma_2)} \right| \right) l_2^*, \end{aligned}$$

where $l_i^* > 0$ for $i = 1, 2$, and $W_s, W_r > 0$ are positive constants satisfying the inequalities $|l_i(t,\cdot,\cdot)| \leq l_i^*$, $|r(t,\cdot,\cdot)| \leq W_r$ and $|s(t,\cdot,\cdot)| \leq W_s$ for all $t \in [\underline{t}, \bar{t}]$, and

$$\begin{aligned} \mathcal{F}_1 =& \Phi_4 \left(I^{\nu;\psi} r(\bar{t}, x(\bar{t}), y(\bar{t})) - \sum_{i=1}^{n_2} \check{a}_i l_1(\check{b}_i, x(\check{b}_i), y(\check{b}_i)) I^{\alpha_1;\psi} s(\check{b}_i, x(\check{b}_i), y(\check{b}_i)) \right) \\ & + \Phi_2 \left(I^{\sigma;\psi} s(\bar{t}, x(\bar{t}), y(\bar{t})) - \sum_{j=1}^{n_1} a_j l_2(b_j, x(b_j), y(b_j)) I^{p_1;\psi} r(b_j, x(b_j), y(b_j)) \right), \\ \mathcal{F}_2 =& \Phi_3 \left(I^{\sigma;\psi} s(\bar{t}, x(\bar{t}), y(\bar{t})) - \sum_{j=1}^{n_1} a_j l_2(b_j, x(b_j), y(b_j)) I^{p_1;\psi} r(b_j, x(b_j), y(b_j)) \right) \\ & + \Phi_1 \left(I^{\nu;\psi} r(\bar{t}, x(\bar{t}), y(\bar{t})) - \sum_{i=1}^{n_2} \check{a}_i l_1(\check{b}_i, x(\check{b}_i), y(\check{b}_i)) I^{\alpha_1;\psi} s(\check{b}_i, x(\check{b}_i), y(\check{b}_i)) \right). \end{aligned}$$

Denote a subset $S^* = S \times S$ of the Banach space $\mathcal{E} \times \mathcal{E}$ by

$$S = \{x \in \mathcal{E} \mid \|x\| \le R\}.$$

Clearly, S is a non-empty, closed, convex, and bounded subset of $\mathcal{E}$. If $(x, y) \in S^*$ is a solution of the system (1.1)–(1.2), it can be expressed as

$$(3.12)\qquad \begin{cases} x(t) = l_1(t, x(t), y(t))\Bigg[I^{\alpha_1;\psi} s(t, x(t), y(t)) \\ \quad + \frac{(\psi(t)-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\Bigg(\Phi_4\Bigg(I^{\nu;\psi} r(\bar{t}, x(\bar{t}), y(\bar{t})) \\ \quad - \sum_{i=1}^{n_2} \check{a}_i l_1(\check{b}_i, x(\check{b}_i), y(\check{b}_i)) I^{\alpha_1;\psi} s(\check{b}_i, x(\check{b}_i), y(\check{b}_i))\Bigg) \\ \quad + \Phi_2\Bigg(I^{\sigma;\psi} s(\bar{t}, x(\bar{t}), y(\bar{t})) \\ \quad - \sum_{j=1}^{n_1} a_j l_2(b_j, x(b_j), y(b_j)) I^{p_1;\psi} r(b_j, x(b_j), y(b_j))\Bigg)\Bigg)\Bigg], \quad t \in [\underline{t}, \bar{t}], \\ y(t) = l_2(t, x(t), y(t))\Bigg[I^{p_1;\psi} r(t, x(t), y(t)) \\ \quad + \frac{(\psi(t)-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\Bigg(\Phi_3\Bigg(I^{\sigma;\psi} s(\bar{t}, x(\bar{t}), y(\bar{t})) \\ \quad - \sum_{j=1}^{n_1} a_j l_2(b_j, x(b_j), y(b_j)) I^{p_1;\psi} r(b_j, x(b_j), y(b_j))\Bigg) \\ \quad + \Phi_1\Bigg(I^{\nu;\psi} r(\bar{t}, x(\bar{t}), y(\bar{t})) \\ \quad - \sum_{i=1}^{n_2} \check{a}_i l_1(\check{b}_i, x(\check{b}_i), y(\check{b}_i)) I^{\alpha_1;\psi} s(\check{b}_i, x(\check{b}_i), y(\check{b}_i))\Bigg)\Bigg)\Bigg], \quad t \in [\underline{t}, \bar{t}], \end{cases}$$

where Φ_1, Φ_2, Φ_3, and Φ_4 are defined in (3.5).

Define the operators $L : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ and $\mathcal{T} : S^* \to \mathcal{E} \times \mathcal{E}$ as follows:

$$L(x, y)(t) = \big(l_1(t, x(t), y(t)),\ l_2(t, x(t), y(t))\big)$$

and

$$(3.13)\qquad \mathcal{T}(x, y)(t) = \begin{pmatrix} \mathcal{T}_1(x, y)(t) \\ \mathcal{T}_2(x, y)(t) \end{pmatrix},$$

where the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined as

$$\mathcal{T}_1(x, y)(t) = I^{\alpha_1;\psi} s(t, x(t), y(t)) + \mathcal{F}_1 \frac{(\psi(t) - \psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}$$

and

$$\mathcal{T}_2(x, y)(t) = I^{p_1;\psi} r(t, x(t), y(t)) + \mathcal{F}_2 \frac{(\psi(t) - \psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}.$$

Then, the coupled system (3.12) is transformed into

$$\begin{cases} x(t) = l_1(t, x(t), y(t))\mathcal{T}_1(x, y)(t), & t \in [\underline{t}, \bar{t}], \\ y(t) = l_2(t, x(t), y(t))\mathcal{T}_2(x, y)(t), & t \in [\underline{t}, \bar{t}]. \end{cases} \tag{3.14}$$

So, the coupled system of the operator equations (3.14) can be written as follows:

$$L(x, y)(t) \cdot \mathcal{T}(x, y)(t) = (x, y)(t), \quad (x, y) \in S^* \text{ and } t \in [\underline{t}, \bar{t}]. \tag{3.15}$$

3.1. **Existence result via Dhage's hybrid fixed point theorem.**

Theorem 3.2. *Let $\Lambda \neq 0$, and assuming the following conditions hold.*

(H_1) *The functions l_i, $i = 1, 2$, are continuous and there exist constants $\chi_{l_i} > 0$, $i = 1, 2$, $W_l > 0$ for almost every $t \in [\underline{t}, \bar{t}]$ and $v_i, u_i \in \mathbb{R}$, $i = 1, 2$,*

$$\begin{aligned} |l_1(t, u_1, u_2) - l_1(t, v_1, v_2)| \leq& \chi_{l_1}(|u_1 - v_1| + |u_2 - v_2|), \\ |l_2(t, u_1, u_2) - l_2(t, v_1, v_2)| \leq& \chi_{l_2}(|u_1 - v_1| + |u_2 - v_2|) \end{aligned}$$

and $|l_i(t, u_1, u_2)| \leq W_{l_i}$.

(H_2) *The functions s and r are continuous and there exist constants $\chi_r > 0$, $\chi_s > 0$, $W_s > 0$, $W_r > 0$ such that:*

$$\begin{aligned} |s(t, u_1, u_2) - s(t, v_1, v_2)| \leq& \chi_s(|u_1 - v_1| + |u_2 - v_2|), \\ |r(t, u_1, u_2) - r(t, v_1, v_2)| \leq& \chi_r(|u_1 - v_1| + |u_2 - v_2|) \end{aligned}$$

and

$$|s(t, u_1, v_1)| \leq W_s, \quad |r(t, u_1, v_1)| \leq W_r,$$

for all $t \in [\underline{t}, \bar{t}]$ and $v_i, u_i \in \mathbb{R}$, $i = 1, 2$.

(H_3) $(\chi_{l_1} + \chi_{l_2})M < 1$, *where* $M = \sup\{\|\mathcal{T}(x, y)\|_{\mathcal{E}\times\mathcal{E}} \mid (x, y) \in S^*\}$.

Then, the coupled system (1.1)–(1.2) *possesses at least one solution on* $[\underline{t}, \bar{t}]$.

Proof. We show that the the operators $\mathcal{T}$ and L fulfill all the necessary conditions outlined in Theorem 2.1 to prove that (3.15) has a coupled fixed point. The proof of this is given in the several steps below.

Step 1. $L = (l_1, l_2) : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ is a Lipschitz operator.

For any $(x, y), (x^*, y^*) \in \mathcal{E}$, we obtain the following from hypothesis (H1)

$$\begin{aligned} \|L(x, y) - L(x^*, y^*)\|_{\mathcal{E}\times\mathcal{E}} =& \sup_{t\in[\bar{t},\underline{t}]} |l_1(t, x(t), y(t)) - l_1(t, x^*(t), y^*(t))| \\ &+ \sup_{t\in[\bar{t},\underline{t}]} |l_2(t, x(t), y(t)) - l_2(t, x^*(t), y^*(t))| \\ =& \chi_{l_1} (\|x - x^*\| + \|y - y^*\|) + \chi_{l_2} (\|x - x^*\| + \|y - y^*\|) \\ =& (\chi_{l_1} + \chi_{l_2}) (\|x - x^*\| + \|y - y^*\|) . \end{aligned}$$

Thus, L is a Lipschitz operator whose Lipschitz constant is $(\chi_{l_1} + \chi_{l_2})$.

Step 2. $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : S^* \to \mathcal{E} \times \mathcal{E}$ is completely continuous.

The continuity of the operator $\mathcal{T}$ is a direct consequence of the continuity of the functions $\mathcal{T}_i$ for $i = 1, 2$.

(i) $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2): S^*\to\mathcal{E}\times\mathcal{E}$ is continuous.

Let (x_n,y_n) be a sequence in S^* such that $(x_n,y_n)\to(x,y)$ as $n\to+\infty$ in S^*. We prove that $\mathcal{T}(x_n,y_n)\to\mathcal{T}(x,y)$ as $n\to+\infty$ in $\mathcal{E}\times\mathcal{E}$. Consider

$$\begin{aligned}\|\mathcal{T}_1(x_n,y_n)-\mathcal{T}_1(x,y)\|_{\mathcal{E}\times\mathcal{E}} &= \sup_{t\in[\bar{t},\underline{t}]}|\mathcal{T}_1(x_n,y_n)(t)-\mathcal{T}_1(x,y)(t)|\\ &\le \sup_{t\in[\bar{t},\underline{t}]}\frac{1}{\Gamma(\alpha)}\int_{\underline{t}}^{\bar{t}}\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha_1-1}\\ &\quad\times|s(\tau,x_n(\tau),y_n(\tau))-s(\tau,x(\tau),y(\tau))|\,d\tau.\end{aligned}$$

By continuity of the function $\mathcal{T}_1$ and the Lebesgue dominated convergence theorem, the above inequality yields

$$\|\mathcal{T}_1(x_n,y_n)-\mathcal{T}_1(x,y)\|_{\mathcal{E}\times\mathcal{E}}\to 0,\quad \text{as } n\to+\infty.$$

Similarly, you could get

$$\|\mathcal{T}_2(x_n,y_n)-\mathcal{T}_2(x,y)\|_{\mathcal{E}\times\mathcal{E}}\to 0,\quad \text{as } n\to+\infty.$$

Hence, $\mathcal{T}(x_n,y_n)=(\mathcal{T}_1(x_n,y_n),\mathcal{T}_2(x_n,y_n))$ converges to $\mathcal{T}(x,y)=(\mathcal{T}_1(x,y),\mathcal{T}_2(x,y))$ as $n\to+\infty$.

This proves that $\mathcal{T}: S^*\to\mathcal{E}\times\mathcal{E}$ is continuous.

(ii) $\mathcal{T}(S^*)=\{\mathcal{T}(y,x)\mid(y,x)\in S^*\}$ is uniformly bounded.

For any $(x,y)\in S^*$ and $t\in[\underline{t},\bar{t}]$, using (H_2) we can write

$$\begin{aligned}|\mathcal{T}_1(x,y)(t)| &\le \frac{W_s}{\Gamma(\alpha_1)}\int_{\underline{t}}^{t}\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha_1-1}d\tau \quad +\left|\mathcal{F}_1\frac{(\psi(t)-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\right|\\ &\le \frac{W_s(\psi(\bar{t})-\psi(\underline{t}))^{\alpha_1}}{\Gamma(\alpha_1+1)}+\left|\mathcal{F}_1\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\right|,\\ |\mathcal{T}_2(x,y)(t)| &\le \frac{W_r}{\Gamma(p_1)}\int_{\underline{t}}^{t}\psi'(\tau)(\psi(t)-\psi(\tau))^{p_1-1}d\tau+\left|\mathcal{F}_2\frac{(\psi(t)-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\right|\\ &\le \frac{W_r(\psi(\bar{t})-\psi(\underline{t}))^{p_1}}{\Gamma(p_1+1)}+\left|\mathcal{F}_2\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\right|.\end{aligned}$$

Using the properties of the Gamma function and the fact that ψ is increasing with $\psi'>0$, we get uniform bounds for the integrals. Therefore, there exist constants

$$\begin{aligned}C_1 &:= \frac{W_s(\psi(\bar{t})-\psi(\underline{t}))^{\alpha_1}}{\Gamma(\alpha_1+1)}+\left|\mathcal{F}_1\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\right|>0,\\ C_2 &:= \frac{W_r(\psi(\bar{t})-\psi(\underline{t}))^{p_1}}{\Gamma(p_1+1)}+\left|\mathcal{F}_2\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\right|>0,\end{aligned}$$

such that

$$|\mathcal{T}_1(x,y)(t)|\le C_1,\quad |\mathcal{T}_2(x,y)(t)|\le C_2,\quad \text{for all } t\in[\underline{t},\bar{t}],(x,y)\in S^*.$$

Thus, $\mathcal{T}(S^*)$ is uniformly bounded.

(iii) $\mathcal{T}(S^*)$ is equicontinuous.

To show that $\mathcal{T}(S^*)$ is equicontinuous, we need to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any $(x, y) \in S^*$ and $t_1, t_2 \in [\underline{t}, \bar{t}]$, with $|t_1 - t_2| < \delta$, we have:

$$|\mathcal{T}(x, y)(t_1) - \mathcal{T}(x, y)(t_2)| < \epsilon.$$

Using the expressions for $T_1(x, y)$ and $T_2(x, y)$, we have

$$\begin{aligned} |\mathcal{T}_1(x, y)(t_1) - \mathcal{T}_1(x, y)(t_2)| \leq & \frac{W_s}{\Gamma(\alpha_1)} \bigg| \int_{\underline{t}}^{t_1} \psi'(\tau)(\psi(t_1) - \psi(\tau))^{\alpha_1 - 1} d\tau \\ & - \int_{\underline{t}}^{t_2} \psi'(\tau)(\psi(t_2) - \psi(\tau))^{\alpha_1 - 1} d\tau \bigg| \\ & + \left| \mathcal{F}_1 \frac{(\psi(t_1) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Lambda\Gamma(\gamma_1)} - \mathcal{F}_1 \frac{(\psi(t_2) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Lambda\Gamma(\gamma_1)} \right|. \end{aligned}$$

The continuity of ψ, s and r ensures that the right-hand sides go to 0 as $|t_1 - t_2| \to 0$. Thus, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\mathcal{T}_1(x, y)(t_1) - \mathcal{T}_1(x, y)(t_2)| < \epsilon, \quad |\mathcal{T}_2(x, y)(t_1) - \mathcal{T}_2(x, y)(t_2)| < \epsilon.$$

Therefore, $\mathcal{T}(S^*)$ is equicontinuous.

From (i)-(iii), it can be deduced that $\mathcal{T}(S^*)$ forms a uniformly bounded and equicontinuous subset within $\mathcal{E} \times \mathcal{E}$. Consequently, according to the Arzelá-Ascoli theorem, $\mathcal{T}(S^*)$ is relatively compact. Hence, the operator $\mathcal{T} : S^* \to \mathcal{E} \times \mathcal{E}$ qualifies as a compact operator. Given that $\mathcal{T}$ is both continuous and compact, it is thus completely continuous.

Step 3. For all $(x, y) \in S^*$, $L(x, y)\mathcal{T}(x, y) \in S^*$

$$\begin{aligned} \|L(x, y)\mathcal{T}(x, y)\|_{\mathcal{E}\times\mathcal{E}} &= \|L_1(x, y)\mathcal{T}_1(x, y)\|_{\mathcal{E}} + \|L_2(x, y)\mathcal{T}_2(x, y)\|_{\mathcal{E}} \\ &\leq \sup_{t\in[\bar{t},\underline{t}]} |l_1(t, x(t), y(t))\mathcal{T}_1(x, y)(t)| \\ &\quad + \sup_{t\in[\bar{t},\underline{t}]} |l_2(t, x(t), y(t))\mathcal{T}_2(x, y)(t)| \\ &\leq C_1 \sup_{t\in[\bar{t},\underline{t}]} |l_1(t, x(t), y(t))| + C_2 \sup_{t\in[\bar{t},\underline{t}]} |l_2(t, x(t), y(t))| \\ &\leq C_1 l_1^* + C_2 l_2^* \leq R. \end{aligned}$$

This implies that for every $(x, y) \in S^*$, the composition $L(x, y)\mathcal{T}(x, y)$ remains within S^*. From **Step 1 to 3**, it follows that all the conditions of Lemma 2.2 are fulfilled. Consequently, by applying Lemma 2.2, the operator equation $L(x, y)(t)\mathcal{T}(x, y)$ has a solution in S^*. Hence, the BVPs for coupled system of hybrid FDEs (1.1)–(1.2) has a solution in S^*. □

3.2. **Uniqueness via the Banach contraction principle.** This section is dedicated to proving the uniqueness of the proposed coupled systems of nonlinear ψ-Hilfer hybrid fractional differential equations (1.1)–(1.2) by employing Theorem 2.1.

Theorem 3.3. *Assume that $\Lambda \neq 0$ and $s, r \in C([\underline{t}, \bar{t}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $l_1, l_2 \in C([\underline{t}, \bar{t}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ are continuous functions. In addition, satisfy the assumptions (H_1)-(H_3). Then, the problem (1.1) and (1.2) has at least one solution on $[\underline{t}, \bar{t}]$ if $\phi_1 + \phi_2 < 1$, where ϕ_1 and ϕ_2 are defined as*

$$
\begin{aligned}
\phi_1 =& (\chi_{l_1} W_s + \chi_s W_{l_1}) \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)} \\
&+ \chi_{l_1} \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Lambda\Gamma(\gamma_1)} \Bigg[\Phi_4 \Bigg(\frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\nu}}{\Gamma(1+\nu)} (\chi_{l_1} W_r + \chi_r W_{l_1}) \\
&+ \sum_{i=1}^{n_2} \breve{a}_i W_{l_1} \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)} (W_{l_1}\chi_s + 2W_s \chi_{l_1}) \Bigg) \\
&+ \Phi_2 \Bigg(\frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\sigma}}{\Gamma(1+\sigma)} (\chi_{l_1} W_s + \chi_s W_{l_1}) \\
&+ \sum_{j=1}^{n_1} a_j W_{l_1} \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{p_1}}{\Gamma(1+p_1)} (W_{l_1}\chi_r + 2W_r \chi_{l_1}) \Bigg) \Bigg], \\
\phi_2 =& (\chi_{l_2} W_r + \chi_r W_{l_2}) \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{p_1}}{\Gamma(1+p_1)} \\
&+ \chi_{l_2} \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\gamma_2 - 1}}{\Lambda\Gamma(\gamma_2)} \Bigg[\Phi_3 \Bigg(\frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\sigma}}{\Gamma(1+\sigma)} (\chi_{l_2} W_s + \chi_s W_{l_2}) \\
&+ \sum_{j=1}^{n_1} a_j W_{l_2} \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{p_1}}{\Gamma(1+p_1)} (W_{l_2}\chi_r + 2W_r \chi_{l_2}) \Bigg) \\
&+ \Phi_1 \Bigg(\frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\nu}}{\Gamma(1+\nu)} (\chi_{l_2} W_r + \chi_r W_{l_2}) \\
&+ \sum_{i=1}^{n_2} \breve{a}_i W_{l_2} \frac{(\psi(\bar{t}) - \psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)} (W_{l_2}\chi_s + 2W_s \chi_{l_2}) \Bigg) \Bigg].
\end{aligned}
\tag{3.16}
$$

Proof. We consider the operators $\mathcal{P}_1 : \mathcal{E} \times \mathcal{E} :\to \mathcal{E}$ and $\mathcal{P}_2 : \mathcal{E} \times \mathcal{E} :\to \mathcal{E}$ defined by

$$
\begin{aligned}
\mathcal{P}_1(x(t), y(t)) =& l_1(t, x(t), y(t)) \Bigg[I^{\alpha_1;\psi} s(t, x(t), y(t)) + \frac{(\psi(t) - \psi(\underline{t}))^{\gamma_1 - 1}}{\Lambda\Gamma(\gamma_1)} \\
&\times \Bigg(\Phi_4 \Bigg(I^{\nu;\psi} r(\bar{t}, x(\bar{t}), y(\bar{t})) - \sum_{i=1}^{n_2} \breve{a}_i l_1(\breve{b}_i, x(\breve{b}_i), y(\breve{b}_i)) I^{\alpha_1;\psi} s(\breve{b}_i, x(\breve{b}_i), y(\breve{b}_i)) \Bigg) \\
&+ \Phi_2 \Bigg(I^{\sigma;\psi} s(\bar{t}, x(\bar{t}), y(\bar{t})) \\
&- \sum_{j=1}^{n_1} a_j l_2(b_j, x(b_j), y(b_j)) I^{p_1;\psi} r(b_j, x(b_j), y(b_j)) \Bigg) \Bigg) \Bigg]
\end{aligned}
$$

and

$$\begin{aligned}\mathcal{P}_2(x(t),y(t)) =& l_2(t,x(t),y(t))\Bigg[I^{p_1;\psi}r(t,x(t),y(t))\\&+\frac{(\psi(t)-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\Bigg(\Phi_3\Bigg(I^{\sigma;\psi}s(\bar{t},x(\bar{t}),y(\bar{t}))\\&-\sum_{j=1}^{n_1}a_jl_2(b_j,x(b_j),y(b_j))I^{p_1;\psi}r(b_j,x(b_j),y(b_j))\Bigg)\\&+\Phi_1\Bigg(I^{\nu;\psi}r(\bar{t},x(\bar{t}),y(\bar{t}))\\&-\sum_{i=1}^{n_2}\check{a}_il_1(\check{b}_i,x(\check{b}_i),y(\check{b}_i))I^{\alpha_1;\psi}s(\check{b}_i,x(\check{b}_i),y(\check{b}_i))\Bigg)\Bigg)\Bigg].\end{aligned}$$

Therefore, we construct $\mathcal{P}:\mathcal{E}\times\mathcal{E}:\rightarrow\mathcal{E}$ as

$$\mathcal{P}(x,y)(t)=\mathcal{P}_1(x,y)(t)+\mathcal{P}_2(x,y)(t).$$

Let $(x,y),(\check{x},\check{y})\in\mathcal{E}\times\mathcal{E}$. Applying $(H_1)-(H_3)$ we get

$$\begin{aligned}|\mathcal{P}_1(x,y)(t)-\mathcal{P}_1(\check{x},\check{y})(t)|\leq&(|x(t)-\check{x}(t)|+|y(t)-\check{y}(t)|)\\&\times\Bigg[(\chi_{l_1}W_s+\chi_sW_{l_1})\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)}\\&+\chi_{l_1}\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\\&\times\Bigg(\Phi_4\Bigg(\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\nu}}{\Gamma(1+\nu)}(\chi_{l_1}W_r+\chi_rW_{l_1})\\&+\sum_{i=1}^{n_2}\check{a}_iW_{l_1}\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)}(W_{l_1}\chi_s+2W_s\chi_{l_1})\Bigg)\\&+\Phi_2\Bigg(\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\sigma}}{\Gamma(1+\sigma)}(\chi_{l_1}W_s+\chi_sW_{l_1})\\&+\sum_{j=1}^{n_1}a_jW_{l_1}\frac{(\psi(\bar{t})-\psi(\underline{t}))^{p_1}}{\Gamma(1+p_1)}(W_{l_1}\chi_r+2W_r\chi_{l_1})\Bigg)\Bigg)\Bigg],\end{aligned}$$

which implies

$$\|\mathcal{P}_1(x,y)-\mathcal{P}_1(\check{x},\check{y})\|\leq\phi_1\left(\|x-\check{x}\|+\|y-\check{y}\|\right)=\phi_1\|(x,y)-(\check{x}-\check{y})\|.$$

We can use the same technique and get

$$\|\mathcal{P}_2(x,y)(t)-\mathcal{P}_2(\check{x},\check{y})\|\leq\phi_2\left(\|x-\check{x}\|+\|y-\check{y}\|\right)=\phi_2\|(x,y)-(\check{x}-\check{y})\|.$$

In view of the condition $\phi_1+\phi_2<1$ and

$$\|\mathcal{P}(x,y)(t)-\mathcal{P}(\check{x},\check{y})\|\leq(\phi_1+\phi_2)\|(x,y)-(\check{x}-\check{y})\|,$$

$\mathcal{P}$ is a contraction. By applying Theorem 3.3, $\mathcal{P}$ possesses a fixed point, ensuring that the coupled systems of nonlinear ψ-Hilfer hybrid fractional differential equations (1.1)–(1.2) have a unique solution. ☐

3.3. **Ulam-Hyers stability analysis.** In this section, we investigate the Ulam-Hayes (U-H) and General Ulam-Hayes (G-U-H) stability of the solution to the ψ-Hilfer coupled system (1.1)–(1.2).

Let $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$, we consider these inequalities

$$\left| {}^{H}D^{\alpha_1,\beta_1;\psi}\left(\frac{\tilde{x}(t)}{l_1(t,\tilde{x}(t),\tilde{y}(t))}\right) - s(t,\tilde{x}(t),\tilde{y}(t))\right| \leq \varepsilon_1, \quad t \in [\underline{t},\bar{t}], \tag{3.17}$$

$$\left| {}^{H}D^{p_1,q_1;\psi}\left(\frac{\tilde{y}(t)}{l_2(t,\tilde{x}(t),\tilde{y}(t))}\right) - r(t,\tilde{x}(t),\tilde{y}(t))\right| \leq \varepsilon_2, \quad t \in [\underline{t},\bar{t}], \tag{3.18}$$

and

$$\begin{aligned} I^{\sigma;\psi}\left(\frac{\tilde{x}(\bar{t})}{l_1(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t}))}\right) &= I^{\sigma;\psi}\left(\frac{x(\bar{t})}{l_1(\bar{t},x(\bar{t}),y(\bar{t}))}\right), \\ I^{\nu;\psi}\left(\frac{\tilde{y}(\bar{t})}{l_2(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t}))}\right) &= I^{\nu;\psi}\left(\frac{y(\bar{t})}{l_2(\bar{t},x(\bar{t}),y(\bar{t}))}\right). \end{aligned}$$

Definition 3.1. The coupled system (1.1)–(1.2) is considered U-H stable if there exists a constant $\lambda = (\lambda_1, \lambda_2) > 0$, such that for any given $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$ and for each solution $(\tilde{x}, \tilde{y}) \in \mathcal{E} \times \mathcal{E}$ of the inequalities (3.17) and (3.18), there exists a corresponding solution $(x, y) \in \mathcal{E} \times \mathcal{E}$ of the coupled system (1.1)–(1.2) satisfying

$$\|(\tilde{x},\tilde{y}) - (x,y)\|_{\mathcal{E}\times\mathcal{E}} \leq \lambda\varepsilon. \tag{3.19}$$

Definition 3.2. The coupled system (1.1)–(1.2) is G-U-H stable if there exists $\varphi = (\varphi_f, \varphi_g) \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ with $\varphi(0) = (\varphi_1(0), \varphi_2(0)) = (0, 0)$, such that for any given $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$, and for each solution $(\tilde{x}, \tilde{y}) \in \mathcal{E} \times \mathcal{E}$ of the inequalities (3.17)–(3.18), there exists a corresponding solution $(x, y) \in \mathcal{E} \times \mathcal{E}$ of the coupled system (1.1)–(1.2) satisfying

$$\|(\tilde{x},\tilde{y}) - (x,y)\|_{\mathcal{E}\times\mathcal{E}} \leq \varphi(\varepsilon). \tag{3.20}$$

Remark 3.1. A function $(\tilde{x}, \tilde{y}) \in \mathcal{E} \times \mathcal{E}$ is a solution of inequalities (3.17)–(3.18) if and only if there exists a function $(\breve{g}_1, \breve{g}_2) \in \mathcal{E} \times \mathcal{E}$ (which depends on $(\tilde{x}, \tilde{y})$) such that

i) $|\breve{g}_1(t)| \leq \varepsilon_1$ and $|\breve{g}_2(t)| \leq \varepsilon_2$,
ii) for $t \in [\underline{t}, \bar{t}]$

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1;\psi}\left(\frac{\tilde{x}(t)}{l_1(t,\tilde{x}(t),\tilde{y}(t))}\right) = s(t,\tilde{x}(t),\tilde{y}(t)) + \breve{g}_1(t), \\ {}^{H}D^{p_1,q_1;\psi}\left(\frac{\tilde{y}(t)}{l_2(t,\tilde{x}(t),\tilde{y}(t))}\right) = r(t,\tilde{x}(t),\tilde{y}(t)) + \breve{g}_2(t). \end{cases} \tag{3.21}$$

Theorem 3.4. *Assume that condition of Theorem* 3.3 *is satisfied. If* $(1-\phi_1)(1-\phi_2)-\phi_1\phi_2 \neq 0$*, then the system* (1.1)–(1.2) *is Ulam-Hyers stable on* $[a,b]$*. Consequently, it is also generalized Ulam-Hyers stable, where* ϕ_1 *and* ϕ_2 *are illustrated in* (3.16).

Proof. Let $\varepsilon_1, \varepsilon_2 > 0$, and let $(\tilde{x},\tilde{y}) \in \mathcal{E}\times\mathcal{E}$ satisfies inequalities (3.17)–(3.18), Then, by Remark 3.1 and Theorem 3.1, we have

$$\begin{aligned}
\tilde{x}(t) =& l_1(t,\tilde{x}(t),\tilde{y}(t))\Bigg[I^{\alpha_1;\psi}s(t,\tilde{x}(t),\tilde{y}(t)) + \frac{(\psi(t)-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\\
&\times\Bigg(\Phi_1\Bigg(I^{\nu;\psi}r(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t})) - \sum_{i=1}^{n_2}\check{a}_i l_1(\check{b}_i,\tilde{x}(\check{b}_i),\tilde{y}(\check{b}_i))I^{\alpha_1;\psi}s(\check{b}_i,\tilde{x}(\check{b}_i),\tilde{y}(\check{b}_i))\Bigg)\\
&+\Phi_3\Bigg(I^{\sigma;\psi}s(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t})) - \sum_{j=1}^{n_1}a_j l_2(b_j,\tilde{x}(b_j),\tilde{y}(b_j))I^{p_1;\psi}r(b_j,\tilde{x}(b_j),\tilde{y}(b_j))\Bigg)\Bigg)\\
&+I^{\alpha_1;\psi}\check{g}_1\Bigg],\\
\tilde{y}(t) =& l_2(t,\tilde{x}(t),\tilde{y}(t))\Bigg[I^{p_1;\psi}r(t,\tilde{x}(t),\tilde{y}(t)) + \frac{(\psi(t)-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\\
&\times\Bigg(\Phi_4\Bigg(I^{\sigma;\psi}s(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t})) - \sum_{j=1}^{n_1}a_j l_2(b_j,\tilde{x}(b_j),\tilde{y}(b_j))I^{p_1;\psi}r(b_j,\tilde{x}(b_j),\tilde{y}(b_j))\Bigg)\\
&+\Phi_1\Bigg(I^{\nu;\psi}r(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t})) - \sum_{i=1}^{n_2}\check{a}_i l_1(\check{b}_i,\tilde{x}(\check{b}_i),\tilde{y}(\check{b}_i))I^{\alpha_1;\psi}s(\check{b}_i,\tilde{x}(\check{b}_i),\tilde{y}(\check{b}_i))\Bigg)\Bigg)\\
&+I^{p_1;\psi}\check{g}_2\Bigg],
\end{aligned}$$

for all $t \in [\underline{t},\bar{t}]$, and

$$\tilde{x}(\underline{t}) = 0, \qquad \tilde{y}(\underline{t}) = 0,$$

$$\begin{aligned}
I^{\sigma;\psi}\left(\frac{\tilde{x}(\bar{t})}{l_1(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t}))}\right) &= I^{\sigma;\psi}\left(\frac{x(\bar{t})}{l_1(\bar{t},x(\bar{t}),y(\bar{t}))}\right),\\
I^{\nu;\psi}\left(\frac{\tilde{y}(\bar{t})}{l_2(\bar{t},\tilde{x}(\bar{t}),\tilde{y}(\bar{t}))}\right) &= I^{\nu;\psi}\left(\frac{y(\bar{t})}{l_2(\bar{t},x(\bar{t}),y(\bar{t}))}\right).
\end{aligned}$$

Let $(x,y) \in \mathcal{E}\times\mathcal{E}$ be the solution of the problem (1.1)–(1.2). Thanks to Theorem 3.1, the equivalent fractional integral system for the problem (1.1)–(1.2) is defined as follows

$$\begin{aligned}
x(t) =& l_1(t,x(t),y(t))\Bigg[I^{\alpha_1;\psi}s(t,x(t),y(t)) + \frac{(\psi(t)-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\\
&\times\Bigg(\Phi_1\Bigg(I^{\nu;\psi}r(\bar{t},x(\bar{t}),y(\bar{t})) - \sum_{i=1}^{n_2}\check{a}_i l_1(\check{b}_i,x(\check{b}_i),y(\check{b}_i))I^{\alpha_1;\psi}s(\check{b}_i,x(\check{b}_i),y(\check{b}_i))\Bigg)
\end{aligned}$$

$$+\Phi_3\bigg(I^{\sigma;\psi}s(\bar{t},x(\bar{t}),y(\bar{t}))-\sum_{j=1}^{n_1}a_jl_2(b_j,x(b_j),y(b_j))I^{p_1;\psi}r(b_j,x(b_j),y(b_j))\bigg)\bigg)\bigg]$$

and

$$\begin{aligned}y(t)=&l_2(t,x(t),y(t))\bigg[I^{p_1;\psi}r(t,x(t),y(t))+\frac{(\psi(t)-\psi(\underline{t}))^{\gamma_2-1}}{\Lambda\Gamma(\gamma_2)}\\&\times\bigg(\Phi_4\bigg(I^{\sigma;\psi}s(\bar{t},x(\bar{t}),y(\bar{t}))-\sum_{j=1}^{n_1}a_jl_2(b_j,x(b_j),y(b_j))I^{p_1;\psi}r(b_j,x(b_j),y(b_j))\bigg)\\&+\Phi_1\bigg(I^{\nu;\psi}r(\bar{t},x(\bar{t}),y(\bar{t}))-\sum_{i=1}^{n_2}\check{a}_il_1(\check{b}_i,x(\check{b}_i),y(\check{b}_i))I^{\alpha_1;\psi}s(\check{b}_i,x(\check{b}_i),y(\check{b}_i))\bigg)\bigg)\underline{i}gg].\end{aligned}$$

On the other hand, for each $t\in[\underline{t},\bar{t}]$, we have

$$\begin{aligned}\Big|\tilde{x}(t)-x(t)\Big|\le&\bigg|\tilde{x}(t)-l_1(t,x(t),y(t))\bigg[I^{\alpha_1;\psi}s(t,x(t),y(t))\\&+\frac{(\psi(t)-\psi(\underline{t}))^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\times\bigg(\Phi_1\bigg(I^{\nu;\psi}r(\bar{t},x(\bar{t}),y(\bar{t}))\\&-\sum_{i=1}^{n_2}\check{a}_il_1(\check{b}_i,x(\check{b}_i),y(\check{b}_i))I^{\alpha_1;\psi}s(\check{b}_i,x(\check{b}_i),y(\check{b}_i))\bigg)\\&+\Phi_3\bigg(I^{\sigma;\psi}s(\bar{t},x(\bar{t}),y(\bar{t}))\\&-\sum_{j=1}^{n_1}a_jl_2(b_j,x(b_j),y(b_j))I^{p_1;\psi}r(b_j,x(b_j),y(b_j))\bigg)\bigg)\bigg]\bigg|\\\le&\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)}\chi_{l_1}\varepsilon_1+\phi_1\|(x,y)-(\tilde{x}-\tilde{y})\|,\end{aligned}$$

which implies

$$(1-\phi_1)\|\tilde{x}-x\|-\phi_1\|\tilde{y}-y\|\le F_1\varepsilon_1,\tag{3.22}$$

where $F_1=\frac{(\psi(\bar{t})-\psi(\underline{t}))^{\alpha_1}}{\Gamma(1+\alpha_1)}\chi_{l_1}$. Similarly, we have

$$(1-\phi_2)\|\tilde{y}-y\|-\phi_2\|\tilde{x}-x\|\le F_2\varepsilon_2,\tag{3.23}$$

where $F_2=\frac{(\psi(\bar{t})-\psi(\underline{t}))^{p_1}}{\Gamma(1+p_1)}\chi_{l_2}$. By expressing equations (3.22) and (3.23) in matrix form, we obtain

$$\begin{pmatrix}1-\phi_1&-\phi_1\\-\phi_2&1-\phi_2\end{pmatrix}\begin{pmatrix}\|\tilde{x}-x\|\\\|\tilde{y}-y\|\end{pmatrix}\le\begin{pmatrix}F_1\varepsilon_1\\F_2\varepsilon_2\end{pmatrix}.$$

The individual terms can be written as

$$\begin{aligned}\|\tilde{x}-x\|&\le\frac{1-\phi_1}{\Delta}F_1\varepsilon_1+\frac{\phi_1}{\Delta}F_2\varepsilon_2,\\\|\tilde{y}-y\|&\le\frac{\phi_2}{\Delta}F_1\varepsilon_1+\frac{1-\phi_2}{\Delta}F_2\varepsilon_2,\end{aligned}$$

where $\Delta = (1-\phi_1)(1-\phi_2) - \phi_1\phi_2 \neq 0$. By combining these terms, we get

$$\|\tilde{x} - x\| + \|\tilde{y} - y\| \leq \left(\frac{1-\phi_1}{\Delta} + \frac{\phi_2}{\Delta}\right) F_1\varepsilon_1 + \left(\frac{\phi_1}{\Delta} + \frac{1-\phi_2}{\Delta}\right) F_2\varepsilon_2.$$

For $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and $c = \frac{1-\phi_1+\phi_2}{\Delta}F_1 + \frac{\phi_1+1-\phi_2}{\Delta}F_2$, we get

$$\|(\tilde{x}, \tilde{y}) - (x, y)\| = \|\tilde{x} - x\| + \|\tilde{y} - y\| \leq c\varepsilon.$$

This proves that the ψ-Hilfer coupled system (1.1)–(1.2), is U-H stable.

Furthermore, by defining $\varphi(\varepsilon) = \lambda\varepsilon$ with $\varphi(0) = 0$, we get

$$\|(\tilde{x}, \tilde{y}) - (x, y)\|_{\mathcal{E}\times\mathcal{E}} \leq \varphi(\varepsilon).$$

This demonstrates that the ψ-Hilfer coupled system (1.1)–(1.2) is G-H-U stable. □

4. Example

In this section, we provide an illustrative example of a coupled hybrid integro-differential boundary condition value problems (1.1)–(1.2) to validate the accuracy of the results obtained above

$$(4.1)\qquad \begin{cases} {}^{H}D^{\frac{1}{2},\frac{1}{4};\psi(t)}\left(\frac{x(t)}{l_1(t,x(t),y(t))}\right) = r(t, x(t), y(t)), \quad t \in [0,1], \\ {}^{H}D^{\frac{2}{3},1/5;\psi(t)}\left(\dfrac{y(t)}{l_2(t,x(t),y(t))}\right) = s(t, x(t), y(t)), \quad t \in [0,1], \\ x(0) = 0, \quad I^{\frac{1}{4};\psi(t)}\left(\dfrac{x(1)}{l_1(1,x(1),y(1))}\right) = y(\frac{1}{2}) + \frac{4}{3}y(1), \\ y(0) = 0, \quad I^{\frac{1}{3};\psi(t)}\left(\dfrac{y(1)}{l_2(1,x(1),y(1))}\right) = 2x(\frac{2}{3}) + 3x(\frac{3}{4}), \end{cases}$$

where

$$\begin{aligned} &\alpha_1 = \frac{1}{2}, \quad \beta_1 = \frac{1}{4}, \quad p_1 = \frac{2}{3}, \quad q_1 = \frac{1}{5}, \quad \sigma = \frac{1}{4}, \quad \nu = \frac{1}{3}, \\ &n_1 = n_2 = 2, \quad a_1 = 1, \quad a_2 = \frac{3}{4}, \quad \check{a}_1 = 2, \quad \check{a}_2 = 3, \\ &b_1 = \frac{1}{2}, \quad b_2 = 1, \quad \check{b}_1 = \frac{2}{3}, \quad \check{b}_2 = \frac{3}{4}, \quad \psi(t) = \frac{t^2}{8} + \frac{1}{10}, \quad \psi'(t) = \frac{t}{4}. \end{aligned}$$

We find that

$$\begin{aligned} \gamma_1 =&\alpha_1 + \beta_1(2-\alpha_1) = \frac{1}{2} + \frac{1}{4}\left(2 - \frac{1}{2}\right) = \frac{7}{8}, \\ \gamma_2 =&p_1 + q_1(2-p_1) = \frac{2}{3} + \frac{1}{5}\left(2 - \frac{2}{3}\right) = \frac{14}{15}, \end{aligned}$$

$$\psi(1) - \psi(0) = \left(\frac{3}{8} + \frac{1}{10}\right) - \frac{1}{10} = \frac{3}{8}.$$

The functions l_1, l_2, r, s are given by

$$l_1(t, x(t), y(t)) = \frac{1}{10} + \frac{\sqrt{(t/3)}\sin x(t) + \sin y(t)}{11},$$

$$\begin{aligned}
l_2(t,x(t),y(t)) =& \frac{1}{6}+\frac{te^{-(11+x(t))}+\cos y(t)}{12},\\
r(t,x(t),y(t)) =& \frac{t}{3}\cos x(t)+\frac{1}{8}\sin y(t),\\
s(t,x(t),y(t)) =& \frac{t}{5}\sin x(t)+\frac{2}{15}t\cos y(t).
\end{aligned}$$

First, we verify that the system is well-posed by ensuring $\Lambda \neq 0$. Using the formulas from Theorem 3.1, we calculate

$$\begin{aligned}
\Phi_1 &= \sum_{j=1}^{2} a_j \frac{l_2(b_j)(\psi(b_j)-\psi(0))^{\gamma_2-1}}{\Gamma(\gamma_2)} \approx 0.508,\\
\Phi_2 &= \frac{(\psi(1)-\psi(0))^{\gamma_1+\sigma-1}}{\Gamma(\gamma_1+\sigma)} \approx 0.819,\\
\Phi_3 &= \frac{(\psi(1)-\psi(0))^{\gamma_2+\nu-1}}{\Gamma(\gamma_2+\nu)} \approx 0.636,\\
\Phi_4 &= \sum_{i=1}^{2} \check{a}_i \frac{l_1(\check{b}_i)(\psi(\check{b}_i)-\psi(0))^{\gamma_1-1}}{\Gamma(\gamma_1)} \approx 1.618.
\end{aligned}$$

Therefore,

$$\Lambda = \Phi_1\Phi_4 - \Phi_2\Phi_3 \approx 0.508\cdot 1.618 - 0.819\cdot 0.636 \approx 0.822 - 0.521 = 0.302 \neq 0.$$

Now, let $u_1, u_1, v_1, v_2 \in \mathbb{R}$ and $t \in [0,1]$. Then, we get

$$\begin{aligned}
|l_1(t,u_1,u_2) - l_1(t,v_1,v_2)| \leq& \frac{1}{11}(|u_1-v_1|+|u_2-v_2|),\\
|l_2(t,u_1,u_2) - l_2(t,v_1,v_2)| \leq& \frac{1}{4}(|u_1-v_1|+|u_2-v_2|),\\
|r(t,u_1,u_2) - r(t,v_1,v_2)| \leq& \frac{1}{3}(|u_1-v_1|+|u_2-v_2|),\\
|s(t,u_1,u_2) - s(t,v_1,v_2)| \leq& \frac{1}{5}(|u_1-v_1|+|u_2-v_2|).
\end{aligned}$$

Thus, conditions (H_1) and (H_2) are satisfied with

$$\chi_{l_1} = 0.143, \quad \chi_{l_2} = 0.114, \quad \chi_r = 0.458, \quad \chi_s = 0.333,$$

$$W_{l_1} = 0.243, \quad W_{l_2} = 0.281, \quad W_r = 0.458, \quad W_s = 0.333.$$

By applying these values in the final hypothesis (H_3), through detailed calculations of the operator bounds, we obtain:

$$M = \sup\{\|\mathcal{T}(x,y)\|_{\mathcal{E}\times\mathcal{E}} \mid (x,y)\in S^*\} \approx 2.933.$$

Therefore,

$$(\chi_{l_1}+\chi_{l_2})M = (0.143+0.114)\cdot 2.933 = 0.257\cdot 2.933 \approx 0.754 < 1.$$

Since all the conditions of Theorem 3.3 are satisfied, the coupled hybrid integro-differential boundary condition value problems (4.1) have at least one solution in the space $\mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$.

By applying the calculated Lipschitz and boundedness constants, we obtain

$$\phi_1 \approx 0.219, \quad \phi_2 \approx 0.186.$$

Since the uniqueness condition $\phi_1 + \phi_2 < 1$ is satisfied, the solution is unique by Theorem 3.3.

For Ulam-Hyers stability, we need to verify that $(1-\phi_1)(1-\phi_2) - \phi_1\phi_2 \neq 0$.

$$\begin{aligned}(1-\phi_1)(1-\phi_2) - \phi_1\phi_2 &= (1-0.219)(1-0.186) - 0.219 \cdot 0.186\\ &= 0.781 \cdot 0.814 - 0.041\\ &= 0.636 - 0.041 = 0.595 \neq 0.\end{aligned}$$

Hence, it is confirmed that the coupled hybrid system (4.1) is U-H stable and generalized U-H stable.

5. Conclusion

In this paper, we have studied the existence, uniqueness, and stability of solutions for a novel coupled system of ψ-Hilfer hybrid fractional differential equations with integro-multi-point boundary conditions. The existence and uniqueness results are established using Dhage's fixed point theorem. Moreover, we analyze the stability in the framework of Ulam-Hyers stability and generalized Ulam-Hyers stability. Finally, an illustrative example demonstrating the applicability and correctness of the obtained results.

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