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HORADAM SEQUENCE THROUGH RECURRENT DETERMINANTS OF TRIDIAGONAL MATRICES

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Abstract. Applying the apparatus of triangular matrices, we proved new recurrent formulas for Horadam numbers with even (odd) subscripts through determinants of tridiagonal matrix.

1. Paradeterminant of Triangular Matrix

In this section, we provide basic notions and results about paradeterminants of triangular matrices that will be used for the proving the main results of the paper.

Definition 1.1. [20] A triangular number table

$$A_n = \begin{pmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ \vdots & \vdots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix} \tag{1.1}$$

is called a *triangular matrix*, and the number n is called its *order*.

Note that a matrix thus defined is not a matrix in the standard sense, because it is a triangular, rather than a rectangular, table of numbers.

The functions of triangular matrices are widely used in algebra, combinatorics, number theory, theory of ordinary differential equations, and other branches of mathematics (see [4, 14, 18–21] for more details and examples).

Key words and phrases. Horadam sequence, triangular matrix, tridiagonal matrix, paradeterminant, Fibonacci sequence.

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Definition 1.2. [20] The *paradeterminant* of the triangular matrix (1.1), denote by $\operatorname{ddet}(A_n)$, is the number

$$\operatorname{ddet}(A_n)=\sum_{r=1}^{n}\sum_{p_1+\cdots+p_r=n}(-1)^{n-r}\prod_{s=1}^{r}\{a_{p_1+\cdots+p_s,\,p_1+\cdots+p_{s-1}+1}\},$$

where the summation is over the set of positive integer solutions of the equality $p_1+\cdots+p_r=n$, and $\{a_{ij}\}=a_{ij}\cdot a_{i,j+1}\cdots a_{ii}$.

For example,

$$\begin{aligned}\operatorname{ddet}(A_4)=&-a_{41}a_{42}a_{43}a_{44}+a_{31}a_{32}a_{33}a_{44}+a_{11}a_{42}a_{43}a_{44}+a_{21}a_{22}a_{43}a_{44}\\&-a_{21}a_{22}a_{33}a_{44}-a_{11}a_{32}a_{33}a_{44}-a_{11}a_{22}a_{43}a_{44}+a_{11}a_{22}a_{33}a_{44}.\end{aligned}$$

The following formula (*decomposition of a paradeterminant by elements of the last row*) holds [20]:

$$\operatorname{ddet}(A_n)=\sum_{s=1}^{n}(-1)^{n-s}\{a_{ns}\}\operatorname{ddet}(A_{s-1}).\tag{1.2}$$

For example,

$$\begin{aligned}\operatorname{ddet}(A_4)=&-a_{41}a_{42}a_{43}a_{44}\operatorname{ddet}(A_0)+a_{42}a_{43}a_{44}\operatorname{ddet}(A_1)\\&-a_{43}a_{44}\operatorname{ddet}(A_2)+a_{44}\operatorname{ddet}(A_3),\end{aligned}$$

where, by definition, $\operatorname{ddet}(A_0)=1$.

R. Zatorsky and I. Lishchynskyy [22] established the relation between the paradeterminants and the lower Hessenberg determinants by formula

$$\operatorname{ddet}(A_n)=\begin{vmatrix}\{a_{11}\}&1&0&\dots&0&0\\\{a_{21}\}&\{a_{22}\}&1&\dots&0&0\\\{a_{31}\}&\{a_{32}\}&\{a_{33}\}&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\\{a_{n-1,1}\}&\{a_{n-1,2}\}&\{a_{n-1,3}\}&\dots&\{a_{n-1,n-1}\}&1\\\{a_{n1}\}&\{a_{n2}\}&\{a_{n3}\}&\dots&\{a_{n,n-1}\}&\{a_{nn}\}\end{vmatrix}.\tag{1.3}$$

2. The Relation Between Horadam Numbers with Even (Odd) Subscripts and Paradeterminants

For $n\geq 0$, the second order linear recurrence sequence $h_n=h_n(a,b;p,q)$ is defined by

$$h_0=a,\quad h_1=b,\quad h_{n+2}=ph_{n+1}-qh_n,\tag{2.1}$$

where a,b,p,q are integers ($q\neq 0$), was introduce by A. F. Horadam. The properties of this sequence were discussed in detail in [1, 2, 5–7, 13, 15, 16].

Sequence (2.1) generalized many well-known number sequences. Examples of such sequences are the Fibonacci, Lucas, Pell, Jacobsthal, Jacobsthal-Lucas, Pell-Lucas sequences, and some others sequences.

Proposition 2.1. *For $n \geq 1$, the following formula holds*:

$$h_{2n-2} = \operatorname{ddet}\begin{pmatrix} a & & & & & & \\ \frac{p}{q}\frac{h_1}{1} & -q & & & & & \\ 0 & \frac{p}{q}\frac{h_3}{h_0} & -q & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & \cdots & -q & & \\ 0 & 0 & 0 & \cdots & \frac{p}{q}\frac{h_{2n-5}}{h_{2n-8}} & -q & \\ 0 & 0 & 0 & \cdots & 0 & \frac{p}{q}\frac{h_{2n-3}}{h_{2n-6}} & -q \end{pmatrix}. \tag{2.2}$$

For example,

$$\begin{aligned} h_0 &= a, \\ h_2 &= \operatorname{ddet}\begin{pmatrix} a & \\ \frac{p}{q}\frac{h_1}{1} & -q \end{pmatrix} \\ &= -aq + bp, \\ h_4 &= \operatorname{ddet}\begin{pmatrix} a & & \\ \frac{p}{q}\frac{h_1}{1} & -q & \\ 0 & \frac{p}{q}\frac{h_3}{h_0} & -q \end{pmatrix} \\ &= aq^2 - bpq + p(bp^2 - apq - bq) \\ &= aq^2 - 2bpq + bp^3 - ap^2q, \end{aligned}$$

and so on.

Proof. Expanding the paradeterminant (2.2) by elements of the last raw (see (1.2)), we have

$$\begin{aligned} h_{2n-2} &= (-q)h_{2n-4} - \frac{p}{q}(-q)\frac{h_{2n-3}}{h_{2n-6}}h_{2n-6} \\ &= ph_{2n-3} - qh_{2n-4}. \end{aligned}$$

Thus, we obtained the recurrent relation (2.1). □

Proposition 2.2. *For $n \geq 1$, the following formula holds*:

$$h_{2n-1} = \operatorname{ddet}\begin{pmatrix} b & & & & & & \\ \frac{p}{q}\frac{h_2}{1} & -q & & & & & \\ 0 & \frac{p}{q}\frac{h_4}{h_1} & -q & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & \cdots & -q & & \\ 0 & 0 & 0 & \cdots & \frac{p}{q}\frac{h_{2n-4}}{h_{2n-7}} & -q & \\ 0 & 0 & 0 & \cdots & 0 & \frac{p}{q}\frac{h_{2n-2}}{h_{2n-5}} & -q \end{pmatrix}. \tag{2.3}$$

For example,

$$
\begin{aligned}
h_1 &= b,\\
h_3 &= \operatorname{ddet}\begin{pmatrix} b & \\ \frac{p}{q}\frac{h_2}{1} & -q \end{pmatrix}\\
&= -bq + p(bp - aq)\\
&= -bq + bp^2 - apq,\\
h_5 &= \operatorname{ddet}\begin{pmatrix} b & & \\ \frac{p}{q}\frac{h_2}{1} & -q & \\ 0 & \frac{p}{q}\frac{h_4}{h_1} & -q \end{pmatrix}\\
&= bq^2 - pq(bp - aq) + p(aq^2 - 2bpq + bp^3 - ap^2q)\\
&= bq^2 - 3bp^2q + 2apq^2 + bp^4 - ap^3q,
\end{aligned}
$$

and so on.

Proof. The proof is similar to the Proposition 2.1. Indeed, using (1.2), we have

$$
\begin{aligned}
h_{2n-1} &= (-q)h_{2n-3} - \frac{p}{q}(-q)\frac{h_{2n-2}}{h_{2n-5}}h_{2n-5}\\
&= ph_{2n-2} - qh_{2n-3}.
\end{aligned}
$$

□

3. Main Results

In this section, we prove new recurrent formulas expressing Horadam numbers h_n with even (odd) subscripts through the determinants of tridiagonal matrix. As a consequence we obtain the corresponding formulas for Fibonacci numbers.

Theorem 3.1. *For $n \geq 1$, the following formulas are hold*:

$$
h_{2n-2} = \frac{1}{\prod\limits_{i=0}^{n-3} h_{2i}} \cdot \begin{vmatrix} a & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ -ph_1 & -q & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & -ph_3 & -qh_0 & h_0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -ph_{2n-5} & -qh_{2n-8} & h_{2n-8}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -ph_{2n-3} & -qh_{2n-6} \end{vmatrix}, \tag{3.1}
$$

and

$$
h_{2n-1} = \frac{1}{\prod\limits_{i=0}^{n-3} h_{2i+1}} \cdot \begin{vmatrix} b & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ -ph_2 & -q & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & -ph_4 & -qh_1 & h_1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -ph_{2n-4} & -qh_{2n-7} & h_{2n-7}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -ph_{2n-2} & -qh_{2n-5} \end{vmatrix}. \tag{3.2}
$$

Proof. We prove the formula (3.1). Using (1.3), from (2.2) we have

$$h_{2n-2}=\begin{vmatrix} a & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ -p\frac{h_1}{1} & -q & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & -p\frac{h_3}{h_0} & -q & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -p\frac{h_{2n-5}}{h_{2n-8}} & -q & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & -p\frac{h_{2n-3}}{h_{2n-6}} & -q \end{vmatrix}.$$

From this, after obvious transformations we get (3.1).

Formula (3.2) follows from (1.3) and (2.3). □

Corollary 3.1. *From* (3.1), (3.2), *for special choices of* a, b, p, q *the following formulas can be obtained*:

- *Fibonacci numbers* $F_n = h_n(1,1;1,-1)$ (*beginning at* $F_1 = 1$) *with even subscripts*:

$$F_{2n}=\frac{1}{F_2F_4\cdots F_{2n-4}}\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ -F_3 & 1 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & -F_5 & F_2 & F_2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -F_{2n-3} & F_{2n-6} & F_{2n-6}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -F_{2n-1} & F_{2n-4} \end{vmatrix};$$

- *Fibonacci numbers* (*beginning at* $F_1 = 1$) *with odd subscripts*:

$$F_{2n-1}=\frac{1}{F_1F_3\cdots F_{2n-5}}\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ -F_2 & 1 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & -F_4 & F_1 & F_1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -F_{2n-4} & F_{2n-7} & F_{2n-7}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -F_{2n-2} & F_{2n-5} \end{vmatrix}.$$

By choosing other suitable values on a, b, p and q, one can also obtain the Lucas, Pell, Jacobsthal, Jacobsthal-Lucas and Pell-Lucas numbers or polynomials in term of recurrent determinants of tridiagonal matrix.

Note that determinants of matrices, elements of which are classical or generalized Fibonacci and Lucas numbers, in particular, studied in [3, 8–12, 17].

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