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SOME GENERALIZED HEISENBERG-TYPE INEQUALITIES ASSOCIATED WITH THE CONTINUOUS OFFSET LINEAR CANONICAL WAVELET TRANSFORM

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Abstract. The Continuous Offset Linear Canonical Wavelet Transform (COLCWT) represents a generalized form of the classical wavelet transform established within the theoretical framework of the offset linear canonical transform. In this work, a natural correspondence between the COLCWT and the standard wavelet transform is constructed. This relationship allows alternative derivations of several fundamental properties of the COLCWT, including the inversion formula, orthogonality condition, and the associated reproducing kernel. Moreover, by exploiting these structural properties and their interconnections, a refined Heisenberg-type uncertainty inequality is established within the framework of the offset linear canonical transform.

1. Introduction

The well-established theory of the classical wavelet transform (WT) provides a powerful foundation for multiresolution analysis and time frequency localization, attributes that have significantly contributed to its widespread adoption in many scientific and engineering areas. Its versatility has enabled important advances ranging from the investigation of multiscale behavior in quantum mechanical systems to practical applications in signal and image processing, where WT plays a central role in decomposition, reconstruction, and the identification of relevant structural features across various scales [1–3].

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In recent years, researchers have shown growing interest in reformulating the wavelet framework through more general integral transforms that provide enhanced adaptability. The offset linear canonical transform (OLCT) represents a significant contribution to this line of inquiry, as it generalizes the traditional Fourier transform by introducing a parametric structure that offers greater analytical versatility and accommodates a wider class of signal representations [4–6]. Along this direction, several extensions of integral transforms and wavelet frameworks have been proposed to further enhance flexibility and analytical capabilities, including generalized canonical and wavelet-type transforms [7–9]. Motivated by this, numerous studies have developed wavelet transforms based on the OLCT and its variants. In particular, several works [10–13] introduced a generalized formulation known as the Continuous Offset Linear Canonical Wavelet Transform (COLCWT), constructed within the framework of the OLCT.

Fundamental properties of this generalized transform, including linearity, orthogonality relations, and reconstruction formulas, have been extensively studied. Moreover, previous research [6, 12] has demonstrated that the OLCT can be reduced to the classical Fourier transform under specific parameter conditions, allowing many of its properties to be derived via its relationship with the Fourier domain. In parallel, uncertainty principles have remained a central topic in time-frequency analysis. Various forms of Heisenberg-type inequalities and their extensions have been established for generalized transforms, including linear canonical transforms and wavelet-type frameworks [7, 14–16].

Furthermore, more advanced extensions of linear canonical transforms, such as quaternion and biquaternion formulations, have recently attracted attention and led to the development of new analytical tools in broader settings [8, 17–19]. Although these approaches are beyond the scope of the present work, they suggest promising directions for future research within generalized transform frameworks.

In this work, we further develop this framework in the setting of the COLCWT. We establish alternative proofs of key theoretical results, such as the orthogonality relation and inversion formula, by leveraging the structural correspondence between the COLCWT and the classical WT. In addition, we construct the reproducing kernel associated with the COLCWT and employ these results to derive Pitt's inequality and its logarithmic variant within the context of the offset linear canonical framework.

The remainder of this paper is organized as follows. Section 2 reviews essential background on the linear canonical transform. Section 3 introduces the OLCT and its fundamental properties. Section 4 develops the COLCWT and its analytical features. Section 5 establishes uncertainty principles for the COLCWT. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we review the definition of the fractional Fourier transform (FRFT) together with several of its fundamental properties (see, [20–22]). We also present the fundamental connection between the fractional Fourier transform and the classical Fourier transform. We begin with the following definitions.

Definition 2.1. Let $1 \leq p < +\infty$. The space $L^p(\mathbb{R})$ consists of all measurable functions f on $\mathbb{R}$ for which the norm

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(\vartheta)|^p \, d\vartheta\right)^{\frac{1}{p}} < +\infty. \tag{2.1}$$

For the particular case $p = 2$, the space $L^2(\mathbb{R})$ forms a Hilbert space equipped with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(\vartheta)\, \overline{g(\vartheta)}\, d\vartheta.$$

The corresponding norm is given by

$$\|f\|_{L^2(\mathbb{R})} = \sqrt{\langle f, f\rangle_{L^2(\mathbb{R})}}.$$

We denote by $C^{+\infty}(\mathbb{R})$ the set of all infinitely differentiable complex-valued functions on $\mathbb{R}$, and by $\mathbb{Z}_+$ the set of all non-negative integers.

Definition 2.2. The Schwartz space $\mathcal{S}(\mathbb{R})$ is defined as

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^{+\infty}(\mathbb{R}) \;\middle|\; \sup_{\vartheta\in\mathbb{R}} \left|\vartheta^\alpha D^\beta f(\vartheta)\right| < +\infty, \text{ for all } \alpha, \beta \in \mathbb{Z}_+ \right\}.$$

Elements of the dual space $\mathcal{S}'(\mathbb{R})$ are called *tempered distributions*. For a function $f \in L^1(\mathbb{R})$, its Fourier transform is defined by

$$\mathcal{F}(f)(\eta) = \hat{f}(\eta) = \int_{\mathbb{R}} f(\vartheta)\, e^{-i\eta\vartheta}\, d\vartheta.$$

3. Offset Liniear Canonical Transform with Properties

This part introduces the definition of the OLCT along with the key properties that will be employed in the subsequent sections of the paper. It will be shown that these properties serve as generalized forms of those of the linear canonical transform (see [23]).

Definition 3.1. For a given function $f \in L^2(\mathbb{R})$, the definition of its OLCT with parameter

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \tau \\ \mu \end{bmatrix}$$

is

$$\mathcal{L}_O^{(M,\mu,\tau)}(f)(u) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(\vartheta)\, k_M(\vartheta, u)\, d\vartheta, & b \neq 0, \\ \sqrt{d}\, e^{\frac{cd}{2}(u-\tau)+iu\mu}\, f\Big(d(u-\tau)\Big), & b = 0, \end{cases}$$

where the kernel function $k_M(\vartheta, u)$ is

$$k_M(\vartheta, u) = \frac{1}{\sqrt{i2\pi}} e^{\frac{ia}{2b}\vartheta^2 - i\frac{\vartheta}{b}(u-\tau) - i\frac{u}{b}(d\tau - bu) + i\frac{d}{b}(u^2+\tau^2)}.$$

The parameters $a, b, c, d, \tau, \mu, u \in \mathbb{R}$, and satisfy $ad - bc = 1$.

The direct connection between the Fourier transform (FT) and the OLCT for the function $f \in L^2(\mathbb{R})$ is

$$\mathcal{F}(\tilde{f})\left(\frac{u}{b}\right) = \sqrt{i2\pi}\,\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)e^{i\frac{u}{b}(d\tau - bu) - i\frac{d}{b}(u^2+\tau^2)},$$

where $\tilde{f}(\vartheta) = f(\vartheta)\,e^{i\frac{a}{2b}\vartheta^2 + i\frac{\vartheta}{b}\tau}$.

Theorem 3.1. *For every $f \in L^1(\mathbb{R})$ and $\mathcal{F}_O^{(\alpha,m,n)}(f) \in L^1(\mathbb{R})$, the inverse of the OFrFT is given by*

$$f(\vartheta) = \int_{\mathbb{R}} \mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\,\overline{k_M(\vartheta, u)}\,d\eta.$$

Lemma 3.1. *For all functions $f, h \in L^2(\mathbb{R})$, the following relation holds:*

$$\int_{\mathbb{R}} f(\vartheta)\,\overline{h(\vartheta)}\,d\vartheta = \left\langle \mathcal{L}_O^{(M,\mu,\tau)}(f)(u),\, \mathcal{L}_O^{(M,\mu,\tau)}(h)(u) \right\rangle,$$

where

$$\|f\|^2_{L^2(\mathbb{R})} = \left\|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right\|^2_{L^2(\mathbb{R})}. \tag{3.1}$$

Now, we derive the Heisenberg uncertainty principle for the offset linear canonical transform (OLCT) using the classical Heisenberg uncertainty principle for the Fourier transform.

Lemma 3.2 (Heisenberg Uncertainty for OLCT)**.** *For any $f \in L^2(\mathbb{R})$, one has*

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\,d\vartheta\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^M(f)(u)\right|^2 du\right)^{1/2} \geq \frac{|b|^{3/2}\sqrt{\pi}}{4\sqrt{2}} \int_{\mathbb{R}} |f(\vartheta)|^2\,d\vartheta. \tag{3.2}$$

Proof. Using the Heisenberg uncertainty principle for the Fourier transform, we have

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\,d\vartheta\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 |\mathcal{F}(f)(u)|^2\,du\right)^{1/2} \geq \frac{\pi}{4} \int_{\mathbb{R}} |f(\vartheta)|^2\,d\vartheta.$$

By replacing $f(\vartheta)$ with $\tilde{f}(\vartheta)$ in the above equation, we obtain

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |\tilde{f}(\vartheta)|^2\,d\vartheta\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 |\mathcal{F}(\tilde{f})(u)|^2\,du\right)^{1/2} \geq \frac{\pi}{4} \int_{\mathbb{R}} |\tilde{f}(\vartheta)|^2\,d\vartheta.$$

Substituting u with u/b, we further get

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |\tilde{f}(\vartheta)|^2\,d\vartheta\right)^{1/2} \left(\frac{1}{b^3}\int_{\mathbb{R}} |u|^2 |\mathcal{F}(\tilde{f})(u/b)|^2\,du\right)^{1/2} \geq \frac{\pi}{4} \int_{\mathbb{R}} |\tilde{f}(\vartheta)|^2\,d\vartheta.$$

Therefore,

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\,d\vartheta\right)^{1/2} \left(\frac{1}{b^3}\int_{\mathbb{R}} |u|^2 \left|\sqrt{i2\pi}\,\mathcal{L}_O^{(M,\mu,\tau)}(f)(u) \quad e^{i\frac{u}{b}(d\tau - bu) - i\frac{d}{b}(u^2+\tau^2)}\right|^2 du\right)^{1/2}$$

$$\geq \frac{\pi}{4} \int_{\mathbb{R}} |f(\vartheta)|^2\,d\vartheta.$$

Or equivalently,

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\, d\vartheta\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 |\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)|^2\, du\right)^{1/2} \geq \frac{|b|^{3/2}\sqrt{\pi}}{4\sqrt{2}} \int_{\mathbb{R}} |f(\vartheta)|^2\, d\vartheta,$$

which finishes the proof. □

4. Continuous Offset Liniear Canonical Wavelet Transform (COLCWT) and Properties

We start this section by briefly reviewing the concept of the wavelet transform (see [12]). Let $b \in \mathbb{R}$ denote the translation parameter and $\gamma \in \mathbb{R}^+$ the scaling parameter. The translation and dilation operations applied to a function f are expressed as

$$\tau_\mu f(\vartheta) = f(\vartheta - \mu), \quad f_\gamma(\vartheta) = \gamma^{-1/2} f\left(\frac{\vartheta}{\gamma}\right),$$

respectively. The associated family of wavelets is given by

$$\varphi_{\gamma,\mu}(\vartheta) = \tau_\mu \varphi_\gamma(\vartheta) = \frac{1}{\sqrt{\gamma}}\, \varphi\left(\frac{\vartheta - \mu}{\gamma}\right), \tag{4.1}$$

which are generated from a mother (or basic) wavelet function $\psi \in L^2(\mathbb{R})$.

Consequently, the continuous wavelet transform of a function $f \in L^2(\mathbb{R})$ with respect to the mother wavelet ψ is defined as

$$(W_\varphi f)(\gamma, \mu) = \frac{1}{\sqrt{\gamma}} \int_{\mathbb{R}} f(\vartheta)\, \overline{\varphi\left(\frac{\vartheta - \mu}{\gamma}\right)}\, d\vartheta.$$

Let us now introduce the class of fractional mother wavelets defined by

$$\varphi_{\gamma,\mu}(\vartheta) = \frac{1}{\sqrt{\gamma}}\, \varphi\left(\frac{\vartheta - \mu}{\gamma}\right).$$

Definition 4.1 (COFRWT)**.** The COFRWT of $f \in L^2(\mathbb{R})$ with respect to a mother wavelet $\psi \in L^2(\mathbb{R})$ associated with the OFRFT is defined by

$$\begin{aligned}(\mathrm{OLW}_\varphi^{(\mu,\tau)} f)(\gamma, \mu) &= \frac{1}{\sqrt{\gamma}} \int_{\mathbb{R}} f(\vartheta)\, \overline{\varphi^\tau_{\gamma,\mu,M}(\vartheta)}\, d\vartheta \\ &= \frac{1}{\sqrt{\gamma}} \int_{\mathbb{R}} f(\vartheta)\, \overline{\varphi\left(\frac{\vartheta - \mu}{\gamma}\right)}\, e^{\frac{ia}{2b}(\vartheta^2 - \mu^2 - \tau^2)}\, d\vartheta. \end{aligned} \tag{4.2}$$

Example 4.1. Consider the function

$$f(\vartheta) = e^{-k\vartheta^2}, \quad \text{with } k > 0,$$

and

$$\varphi(\vartheta) = \begin{cases} 1, & |\vartheta| < 1, \\ 0, & \vartheta \text{ otherwise.} \end{cases}$$

So,

$$\varphi\left(\frac{\vartheta-\mu}{\gamma}\right)=\begin{cases}1, & -\gamma+\mu<\vartheta<\gamma+\mu,\\ 0, & \vartheta \text{ otherwise.}\end{cases}$$

Based on the (4.2) we have

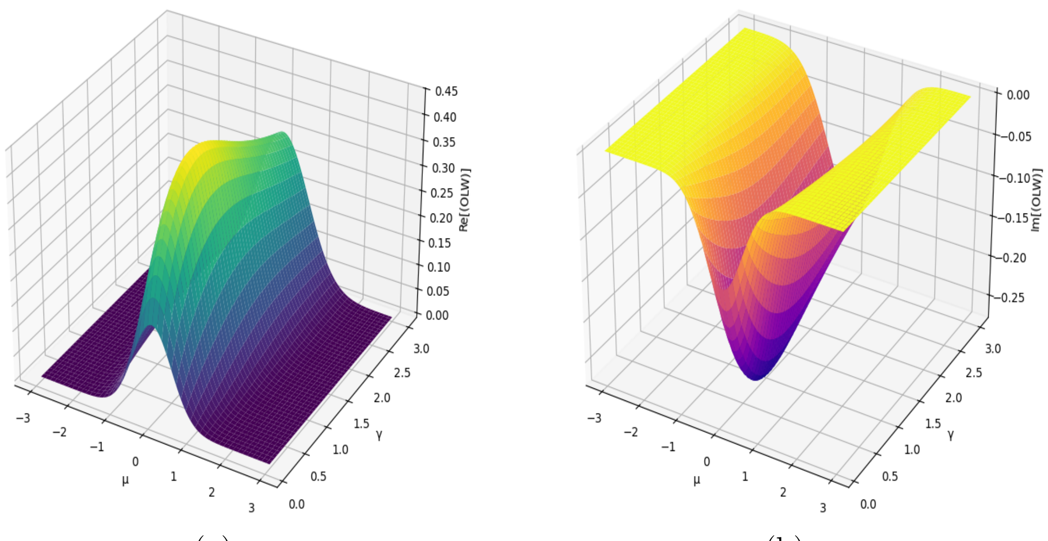
$$\begin{aligned}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)&=\frac{1}{\sqrt{\gamma}}\int_{\mu-\gamma}^{\mu+\gamma}e^{-k\vartheta^2}e^{\frac{ia}{b}(\vartheta^2-\mu^2+\tau^2)}d\vartheta\\ &=\frac{1}{\sqrt{\gamma}}\int_{\mu-\gamma}^{\mu+\gamma}e^{-(k+\frac{ia}{b})\vartheta^2-\mu^2-\tau^2}d\vartheta\\ &=\frac{1}{\sqrt{\gamma}}e^{-\mu^2-\tau^2}\int_{\mu-\gamma}^{\mu+\gamma}e^{-(k+\frac{ia}{b})\vartheta^2}d\vartheta.\end{aligned}$$

Therefore,

$$(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)=\frac{\sqrt{\pi}}{2\sqrt{\gamma(k+\frac{ia}{b})}}e^{-\mu^2-\tau^2}[\mathrm{erf}(\mu+\gamma)-\mathrm{erf}(\mu-\gamma)],$$

where

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt.$$



(a) (b)

FIGURE 1. (a) real part and (b) imaginary part of the COLCWT of Example 4.1 with $a=1$, $b=1$, $\tau=0.5$, $k=1$, $\mu=[-3,3]$, $\gamma=[0.1,3]$.

Lemma 4.1. *Let $\varphi, f\in L^2(\mathbb{R})$. Then,*

$$(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)=e^{-\frac{ia}{2b}(\mu^2+\tau^2)}\left(W_{\varphi}\tilde{f}\right)(a,b), \tag{4.3}$$

where

$$\tilde{f}(\vartheta) = f(\vartheta)\, e^{\frac{ia}{2b}\vartheta^2}. \tag{4.4}$$

Definition 4.2. We say a mother wavelet $\psi \in L^2(\mathbb{R})$ associated with the OFrFT is *admissible* if and only if the following admissibility constant is satisfied:

$$0 < C_{\varphi}^{(\mu,\tau)} = \int_{\mathbb{R}^+} \left| L_O^{(M,\mu,\gamma\tau)}\left[\varphi(y)\, e^{-\frac{ia}{2b}\vartheta^2}\right](\gamma u)\right|^2, \quad \frac{d\gamma}{\gamma} < +\infty. \tag{4.5}$$

In this case, $C_{\varphi}^{(\mu,\tau)}$ is a real positive constant independent of η satisfying $|u| = 1$.

Lemma 4.2. *Let ψ be a mother wavelet. Then, the family of the fractional mother wavelets can be expressed in terms of the OLCT as*

$$\begin{aligned} L_O^{(M,\mu,\tau)}(\varphi^{\tau}_{\gamma,\mu,M})(u) =& \frac{1}{\sqrt{i2\pi}}\sqrt{\gamma}\, e^{\frac{ia}{2b}(\mu^2+\tau^2)}\, e^{-\frac{i\mu}{b}(u-\tau)} \\ & \times e^{-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)+\frac{iu}{b}(d(\gamma\tau)-b(\gamma u))-\frac{id}{b}((u\gamma)^2+(\tau\gamma)^2)} \\ & \times L_O^{(M,\mu,\gamma\tau)}\left[\varphi(y)\, e^{-\frac{ia}{2b}\vartheta^2}\right](\gamma u). \end{aligned} \tag{4.6}$$

Proof. By direct computation, we have

$$\begin{aligned} & L_O^{(M,\mu,\tau)}(\varphi^{\tau}_{\gamma,\mu,M})(u) \\ =& \int_{\mathbb{R}} \varphi^{\tau}_{\gamma,\mu,M}(\vartheta)\, k_M(\vartheta,u)\, d\vartheta \\ =& \int_{\mathbb{R}} \frac{1}{\sqrt{i2\pi}}\frac{1}{\sqrt{\gamma}}\,\varphi\left(\frac{\vartheta-\mu}{\gamma}\right) e^{-\frac{ia}{2b}(\vartheta^2-\mu^2-\tau^2)} e^{\frac{ia}{2b}\vartheta^2-\frac{i\vartheta}{b}(u-\tau)-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)}\, d\vartheta \\ =& \frac{1}{\sqrt{i2\pi}\sqrt{\gamma}}\int_{\mathbb{R}} \varphi\left(\frac{\vartheta-\mu}{\gamma}\right) e^{\frac{ia}{2b}(\mu^2+\tau^2)} e^{-\frac{i\vartheta}{b}(u-\tau)-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)}\, d\vartheta. \end{aligned}$$

Now, substituting $y = \frac{\vartheta-\mu}{\gamma}$ (i.e., $\vartheta = y\gamma+\mu$), we obtain

$$\begin{aligned} & L_O^{(M,\mu,\tau)}(\varphi^{\tau}_{\gamma,\mu,M})(u) \\ =& \frac{1}{\sqrt{i2\pi}}\sqrt{\gamma}\int_{\mathbb{R}} \varphi(y) e^{\frac{ia}{2b}(\mu^2+\tau^2)} e^{-\frac{i(y\gamma+\mu)}{b}(u-\tau)-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)}\, dy \\ =& \frac{1}{\sqrt{i2\pi}}\sqrt{\gamma}\, e^{\frac{ia}{2b}(\mu^2+\tau^2)}\, e^{-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)+\frac{iu}{b}(d(\gamma\tau)-b(\gamma u))-\frac{id}{b}((u\gamma)^2+(\tau\gamma)^2)} \\ & \times e^{-\frac{i\mu}{b}(u-\tau)} \times \int_{\mathbb{R}} \left[\varphi(y) e^{-\frac{ia}{2b}\vartheta^2}\right] e^{\frac{ia}{2b}\vartheta^2-\frac{iy}{b}(\gamma u-\gamma\tau)-\frac{iu}{b}(d(\gamma\tau)-b(\gamma u))+\frac{id}{b}((u\gamma)^2+(\tau\gamma)^2)}\, dy. \end{aligned}$$

Therefore,

$$\begin{aligned} L_O^{(M,\mu,\tau)}(\varphi^{\tau}_{\gamma,\mu,M})(u) =& \frac{1}{\sqrt{i2\pi}}\sqrt{\gamma}\, e^{-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)+\frac{iu}{b}(d(\gamma\tau)-b(\gamma u))-\frac{id}{b}((u\gamma)^2+(\tau\gamma)^2)} \\ & \times e^{\frac{ia}{2b}(\mu^2+\tau^2)}\, e^{-\frac{i\mu}{b}(u-\tau)} \times L_O^{(M,\mu,\gamma\tau)}\left[\varphi(y) e^{-\frac{ia}{2b}\vartheta^2}\right](\gamma u), \end{aligned}$$

which finishes the proof. □

Lemma 4.3. *Suppose that the admissibility condition in classical wavelet transform is*

$$C_\varphi = \int_{\mathbb{R}^+} \frac{|\mathcal{F}(\varphi)(\gamma u)|^2}{\gamma}\, d\gamma < +\infty. \tag{4.7}$$

The relationship between (4.7) *and* (4.10) *is described by*

$$C_\varphi = 2\pi\, C_\varphi^{(\mu,\tau)}. \tag{4.8}$$

Proof. Based on Lemma 4.6, we get

$$C_\varphi^{(\mu,\tau)} = \int_{\mathbb{R}^+} \Bigg| \frac{1}{\sqrt{i2\pi}} \sqrt{\gamma}\, e^{-\frac{iu}{b}(d\tau - bu) + \frac{id}{b}(u^2+\tau^2) + \frac{iu}{b}(d(\gamma\tau) - b(\gamma u)) - \frac{id}{b}((u\gamma)^2 + (\tau\gamma)^2)} \times e^{\frac{ia}{2b}(\mu^2+\tau^2)} e^{-\frac{i\mu}{b}(u-\tau)} \Bigg|^2 \cdot \left| L_O^{(M,\mu,\gamma\tau)} \left\{ \varphi(y) e^{-\frac{ia}{2b}\vartheta^2} \right\} (\gamma u) \right|^2 \frac{d\gamma}{\gamma^2}.$$

Since the exponential terms have modulus one, it follows that

$$C_\varphi^{(\mu,\tau)} = \frac{1}{2\pi} \int_{\mathbb{R}^+} \left| L_O^{(M,\mu,\gamma\tau)} \left[\varphi(y) e^{-\frac{ia}{2b}\vartheta^2} \right] (\gamma u) \right|^2 \frac{d\gamma}{\gamma}. \tag{4.9}$$

The proof is complete. □

Remark 4.1. If $\mu = \tau = 0$, (4.9) becomes

$$C_\varphi = \frac{1}{2\pi} \int_{\mathbb{R}^+} \left| L_O^{M} \left[\varphi(y) e^{-\frac{ia}{2b}\vartheta^2} \right] (\gamma u) \right|^2 \frac{d\gamma}{\gamma}. \tag{4.10}$$

Note that the Fourier transform of (3.1) takes the form

$$\varphi_{\gamma,\mu} = \sqrt{\gamma}\, e^{-i\mu u}\, \mathcal{F}(\varphi)(\gamma u).$$

Lemma 4.4. *Let* $\psi \in L^2(\mathbb{R})$. *Then, the COFRWT* (4.2) *has a fractional Fourier representation form*

$$\begin{aligned} L_O^{(M,\mu,\tau)} \left(\mathrm{OLW}_\varphi^{(\mu,\tau)} f \right)(u) =& \sqrt{\gamma} \left(\sqrt{i2\pi} \right) e^{\frac{ia}{2b}\tau^2 - \frac{iu}{b}(d(\gamma\tau) - b(\gamma u)) + \frac{id}{b}((u\gamma)^2 + (\tau\gamma)^2)} \\ & \times \overline{L_O^{(M,\mu,\gamma\tau)} \left[\varphi(y) e^{-\frac{ia}{2b}\vartheta^2} \right] (\gamma u)} L_O^{(M,\mu,\tau)}(f)(u). \end{aligned} \tag{4.11}$$

Proof. By direct computation, we get

$$\begin{aligned} L_O^{(M,\mu,\tau)} \left(\mathrm{OLW}_\varphi^{(\mu,\tau)} f \right)(u) &= \int_{\mathbb{R}} \left(\mathrm{OLW}_\varphi^{(\mu,\tau)} f \right)(\gamma,\mu)\, k_M(\vartheta,u)\, d\vartheta \\ &= \frac{1}{\sqrt{\gamma}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\vartheta)\, \overline{\varphi\left(\frac{\vartheta - \mu}{\gamma} \right)} \\ &\quad \times e^{-\frac{ia}{2b}(\vartheta^2 - \mu^2 - \tau^2)} e^{\frac{ia}{2b}\vartheta^2 - \frac{i\vartheta}{b}(u-\tau) - \frac{iu}{b}(d\tau - bu) + \frac{id}{b}(u^2+\tau^2)}\, d\mu\, d\vartheta \\ &= \frac{1}{\sqrt{\gamma}\sqrt{i2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\vartheta)\, \overline{\varphi\left(\frac{\vartheta - \mu}{\gamma} \right)} \\ &\quad \times e^{\frac{ia}{2b}(\mu^2+\tau^2)} e^{-\frac{i\vartheta}{b}(u-\tau) - \frac{iu}{b}(d\tau - bu) + \frac{id}{b}(u^2+\tau^2)}\, d\mu\, d\vartheta. \end{aligned}$$

Substituting $y = \frac{\vartheta-\mu}{\gamma}$, $\vartheta = y\gamma + \mu$, yields

$$L_O^{(M,\mu,\tau)}\left[\mathrm{OLW}_\varphi^{(\mu,\tau)} f\right](u) = \frac{1}{\sqrt{\gamma}\sqrt{i2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} f(\mu)\,\overline{\varphi(y)} e^{\frac{ia}{2b}(\mu^2+\tau^2)} \times e^{-\frac{i(y\gamma+\mu)}{b}(u-\tau)-\frac{iu}{b}(d\tau-bu)+\frac{id}{b}(u^2+\tau^2)}\,dy\,d\mu.$$

Therefore,

$$L_O^{(M,\mu,\tau)}\left[\mathrm{OLW}_\varphi^{(\mu,\tau)} f\right](u) = \sqrt{\gamma}\,\overline{\left(\sqrt{i2\pi}\right)} e^{\frac{ia}{2b}\tau^2-\frac{iu}{b}(d(\gamma\tau)-b(\gamma u))+\frac{id}{b}((u\gamma)^2+(\tau\gamma)^2)} \times \overline{L_O^{(M,\mu,\gamma\tau)}\left[\varphi(y)e^{-\frac{ia}{2b}\vartheta^2}\right](\gamma u)}\,L_O^{(M,\mu,\tau)}(f)(u),$$

which finishes the proof. □

The following discussion is aimed at deriving the inversion formula and establishing the orthogonality relation for the COLCWT by employing the direct correspondence between the offset linear canonical wavelet transform and the standard wavelet transform.

Theorem 4.1. *Let $\varphi \in L^2(\mathbb{R})$ satisfy the admissibility condition defined in* (4.5)*. Then, for any $f \in L^2(\mathbb{R})$, the inversion formula is given by*

$$f(\vartheta) = \frac{1}{4\pi^2\,\mathbb{C}_\varphi^{(\mu,\tau)}} \int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{-\frac{i}{2}(z^2-b^2-m^2)}\,\varphi_{(\gamma,\mu)}(\vartheta)\left(\mathrm{OLW}_\varphi^{(\mu,\tau)} f\right)(\gamma,\mu)\,\frac{d\mu\,d\gamma}{\gamma^2}.$$

Proof. Since $\tilde{f}(\vartheta) \in L^2(\mathbb{R})$, the inversion formula for the WFT can be expressed as

$$\tilde{f}(\vartheta) = \frac{1}{2\pi\,\mathbb{C}_\psi} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi_{(\gamma,\mu)}(\vartheta)\left(W_\varphi \tilde{f}\right)(\gamma,\mu)\,\frac{d\mu\,d\gamma}{\gamma^2}.$$

The expression above equals to

$$f(\vartheta)\,e^{\frac{ia}{2b}\vartheta^2} = \frac{1}{2\pi\,\mathbb{C}_\varphi} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi_{(\gamma,\mu)}(\vartheta)\left(W_\varphi \tilde{f}\right)(\gamma,\mu)\,\frac{d\mu\,d\gamma}{\gamma^2}.$$

We can rewrite this equation in the form

$$f(\vartheta) = \frac{1}{2\pi\,\mathbb{C}_\varphi} \int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{-\frac{ia}{2b}(\vartheta^2-\mu^2-\tau^2)}\,\varphi_{(\gamma,\mu)}(\vartheta)\,e^{-\frac{ia}{2b}(\mu^2+\tau^2)}\left(W_\varphi \tilde{f}\right)(\gamma,\mu)\,\frac{d\mu\,d\gamma}{\gamma^2}.$$

Using (4.3) and (4.5), we can see that

$$f(\vartheta) = \frac{1}{4\pi^2\,\mathbb{C}_\varphi^{(\mu,\tau)}} \int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{-\frac{i}{2}(z^2-b^2-m^2)}\,\varphi_{(\gamma,\mu)}(\vartheta)\left(\mathrm{OLW}_\varphi^{(\mu,\tau)} f\right)(\gamma,\mu)\,\frac{d\mu\,d\gamma}{\gamma^2},$$

which finishes the proof. □

Theorem 4.2. *Let the basic wavelet $\psi \in L^2(\mathbb{R})$, associated with the Offset Fractional Fourier Transform (OFrFT), satisfy the admissibility condition given in (22). Then, for any functions $f, g \in L^2(\mathbb{R})$, the following orthogonality relations hold:*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\mathrm{OLW}_\varphi^{(\mu,\tau)} f\right)(\gamma,\mu)\overline{\left(\mathrm{OLW}_\varphi^{(\mu,\tau)} g\right)(\gamma,\mu)}\,\frac{d\mu\,d\gamma}{\gamma^2} = 4\pi^2\,\mathbb{C}_\varphi^{(\mu,\tau)}\,\langle f, g\rangle_{L^2(\mathbb{R})}$$

and

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\left|(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)\right|^2\frac{d\mu\,d\gamma}{\gamma^2}=4\pi^2\,\mathbb{C}_{\varphi}^{(\mu,\tau)}\,\|f\|^2_{L^2(\mathbb{R})}.$$

Proof. Because $\tilde{f}(\vartheta)$ and $\tilde{h}(\vartheta)$ defined by (4.4) are in $L^1(\mathbb{R})$, then the orthogonality for the classical wavelet transform implies that

$$\int_{\mathbb{R}^+}\left(\int_{\mathbb{R}}(W_{\varphi}\tilde{f})(\gamma,\mu)\,\overline{(W_{\varphi}\tilde{h})(\gamma,\mu)}\,d\mu\right)\frac{d\gamma}{\gamma^2}=2\pi\,\mathbb{C}_{\varphi}\,\langle\tilde{f},\tilde{h}\rangle_{L^2(\mathbb{R})},$$

where $\mathbb{C}_{\varphi}$ is given by (4.7). The above identity can be written in the form

$$\int_{\mathbb{R}^+}\left(\int_{\mathbb{R}}e^{-\frac{ia}{2b}(\mu^2+\tau^2)}(W_{\varphi}\tilde{f})(\gamma,\mu)\,\overline{e^{-\frac{ia}{2b}(\mu^2+\tau^2)}(W_{\varphi}\tilde{h})(\gamma,\mu)}\,d\mu\right)\frac{d\gamma}{\gamma^2}$$
$$=2\pi\,\mathbb{C}_{\varphi}\,\langle\tilde{f},\tilde{h}\rangle_{L^2(\mathbb{R})},$$

which gives

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)\,\overline{(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}g)(\gamma,\mu)}\,\frac{d\mu\,d\gamma}{\gamma^2}=2\pi\,\mathbb{C}_{\varphi}\int_{\mathbb{R}}f(z)\,\overline{g(z)}\,dz$$
$$=4\pi^2\,\mathbb{C}_{\varphi}^{(\mu,\tau)}\,\langle f,g\rangle_{L^2(\mathbb{R})},$$

which finishes the proof. □

5. Uncertainty Principle for Continuous Offset Linear Canonical Wavelet Transform (COLCWT)

In this section, our attention is directed toward formulating several uncertainty principles linked to the offset fractional Fourier transform. We further establish a Heisenberg-type uncertainty principle for the COLCWT.

5.1. **Heisenberg-Type Uncertainty Principle of the COLCWT.** In this part, we derive an analogue of Heisenberg-type uncertainty principle for the COFRFT. For this purpose, we need to introduce the following lemma.

Lemma 5.1 (Heisenberg Uncertainty Principle for COFRFT)**.** *Suppose that the functions $f,\varphi\in L^2(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(u)\right|^2\frac{d\mu\,d\gamma}{\gamma^2}\leq 4\pi^2\,\mathbb{C}_{\varphi}^{(\mu,\tau)}\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2du.$$

Proof. In fact, we have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(u)\right|^2\frac{d\mu\,d\gamma}{\gamma^2}$$
$$\leq\int_{\mathbb{R}^+}\int_{\mathbb{R}}|u|^2\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(u)\,\overline{\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(u)}\frac{d\mu\,d\gamma}{\gamma^2}.$$

The use of (4.11), (4.5), and (3.1) gives

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(u)\right|^2\frac{d\mu\,d\gamma}{\gamma^2}$$

$$\leq 2\pi \int_{\mathbb{R}^+}\int_{\mathbb{R}} \gamma |u|^2 \overline{\mathcal{L}_O^{(M,\mu,\gamma\tau)}(\varphi(y)e^{-i\frac{a}{2b}\vartheta^2})(\gamma u)}\, \mathcal{L}_O^{(M,\mu,\tau)}(f)(u)$$
$$\times \overline{\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)}\, \mathcal{L}_O^{(M,\mu,\gamma\tau)}(\varphi(y)e^{-i\frac{a}{2b}\vartheta^2})(\gamma u) \frac{d\mu\, d\gamma}{\gamma^2}.$$

Hence,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_\varphi^{(\mu,\tau)} f)(u)\right|^2 \frac{d\mu\, d\gamma}{\gamma^2}$$
$$\leq 2\pi \int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2 \left(\int_{\mathbb{R}^+} \left|\mathcal{L}_O^{(M,\mu,\gamma\tau)}(\varphi(y)e^{-i\frac{a}{2b}\vartheta^2})(\gamma u)\right|^2 \frac{d\gamma}{\gamma}\right) du.$$

Finally,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(\mathrm{OLW}_\varphi^{(\mu,\tau)} f)(u)\right|^2 \frac{d\mu\, d\gamma}{\gamma^2} \leq 4\pi^2\, \mathbb{C}_\varphi^{(\mu,\tau)} \int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2 du,$$

which is the desired result. □

Theorem 5.1. *Suppose that the functions $f, \varphi \in L^2(\mathbb{R})$. Then, one has*

$$\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\vartheta|^2 \left|(\mathrm{OLW}_\varphi^{(\mu,\tau)} f)(\gamma,\mu)\right|^2 d\vartheta\, \frac{d\gamma}{\gamma^2}\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2 du\right)^{1/2}$$
$$\geq \frac{|b|^{3/2}\pi^2}{2} \sqrt{\mathbb{C}_\varphi^{(\mu,\tau)}}\, \|f\|_{L^2(\mathbb{R})}.$$

Proof. With the aid of the Heisenberg uncertainty principle for the OLCT in Equation (3.2), we have

$$\left(\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\, d\vartheta\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2 du\right)^{1/2} \geq \frac{|b|^{3/2}\sqrt{\pi}}{4\sqrt{2}} \int_{\mathbb{R}} |f(\vartheta)|^2\, d\vartheta. \tag{5.1}$$

Integrating both sides of (5.1) with respect to the measure $d\gamma/\gamma^2$, we immediately obtain

$$\int_{\mathbb{R}^+} \left\{ \left(\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\, d\vartheta\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2 du\right)^{1/2} \right\} \frac{d\gamma}{\gamma^2}$$
$$\geq \frac{|b|^{3/2}\sqrt{\pi}}{4\sqrt{2}} \int_{\mathbb{R}^+}\int_{\mathbb{R}} |f(\vartheta)|^2\, d\vartheta\, \frac{d\gamma}{\gamma^2}.$$

Therefore,

$$\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2\, d\vartheta\, \frac{d\gamma}{\gamma^2}\right)^{1/2} \left(\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u|^2 \left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2 du\, \frac{d\gamma}{\gamma^2}\right)^{1/2}$$
$$\geq \frac{|b|^{3/2}\sqrt{\pi}}{4\sqrt{2}} \int_{\mathbb{R}^+}\int_{\mathbb{R}} |f(\vartheta)|^2\, d\vartheta\, \frac{d\gamma}{\gamma^2}. \tag{5.2}$$

Now, replacing the function $f(\cdot)$ by $(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)$ on both sides of (5.2), we get

$$\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)})(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu))(u)\right|^2du\,\frac{d\gamma}{\gamma^2}\right)^{1/2}$$

$$\times\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}}|\vartheta|^2\left|(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)\right|^2d\vartheta\,\frac{d\gamma}{\gamma^2}\right)^{1/2}$$

$$\geq\frac{|b|^{3/2}\sqrt{\pi}}{4\sqrt{2}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\left|(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)\right|^2d\vartheta\,\frac{d\gamma}{\gamma^2}. \tag{5.3}$$

Using (4.11) and (4.8) into equation (5.3), we get

$$\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}}|\vartheta|^2\left|(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)\right|^2d\vartheta\,\frac{d\gamma}{\gamma^2}\right)^{1/2}$$

$$\times\left(2\pi\mathbb{C}_{\varphi}^{(\mu,\tau)}\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2du\right)^{1/2}$$

$$\geq\frac{|b|^{3/2}\pi^{5/2}}{\sqrt{2}}\mathbb{C}_{\varphi}^{(\mu,\tau)}\|f\|_{L^2(\mathbb{R})}.$$

This equation is equivalent to

$$\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}}|\vartheta|^2\left|(\mathrm{OLW}_{\varphi}^{(\mu,\tau)}f)(\gamma,\mu)\right|^2d\vartheta\,\frac{d\gamma}{\gamma^2}\right)^{1/2}$$

$$\times\left(\int_{\mathbb{R}}|u|^2\left|\mathcal{L}_O^{(M,\mu,\tau)}(f)(u)\right|^2du\right)^{1/2}$$

$$\geq\frac{|b|^{3/2}\pi^2}{2}\sqrt{\mathbb{C}_{\varphi}^{(\mu,\tau)}}\,\|f\|_{L^2(\mathbb{R})}.$$

Thus, the proof is complete. □

A simple modification of theorem yields the following variant.

Lemma 5.2. *For any $f\in L^2(\mathbb{R})$ and any $p,q\geq 1$, one has*

$$\left(\int_{\mathbb{R}}|u|^{2p}|L_O^{(M,\mu,\tau)}(f)(u)|^2\,du\right)^{\frac{q}{q+p}}\left(\int_{\mathbb{R}}|\vartheta|^{2q}|f(\vartheta)|^2\,d\vartheta\right)^{\frac{p}{q+p}}\geq\left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}}\|f\|^2_{L^2(\mathbb{R})}.$$

Proof. It follows from Hölder inequality and Plancherel theorem for OFRFT (4.1) that

$$\begin{aligned}\int_{\mathbb{R}}|u|^2|L_O^{(M,\mu,\tau)}(f)(u)|^2\,du&=\int_{\mathbb{R}}|u|^2|L_O^{(M,\mu,\tau)}(f)(u)|^{2/p}|L_O^{(M,\mu,\tau)}(f)(u)|^{2-2/p}\,du\\&\leq\left(\int_{\mathbb{R}}(|u|^2|L_O^{(M,\mu,\tau)}(f)(u)|^{2/p})^p\,d\eta\right)^{1/p}\\&\quad\times\left(\int_{\mathbb{R}}(|L_O^{(M,\mu,\tau)}(f)(u)|^{2-2/p})^{p/(p-1)}\,d\eta\right)^{(p-1)/p}\end{aligned}$$

$$= \left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \times \left(\int_{\mathbb{R}} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, d\eta \right)^{(p-1)/p}$$
$$= \left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \left(\int_{\mathbb{R}} |f(\vartheta)|^2 \, d\vartheta \right)^{(p-1)/p}$$
$$= \left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \|f\|_{L^2(\mathbb{R})}^{2((p-1)/p)}.$$

Hence,

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \geq \frac{\int_{\mathbb{R}} |u|^2 |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du}{\|f\|_{L^2(\mathbb{R})}^{2((p-1)/p)}}. \tag{5.4}$$

In a similar way, we get

$$\left(\int_{\mathbb{R}} |\vartheta|^{2q} |f(\vartheta)|^2 \, d\vartheta \right)^{1/q} \geq \frac{\int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2 \, d\vartheta}{\|f\|_{L^2(\mathbb{R})}^{2((q-1)/q)}}. \tag{5.5}$$

By combining (5.4) and (5.5) we obtain

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \left(\int_{\mathbb{R}} |\vartheta|^{2q} |f(\vartheta)|^2 \, d\vartheta \right)^{1/q}$$
$$\geq \frac{\int_{\mathbb{R}} |u|^2 |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \int_{\mathbb{R}} |\vartheta|^2 |f(\vartheta)|^2 \, d\vartheta}{\|f\|_{L^2(\mathbb{R})}^{2((p-1)/p)+2((q-1)/q)}}. \tag{5.6}$$

Applying (2.1) in the right-hand side of equation (5.6), we immediately get

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \left(\int_{\mathbb{R}} |\vartheta|^{2q} |f(\vartheta)|^2 d\vartheta \right)^{1/q}$$
$$\geq \frac{(|b|^3 \sqrt{\pi})/32 \left(\int_{\mathbb{R}} |f(\vartheta)|^2 d\vartheta \right)^2}{\|f\|_{L^2(\mathbb{R})}^{2((p-1)/p)+2((q-1)/q)}}$$
$$= \frac{|b|^3 \sqrt{\pi}}{32} \|f\|_{L^2(\mathbb{R})}^{4-2((p-1)/p)-2((q-1)/q)}.$$

Therefore,

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{1/p} \left(\int_{\mathbb{R}} |\vartheta|^{2q} |f(\vartheta)|^2 \, d\vartheta \right)^{1/q} \geq \frac{|b|^3 \sqrt{\pi}}{32} \|f\|_{L^2(\mathbb{R})}^{2((p+q)/pq)}.$$

Or, equivalently,

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du \right)^{\frac{q}{q+p}} \left(\int_{\mathbb{R}} |\vartheta|^{2q} |f(\vartheta)|^2 \, d\vartheta \right)^{\frac{p}{q+p}} \geq \left(\frac{|b|^3 \sqrt{\pi}}{32} \right)^{\frac{pq}{p+q}} \|f\|_{L^2(\mathbb{R})}^2.$$

Thus, the proof is complete. □

The author in [20] introduced a Heisenberg-type certainty principle formulated within the framework of the shearlet transform. The following theorem provides an analogous result for the case of the COLCWT.

Theorem 5.2. *Let $\psi \in L^2(\mathbb{R})$ be an offset fractional admissible wavelet. Then, for every $f \in L^2(\mathbb{R})$ and for all $p \in [1, 2]$, the following inequality holds:*

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du\right)^{\frac{q}{q+p}} \left(\int_{\mathbb{R}^+} \int_{\mathbb{R}} |\vartheta|^{2q} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)|^2 \, d\vartheta \frac{d\gamma}{\gamma^2}\right)^{\frac{p}{q+p}}$$
$$\geq \left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \left(2\pi \mathbb{C}_\varphi^{(\mu,\tau)}\right)^{\frac{p}{q+p}} \|f\|_{L^2(\mathbb{R})}.$$

Proof. Based on (3.2), we have

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du\right)^{\frac{q}{q+p}} \left(\int_{\mathbb{R}} |\vartheta|^{2q} |f(\vartheta)|^2 \, d\vartheta\right)^{\frac{p}{q+p}}$$
$$\geq \left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \int_{\mathbb{R}} |f(\vartheta)|^2 \, d\vartheta. \tag{5.7}$$

Replacing the function $f(\cdot)$ by $(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)$ on both sides of Equation (5.7), we get

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}((OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu))(u)|^2 \, du\right)^{\frac{q}{q+p}}$$
$$\times \left(\int_{\mathbb{R}} |\vartheta|^{2q} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)|^2 \, d\vartheta\right)^{\frac{p}{q+p}}$$
$$\geq \left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \int_{\mathbb{R}} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)|^2 \, d\vartheta. \tag{5.8}$$

Integrating both sides of equation (5.8) with respect to the measure $d\gamma/\gamma^2$, we obtain

$$\left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)|^2 \, d\vartheta \frac{d\gamma}{\gamma^2}$$
$$\leq \int_{\mathbb{R}^+} \left\{ \left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}((OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu))(u)|^2 \, du\right)^{\frac{q}{q+p}} \right.$$
$$\left. \times \left(\int_{\mathbb{R}} |\vartheta|^{2q} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)|^2 \, d\vartheta\right)^{\frac{p}{q+p}} \right\} \frac{d\gamma}{\gamma^2}.$$

Therefore,

$$\left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma, \mu)|^2 \, d\vartheta \frac{d\gamma}{\gamma^2}$$

$$\leq \left(\int_{\mathbb{R}} |u|^{2p} \left(\int_{\mathbb{R}^+} |L_O^{(M,\mu,\tau)}((OLW_\varphi^{(\mu,\tau)} f)(\gamma,\mu))(u)|^2 \frac{d\gamma}{\gamma^2}\right) du\right)^{\frac{q}{q+p}}$$
$$\times \left(\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\vartheta|^{2q} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma,\mu)|^2 \, d\vartheta \frac{d\gamma}{\gamma^2}\right)^{\frac{p}{q+p}}.$$

Using (4.11) and (4.8) into the above, we get

$$2\pi \left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \mathbb{C}_\varphi^{(\mu,\tau)} \|f\|_{L^2(\mathbb{R})} \leq \left(2\pi \mathbb{C}_\varphi^{(\mu,\tau)} \int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du\right)^{\frac{q}{q+p}}$$
$$\times \left(\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\vartheta|^{2q} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma,\mu)|^2 \, d\vartheta \frac{d\gamma}{\gamma^2}\right)^{\frac{p}{q+p}}.$$

Hence,

$$\left(\int_{\mathbb{R}} |u|^{2p} |L_O^{(M,\mu,\tau)}(f)(u)|^2 \, du\right)^{\frac{q}{q+p}} \left(\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\vartheta|^{2q} |(OLW_\varphi^{(\mu,\tau)} f)(\gamma,\mu)|^2 \, d\vartheta \frac{d\gamma}{\gamma^2}\right)^{\frac{p}{q+p}}$$
$$\geq \left(\frac{|b|^3\sqrt{\pi}}{32}\right)^{\frac{pq}{p+q}} \left(2\pi \mathbb{C}_\varphi^{(\mu,\tau)}\right)^{\frac{p}{q+p}} \|f\|_{L^2(\mathbb{R})}.$$

This completes the proof. □

6. Conclusion

In this work, we have investigated some fundamental properties of the offset linear canonical transform like the orthogonality relation and the inversion formula using direct interaction between the COLCWT and the WT. Based on some Heisenberg uncertainty principle related to the OLCT we have derived the Heisenberg uncertainty principle in context of the COLCWT.

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