

APPLICATIONS POISSON DISTRIBUTION AND RUSCHEWEYH DERIVATIVE OPERATOR FOR BI-UNIVALENT FUNCTIONS

ABBAS KAREEM WANAS¹ AND JANUSZ SOKÓŁ²

ABSTRACT. In this paper we establish upper bounds for the second and third coefficients of holomorphic and bi-univalent functions in a new family which involve the Bazilevič functions and β -pseudo-starlike functions under a new operator joining Poisson distribution with Ruscheweyh derivative operator. Also, we discuss Fekete-Szegő problem of functions in this family.

1. INTRODUCTION

Let \mathcal{A} be the collection of functions f that are holomorphic in the unit disk $\mathbb{D} = \{|z| < 1\}$ in the complex plane \mathbb{C} and that have the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Further, let \mathcal{S} be the sub-collection of \mathcal{A} containing of functions which are univalent in \mathbb{D} . According to the Koebe one-quarter theorem (see [3]), every function $f \in \mathcal{S}$ has an inverse f^{-1} such that $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and $f(f^{-1}(w)) = w$, $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. If f is of the form (1.1), then

$$(1.2) \quad f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots, \quad |w| < r_0(f).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{D} if both f and f^{-1} are univalent in \mathbb{D} . We denote by Σ the set of bi-univalent functions in \mathbb{D} . Srivastava et al. [19] have apparently resuscitated the study of holomorphic and bi-univalent functions in

Key words and phrases. Bi-univalent function, (M, N) -Lucas polynomial, coefficient bound, Fekete-Szegő problem, Poisson distribution, subordination, Ruscheweyh derivative.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30C80.

DOI

Received: May 03, 2020.

Accepted: March 05, 2021.

recent years. It was followed by such works as those by Frasin and Aouf [5], Goyal and Goswami [6], Srivastava and Bansal [15] and others (see, for example [2, 16–18, 20]).

For the polynomials $M(x)$ and $N(x)$ with real coefficients, the (M, N) -Lucas Polynomials $L_{M,N,k}(x)$ are defined by the following recurrence relation (see [8]):

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x), \quad k \geq 2,$$

with

$$(1.3) \quad L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x) \quad \text{and} \quad L_{M,N,2}(x) = M^2(x) + 2N(x).$$

The Lucas Polynomials play an important role in a diversity of disciplines in the mathematical, statistical, physical and engineering sciences (see, for example [4, 9, 21]). The generating function of the (M, N) -Lucas Polynomial $L_{M,N,k}(x)$ (see [9]) is given by

$$(1.4) \quad T_{M(x),N(x)}(z) = \sum_{k=2}^{\infty} L_{M,N,k}(x)z^k = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}.$$

Let the functions f and g be holomorphic in \mathbb{D} , we say that the function f is subordinate to g , if there exists a function w , holomorphic in \mathbb{D} with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathbb{D}$. Furthermore, if the function g is univalent in \mathbb{D} , then we have the following equivalence (see [10])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

A function $f \in \mathcal{A}$ is called Bazilevič function of order α , $\alpha \geq 0$, if (see [14])

$$\operatorname{Re} \left\{ \frac{z^{1-\alpha} f'(z)}{(f(z))^{1-\alpha}} \right\} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called β -pseudo-starlike function of order β , $\beta \geq 1$, if (see [1])

$$\operatorname{Re} \left\{ \frac{z (f'(z))^\beta}{f(z)} \right\} > 0, \quad z \in \mathbb{D}.$$

Recall that a random variable X has the Poisson distribution with parameter θ , if

$$P(X = r) = \frac{\theta^r e^{-\theta}}{r!}, \quad r = 0, 1, 2, 3, \dots$$

Recently, Porwal [11] introduced a power series whose coefficients are probabilities of "Poisson distribution"

$$K(\theta, z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}}{(n-1)!} e^{-\theta} z^n, \quad z \in \mathbb{D},$$

where $\theta > 0$. By ratio test the radius of convergence of the above series is infinity.

In 2016, Porwal and Kumar [12] introduced and investigated a linear operator $I(\theta, z) : \mathcal{A} \rightarrow \mathcal{A}$, $\theta > 0$, by using the Hadamard product (or convolution) and defined as follows

$$I(\theta, z)f(z) = K(\theta, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}}{(n-1)!} e^{-\theta} a_n z^n, \quad z \in \mathbb{D},$$

where " $*$ " indicate the Hadamard product (or convolution) of two power series.

In this paper, for $f \in \mathcal{A}$ we introduce a new linear operator $\mathcal{J}_\theta^\delta : \mathcal{A} \rightarrow \mathcal{A}$ by

$$(1.5) \quad \mathcal{J}_\theta^\delta f(z) = I(\theta, z) * \mathcal{R}^\delta,$$

where \mathcal{R}^δ , $\delta \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, denote the Ruscheweyh derivative operator [13] given by

$$\mathcal{R}^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\delta + n)}{\Gamma(\delta + 1)\Gamma(n)} a_n z^n, \quad z \in \mathbb{D}.$$

It is easy to obtain from (1.5) that

$$\mathcal{J}_\theta^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}\Gamma(\delta + n)}{\Gamma(\delta + 1)(\Gamma(n))^2} e^{-\theta} a_n z^n, \quad z \in \mathbb{D},$$

where $\theta > 0$, $\delta \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

2. MAIN RESULTS

We begin this section by defining the family $\Upsilon_\Sigma(\lambda, \alpha, \beta, \delta, \theta; h)$ as follows.

Definition 2.1. Assume that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\theta > 0$ and h is analytic in \mathbb{D} , $h(0) = 1$. The function $f \in \Sigma$ is in the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ if it fulfills the subordinations:

$$(1 - \lambda) \frac{z^{1-\alpha} \left(\mathcal{J}_\theta^\delta f(z)\right)'}{\left(\mathcal{J}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda \frac{z \left(\left(\mathcal{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathcal{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$(1 - \lambda) \frac{w^{1-\alpha} \left(\mathcal{J}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathcal{J}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda \frac{w \left(\left(\mathcal{J}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathcal{J}_\theta^\delta f^{-1}(w)} \prec 1 + e_1 z + e_2 z^2 + \dots,$$

where f^{-1} is given by (1.2).

In particular, if we choose $\lambda = 1$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ reduces to the family $\mathcal{L}_\Sigma(\beta, \delta, \theta; h)$ of β -pseudo bi-starlike functions which satisfying the following subordinations:

$$\frac{z \left(\left(\mathcal{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathcal{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$\frac{w \left(\left(\mathcal{J}_\theta^\delta f^{-1}(w) \right)' \right)^\beta}{\mathcal{J}_\theta^\delta f^{-1}(w)} \prec h(w).$$

If we choose $\lambda = 0$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ reduces to the family $\mathcal{B}_\Sigma(\alpha, \delta, \theta; h)$ of Bazilevič bi-univalent functions which satisfying the following subordinations:

$$\frac{z^{1-\alpha} \left(\mathcal{J}_\theta^\delta f(z) \right)'}{\left(\mathcal{J}_\theta^\delta f(z) \right)^{1-\alpha}} \prec h(z)$$

and

$$\frac{w^{1-\alpha} \left(\mathcal{J}_\theta^\delta f^{-1}(w) \right)'}{\left(\mathcal{J}_\theta^\delta f^{-1}(w) \right)^{1-\alpha}} \prec h(w).$$

If we choose $\lambda = \beta = 1$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ reduces to the family $\mathcal{S}_\Sigma(\delta, \theta; h)$ of bi-starlike functions which satisfying the following subordinations:

$$\frac{z \left(\mathcal{J}_\theta^\delta f(z) \right)'}{\mathcal{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$\frac{w \left(\mathcal{J}_\theta^\delta f^{-1}(w) \right)'}{\mathcal{J}_\theta^\delta f^{-1}(w)} \prec h(w).$$

Theorem 2.1. *Assume that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$ and $\theta > 0$. If $f \in \Sigma$ of the form (1.1) is in the class $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$, with $h(z) = 1 + e_1 z + e_2 z^2 + \dots$, then*

$$(2.1) \quad |a_2| \leq \frac{|e_1|}{[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)](\delta+1)\theta e^{-\theta}} = \frac{|e_1|}{A}$$

and

$$(2.2) \quad |a_3| \leq \min \left\{ \max \left\{ \left| \frac{e_1}{B} \right|, \left| \frac{e_2}{B} - \frac{C e_1^2}{A^2 B} \right| \right\}, \max \left\{ \left| \frac{e_1}{B} \right|, \left| \frac{e_2}{B} - \frac{(2B+C)e_1^2}{A^2 B} \right| \right\} \right\},$$

where

$$(2.3) \quad \begin{aligned} A &= [(1-\lambda)(\alpha+1) + \lambda(2\beta-1)](\delta+1)\theta e^{-\theta}, \\ B &= \frac{1}{4} [(1-\lambda)(\alpha+2) + \lambda(3\beta-1)](\delta^2 + 3\delta + 2)\theta^2 e^{-\theta}, \\ C &= \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1) + \lambda(2\beta(\beta-2) + 1) \right] (\delta+1)^2 \theta^2 e^{-2\theta}. \end{aligned}$$

Proof. Suppose that $f \in \Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; ; e_1; e_2)$. Then there exist two holomorphic functions $\phi, \psi : \mathbb{D} \rightarrow \mathbb{D}$ given by

$$(2.4) \quad \phi(z) = r_1 z + r_2 z^2 + r_3 z^3 + \dots, \quad z \in \mathbb{D},$$

and

$$(2.5) \quad \psi(w) = s_1 w + s_2 w^2 + s_3 w^3 + \dots, \quad w \in \mathbb{D},$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$(2.6) \quad (1 - \lambda) \frac{z^{1-\alpha} (\mathcal{J}_\theta^\delta f(z))'}{(\mathcal{J}_\theta^\delta f(z))^{1-\alpha}} + \lambda \frac{z \left((\mathcal{J}_\theta^\delta f(z))' \right)^\beta}{\mathcal{J}_\theta^\delta f(z)} = 1 + e_1 \phi(z) + e_2 \phi^2(z) + \dots$$

and

$$(2.7) \quad (1 - \lambda) \frac{w^{1-\alpha} (\mathcal{J}_\theta^\delta f^{-1}(w))'}{(\mathcal{J}_\theta^\delta f^{-1}(w))^{1-\alpha}} + \lambda \frac{w \left((\mathcal{J}_\theta^\delta f^{-1}(w))' \right)^\beta}{\mathcal{J}_\theta^\delta f^{-1}(w)} = 1 + e_1 \psi(w) + e_2 \psi^2(w) + \dots$$

Combining (2.4), (2.5), (2.6) and (2.7), yield

$$(2.8) \quad (1 - \lambda) \frac{z^{1-\alpha} (\mathcal{J}_\theta^\delta f(z))'}{(\mathcal{J}_\theta^\delta f(z))^{1-\alpha}} + \lambda \frac{z \left((\mathcal{J}_\theta^\delta f(z))' \right)^\beta}{\mathcal{J}_\theta^\delta f(z)} = 1 + e_1 r_1 z + [e_1 r_2 + e_2 (x) r_1^2] z^2 + \dots$$

and

$$(2.9) \quad (1 - \lambda) \frac{w^{1-\alpha} (\mathcal{J}_\theta^\delta f^{-1}(w))'}{(\mathcal{J}_\theta^\delta f^{-1}(w))^{1-\alpha}} + \lambda \frac{w \left((\mathcal{J}_\theta^\delta f^{-1}(w))' \right)^\beta}{\mathcal{J}_\theta^\delta f^{-1}(w)} = 1 + e_1 s_1 w + [e_1 s_2 + e_2 s_1^2] w^2 + \dots$$

It is quite well-known that if $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z, w \in \mathbb{D}$, we get

$$(2.10) \quad |r_j| \leq 1 \quad \text{and} \quad |s_j| \leq 1, \quad j \in \mathbb{N}.$$

In the light of (2.8) and (2.9), after simplifying, we find that

$$(2.11) \quad [(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)] (\delta + 1) \theta e^{-\theta} a_2 = e_1 r_1,$$

$$\frac{1}{4} [(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)] (\delta^2 + 3\delta + 2) \theta^2 e^{-\theta} a_3$$

$$+ \left[\frac{1}{2} (1 - \lambda)(\alpha + 2)(\alpha - 1) + \lambda(2\beta(\beta - 2) + 1) \right] (\delta + 1)^2 \theta^2 e^{-2\theta} a_2^2$$

$$(2.12) \quad = e_1 r_2 + e_2 r_1^2,$$

$$(2.13) \quad - [(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)] (\delta + 1) \theta e^{-\theta} a_2 = e_1 s_1$$

and

$$\begin{aligned}
 & \frac{1}{4} [(1-\lambda)(\alpha+2) + \lambda(3\beta-1)] (\delta^2 + 3\delta + 2) \theta^2 e^{-\theta} (2a_2^2 - a_3) \\
 & + \left[\frac{1}{2} (1-\lambda)(\alpha+2)(\alpha-1) + \lambda(2\beta(\beta-2) + 1) \right] (\delta+1)^2 \theta^2 e^{-2\theta} a_2^2 \\
 (2.14) \quad & = e_1 s_2 + e_2 s_1^2.
 \end{aligned}$$

Inequality (2.1) follows from (2.11) and (2.13). If we apply notation (2.3), then (2.11) and (2.12) become

$$(2.15) \quad Aa_2 = e_1 r_1, \quad Ba_3 + Ca_2^2 = e_1 r_2 + e_2 r_1^2.$$

This gives

$$(2.16) \quad \frac{B}{e_1} a_3 = r_2 + \left(\frac{e_2}{e_1} - \frac{Ce_1}{A^2} \right) r_1^2,$$

and on using the known sharp result [7, page 10]:

$$(2.17) \quad |r_2 - \mu r_1^2| \leq \max \{1, |\mu|\},$$

for all $\mu \in \mathbb{C}$, we obtain

$$(2.18) \quad \left| \frac{B}{e_1} \right| |a_3| \leq \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{Ce_1}{A^2} \right| \right\}.$$

In the same way, (2.13) and (2.14) become

$$(2.19) \quad -Aa_2 = e_1 s_1, \quad B(2a_2^2 - a_3) + Ca_2^2 = e_1 s_2 + e_2 s_1^2.$$

This gives

$$(2.20) \quad -\frac{B}{e_1} a_3 = s_2 + \left(\frac{e_2}{e_1} - \frac{(2B+C)e_1}{A^2} \right) s_1^2.$$

Applying (2.17), we obtain

$$(2.21) \quad \left| \frac{B}{e_1} \right| |a_3| \leq \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{(2B+C)e_1}{A^2} \right| \right\}.$$

Inequality (2.2) follows from (2.18) and (2.21). \square

If we take the generating function (1.4) of the (M, N) -Lucas polynomials $L_{M,N,k}(x)$ as $h(z) + 1$, then from (1.3), we have $e_1 = M(x)$ and $e_2 = M^2(x) + 2N(x)$ and Theorem 2.1 becomes the following corollary.

Corollary 2.1. *If $f \in \Sigma$ of the form (1.1) is in the class $\Upsilon_{\Sigma}(\alpha, \beta, \delta, \lambda, \theta; T_{M(x), N(x)} - 1)$, then*

$$|a_2| \leq \frac{|M(x)|}{[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)] (\delta+1) \theta e^{-\theta}}$$

and

$$|a_3| \leq \min \left\{ \max \left\{ \left| \frac{M(x)}{B} \right|, \left| \frac{M^2(x) + 2N(x)}{B} - \frac{CM^2(x)}{A^2B} \right| \right\}, \max \left\{ \left| \frac{M(x)}{B} \right|, \left| \frac{M^2(x) + 2N(x)}{B} - \frac{(2B+C)M^2(x)}{A^2B} \right| \right\} \right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, x$ such that $\alpha \geq 0, \beta \geq 1, \delta \in \mathbb{N}_0, 0 \leq \lambda \leq 1, \theta > 0$ and $x \in \mathbb{R}$, where A, B, C are given by (2.3) and $T_{M(x), N(x)}$ is given by (1.4).

In the next theorem, we discuss a bound for $|a_3 - \eta a_2^2|$ called "the Fekete-Szegő problem".

Theorem 2.2. *If $f \in \Sigma$ of the form (1.1) is in the class $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$, then*

(2.22)

$$\begin{aligned} & |a_3 - \eta a_2^2| \\ & \leq \frac{|e_1|}{B} \min \left\{ \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{(C - \eta B)e_1}{A^2} \right| \right\}, \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{(2B + C - \eta B)e_1}{A^2} \right| \right\} \right\}, \end{aligned}$$

for all $\alpha, \beta, \delta, \lambda, \theta, \eta$ such that $\alpha \geq 0, \beta \geq 1, \delta \in \mathbb{N}_0, 0 \leq \lambda \leq 1, \theta > 0$ and $\eta \in \mathbb{C}$, where A, B, C are given by (2.3).

Proof. We apply the notations from the proof of Theorem 2.1. From (2.15) and from (2.16), we have

$$(2.23) \quad a_3 - \eta a_2^2 = \frac{e_1}{B} \left(r_2 + \left(\frac{e_2}{e_1} - \frac{(C - \eta B)e_1}{A^2} \right) r_1^2 \right)$$

and on using the known sharp result $|r_2 - \mu r_1^2| \leq \max \{1, |\mu|\}$, we get

$$(2.24) \quad |a_3 - \eta a_2^2| \leq \frac{|e_1|}{B} \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{(C - \eta B)e_1}{A^2} \right| \right\}.$$

In the same way, from (2.19) and from (2.20), we have

$$(2.25) \quad a_3 - \eta a_2^2 = -\frac{e_1}{B} \left(s_2 + \left(\frac{e_2}{e_1} - \frac{(2B + C - \eta B)e_1}{A^2} \right) s_1^2 \right)$$

and on using $|s_2 - \mu s_1^2| \leq \max \{1, |\mu|\}$, we get

$$(2.26) \quad |a_3 - \eta a_2^2| \leq \frac{|e_1|}{B} \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{(2B + C - \eta B)e_1}{A^2} \right| \right\}.$$

Inequality (2.22) follows from (2.24) and (2.26). □

Corollary 2.2. *If $f \in \Sigma$ of the form (1.1) is in the class $\Upsilon_{\Sigma}(\alpha, \beta, \delta, \lambda, \theta; T_{M(x), N(x)} - 1)$, then*

$$\left| a_3 - \eta a_2^2 \right| \leq \frac{|M(x)|}{B} \min \left\{ \max \left\{ 1, \left| \frac{M^2(x) + 2N(x)}{M(x)} - \frac{(C - \eta B)M(x)}{A^2} \right| \right\}, \right. \\ \left. \max \left\{ 1, \left| \frac{M^2(x) + 2N(x)}{M(x)} - \frac{(2B + C - \eta B)M(x)}{A^2} \right| \right\} \right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, \eta, x$ such that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\theta > 0$, $\eta \in \mathbb{C}$ and $x \in \mathbb{R}$, where A, B, C are given by (2.3) and $T_{M(x), N(x)}$ is given by (1.4).

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¹DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE,
UNIVERSITY OF AL-QADISIYAH,
IRAQ
Email address: abbas.kareem.w@qu.edu.iq

²COLLEGE OF NATURAL SCIENCES,
UNIVERSITY OF RZESZÓW,
UL. PROF. PIGONIA 1, 35-310 RZESZÓW, POLAND
Email address: jsokol@ur.edu.pl