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DISCRETE LOCAL FRACTIONAL HILBERT-TYPE INEQUALITIES

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Abstract. The main objective of this paper is a study of some new discrete local fractional Hilbert-type inequalities. We apply our general results to homogeneous kernels. Also, the obtained results have the best possible constants.

1. Introduction

If $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^{+\infty} f^2(x)dx < +\infty$ and $0 < \int_0^{+\infty} g^2(x)dx < +\infty$, then we have (see [1]):

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}dxdy \leq \pi\left(\int_0^{+\infty} f^2(x)dx\int_0^{+\infty} g^2(y)dy\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant π is the best possible. The inequality (1.1) is well known as Hilbert's integral inequality, which is important in mathematical analysis and its applications.

Over the last ten years, by using the kinds of generalized fractional integral operators, a great deal of fractional integral inequalities have been presented [2–5]. Recently, local fractional calculus has caused widespread attention from many scholars, we give basic definitions and results of the local fractional calculus (see [6–13]). Based on the local fractal identity and the generalized p-convexity, some novel Newton's type variants for the local differentiable functions were obtained in the paper [14]. Sarikaya et al. [15] established the generalized Grüss type inequality and some generalized Čebyšev type inequalities for local fractional integrals on fractal sets. Acorrding to the identity

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involving local fractional integrals, Iftikhar et al. [16] presented some new Newton-type inequalities for functions with the local fractional derivatives. By employing the local fractional integrals, Akkurt et al. [17] investigated the generalized Ostrowski type integral inequalities involving moments of continuous random variables. Sarikaya and Budak [18] gave a generalized Ostrowski inequality and some new inequalities using the generalized convex function for local fractional integrals on fractal sets. Based on two local fractional integral operators with Mittag-Leffler kernel, Sun [19] obtained some Hermite-Hadamard and Hermite-Hadamard-Fejér-type local fractional integral inequalities for generalized preinvex functions on Yang's fractal sets.

For the sake of convenience, we recall Yang's fractal set Ω^α, where the set Ω is called base set of fractional set, and α denotes the dimension of cantor set, $0 < \alpha \leq 1$. The α-type set of integers $\mathbb{Z}^\alpha$ is defined by (see [6–8])

$$\mathbb{Z}^\alpha := \{0^\alpha\} \cup \{\pm m^\alpha : m \in \mathbb{N}\}.$$

The α-type set of rational numbers $\mathbb{Q}^\alpha$ is defined by

$$\mathbb{Q}^\alpha := \{q^\alpha : q \in \mathbb{Q}\} = \left\{ \left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

The α-type set of irrational numbers $\mathbb{J}^\alpha$ is defined by

$$\mathbb{J}^\alpha := \{r^\alpha : r \in \mathbb{J}\} = \left\{ r^\alpha \neq \left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

The α-type set of real line numbers $\mathbb{R}^\alpha$ is defined by

$$\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha.$$

Some basic operation rules on $\mathbb{R}^\alpha$ are presented as follows: If a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$, then

(a1) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$, $a^\alpha b^\alpha \in \mathbb{R}^\alpha$;
(a2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
(a3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
(a4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
(a5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
(a6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
(a7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$;
(a8) for each $a^\alpha \in \mathbb{R}^\alpha$, its inverse element $(-a^\alpha)$ may be written as $-a^\alpha$; for each $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, its inverse element $(1/b)^\alpha$ may be written as $1^\alpha / b^\alpha$ but not as $1/b^\alpha$;
(a9) $a^\alpha < b^\alpha$ if and only if $a < b$;
(a10) $a^\alpha = b^\alpha$ if and only if $a = b$.

Further, we define the local fractional derivative and integral.

Definition 1.1. A non-differentiable function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each $\varepsilon > 0$, there exists for $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha,$$

holds for $0 < |x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote $f \in C_\alpha(a, b)$.

Definition 1.2. Let $f(x) \in C_\alpha[a, b]$. Local fractional derivative of the function $f(x)$ at $x = x_0$ is given by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^\alpha}.$$

Definition 1.3. Let $f(x) \in C_\alpha[a, b]$ and let $P = \{t_0, t_1, \ldots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Further, for this partition P, let $\Delta t_j = t_{j+1} - t_j$, $j = 0, \ldots, N-1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\}$. Then the local fractional integral of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^\alpha f(x)$) is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha.$$

The above definition implies that ${}_aI_b^{(\alpha)} f(x) = 0$ if $a = b$, and ${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{(\alpha)} f(x)$, then we denote by $f(x) \in I_x^{(\alpha)}[a, b]$.

At the end of this summary, we give some useful formulas:

(b1) $\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$, $k > 0$;
(b2) $\frac{d^\alpha E_\alpha((cx)^\alpha)}{dx^\alpha} = c^\alpha E_\alpha((cx)^\alpha)$, where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function given by $E_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$;
(b3) If $y(x) = (f \circ g)(x)$, then $\frac{d^\alpha y(x)}{dx^\alpha} = f^{(\alpha)}(g(x))(g'(x))^\alpha$;
(b4) $\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha)$;
(b5) $\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k > 0$;
(b6) $B_\alpha(a, b) = \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} \frac{x^{\alpha(b-1)}}{(1^\alpha + x^\alpha)^{a+b}} (dx)^\alpha$, where $B_\alpha(a, b)$ denotes local fractional Beta function.

In this paper, by using the way of weight functions and the technique of local fractional calculus, a new Hilbert-type discrete inequality with homogeneous kernel and a best constant is built. As applications, the equivalent form and some particular cases are obtained.

2. Main Results

The starting point in the researching Hilbert-type inequalities is the well-known Hölder's inequality. A fractal version of Hölder's inequality is presented in the following lemma.

Lemma 2.1 ([8])**.** *Let* $1/p + 1/q = 1$*,* $p > 1$*, and let* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be non-negative real sequences. Then*

$$\sum_{i=1}^{n} a_i^\alpha b_i^\alpha \leq \left(\sum_{i=1}^{n} a_i^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^{\alpha q}\right)^{\frac{1}{q}}.$$

If $\sum_{i=1}^{+\infty} a_i^{\alpha p} < +\infty$ *and* $\sum_{i=1}^{+\infty} b_i^{\alpha q} < +\infty$*, then the following inequalitiy holds*

$$\sum_{i=1}^{+\infty} a_i^\alpha b_i^\alpha \leq \left(\sum_{i=1}^{+\infty} a_i^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{+\infty} b_i^{\alpha q}\right)^{\frac{1}{q}}.$$

In particular, a two-variable version of the fractal Hölder's inequality is given in the next lemma.

Lemma 2.2. *Let* $1/p + 1/q = 1$*,* $p > 1$*, and let* h*,* F*,* $G \in C_\alpha(\mathbb{R}_+^2)$ *be non-negative functions. If*

$$0 < \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n) < +\infty, \quad 0 < \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n) < +\infty,$$

then the following inequality holds

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F(m,n)G(m,n) \leq \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)\right)^{\frac{1}{p}} \times \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)\right)^{\frac{1}{p}}. \tag{2.1}$$

Proof. The inequality (2.1) is trivially true in the case when h or F or G is identically zero. Suppose that

$$\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)\right)\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)\right) \neq 0.$$

Applying the following α-Young's inequality

$$x_i^{\frac{\alpha}{p}} y_i^{\frac{\alpha}{q}} \leq \frac{x_i^\alpha}{p^\alpha} + \frac{y_i^\alpha}{q^\alpha}, \quad x_i, y_i \geq 0, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

to

$$x^\alpha := \frac{h(m,n)F^p(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)}$$

and

$$y^\alpha := \frac{h(m,n)G^q(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)},$$

we can obtain

$$\begin{aligned}&\frac{[h(m,n)]^{\frac{1}{p}}\,F(m,n)\,[h(m,n)]^{\frac{1}{q}}\,G(m,n)}{\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)F^p(m,n)\right)^{\frac{1}{p}}\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)G^q(m,n)\right)^{\frac{1}{p}}}\\ \le&\frac{1}{p^\alpha}\cdot\frac{h(m,n)F^p(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)F^p(m,n)}+\frac{1}{q^\alpha}\cdot\frac{h(m,n)G^q(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)G^q(m,n)}.\end{aligned}$$

Summarizing both side of the obtained inequality, we have

$$\begin{aligned}&\frac{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)F(m,n)G(m,n)}{\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)F^p(m,n)\right)^{\frac{1}{p}}\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)G^q(m,n)\right)^{\frac{1}{p}}}\\ \le&\frac{1}{p^\alpha}\cdot\frac{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)F^p(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)F^p(m,n)}+\frac{1}{q^\alpha}\cdot\frac{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)G^q(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}h(m,n)G^q(m,n)}\\ =&\frac{1}{p^\alpha}+\frac{1}{q^\alpha}=1^\alpha.\end{aligned}$$

This directly gives the desired inequality (2.1). The proof is completed. □

Besides, we introduce the following notation and definition (see [21]).

Definition 2.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. If the following inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\alpha f(x_1) + (1-\lambda)^\alpha f(x_2) \tag{2.2}$$

holds, for any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, then f is said to be a generalized convex function on I.

Mo et al. [21] proved the following generalized Hermite-Hadamard inequality for local fractional integral. Let $f \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^{(\alpha)}f \le \frac{f(a)+f(b)}{2^\alpha}. \tag{2.3}$$

Applying above inequality we can prove next lemma.

Lemma 2.3. *If* $f \in I_x^{(\alpha)}(\mathbb{R}_+)$, $f^{(\alpha)}(t) < 0$, $f^{(2\alpha)}(t) > 0$, $t \in (1/2, +\infty)$, *then we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} f(t)(dt)^\alpha \le \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{+\infty} f(n) \le \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{+\infty} f(t)(dt)^\alpha. \tag{2.4}$$

Proof. Setting $a = n - \frac{1}{2}$, $b = n + \frac{1}{2}$, the generalized Hermite-Hadamard inequality (2.3) yields

$$\frac{f(n)}{\Gamma(1+\alpha)} \le \frac{1}{\Gamma(1+\alpha)}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)(dt)^\alpha. \tag{2.5}$$

Similarly, for $a = n$, $b = n+1$, from (2.3) we get

$$\frac{1}{\Gamma(1+\alpha)}\int_n^{n+1} f(t)(dt)^\alpha \le \frac{f(n)}{\Gamma(1+\alpha)}. \tag{2.6}$$

Now, from (2.5) and (2.6) we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_n^{n+1} f(t)(dt)^\alpha \leq \frac{f(n)}{\Gamma(1+\alpha)} \leq \frac{1}{\Gamma(1+\alpha)}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)(dt)^\alpha.$$

Furthermore, we can obtain

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{1}{\Gamma(1+\alpha)}\int_n^{n+1} f(t)(dt)^\alpha &= \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} f(t)(dt)^\alpha\\ \leq \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{+\infty} f(t) \leq \sum_{n=1}^{+\infty}\frac{1}{\Gamma(1+\alpha)}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)(dt)^\alpha &= \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{+\infty} f(t)(dt)^\alpha,\end{aligned}$$

which implies (2.4) holds. This completes the proof. ☐

Suppose that $r > 0$ and $K(x,y)$ is strictly decreasing and generalized convex function in both variables on $\mathbb{R}_+$. Using chain rule for local fractional derivative (the formula (b3) from Introduction) yields

$$\frac{\partial^\alpha}{\partial x^\alpha}K(x,n)x^{-\alpha r} = \frac{1}{x^{\alpha r}}\cdot\frac{\partial^\alpha}{\partial x^\alpha}[K(x,n)] - \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r-1)\alpha)}\cdot\frac{K(x,n)}{x^{\alpha(r+1)}} < 0$$

and

$$\begin{aligned}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}K(x,n)x^{-\alpha r} =& \frac{1}{x^{\alpha r}}\cdot\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}[K(x,n)] - \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r-1)\alpha)}\cdot\frac{K(x,n)}{x^{\alpha(r+1)}}\\ &\times\frac{\partial^\alpha}{\partial x^\alpha}[K(x,n)] > 0,\end{aligned}$$

for $x > 0$ and $n \in \mathbb{N}$. In this way (see also [22], Corollary 1) we obtain the following result.

Lemma 2.4. *Let $r > 0$, $m, n \in \mathbb{N}$, and $K(x,y)$ be strictly decreasing and generalized convex function in both variables on $\mathbb{R}_+$. Then*

$$K(m,y)y^{-\alpha r} \quad \text{and} \quad K(x,n)x^{-\alpha r}$$

are strictly decreasing and generalized convex function on $\mathbb{R}_+$.

In what follows we suppose that $K \in C_\alpha(\mathbb{R}^2_+)$ is a non-negative homogeneous function of degree $-\alpha s$, $s > 0$. Further, we define

$$k(\beta) = \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} K(1,t)t^{-\alpha\beta}(dt)^\alpha, \tag{2.7}$$

under assumption $k(\beta) < +\infty$.

To prove our main results we need some technical lemma.

Lemma 2.5. *Let $1/p + 1/q = 1$, $p > 1$, and let $K \in C_\alpha(\mathbb{R}^2_+)$ be a non-negative homogeneous function of degree $-\alpha s$, $s > 0$. If K is strictly decreasing and generalized* *convex function in both variables on* $\mathbb{R}_+$, *then*

$$\omega_m(pA_2) := \sum_{n=1}^{+\infty} K(m,n)\left(\frac{m}{n}\right)^{\alpha p A_2} \leq \Gamma(1+\alpha) m^{\alpha(1-s)} k(pA_2) \tag{2.8}$$

and

$$\overline{\omega}_n(qA_1) := \sum_{n=1}^{+\infty} K(m,n)\left(\frac{n}{m}\right)^{\alpha q A_1} \leq \Gamma(1+\alpha) n^{\alpha(1-s)} k(2-s-qA_1), \tag{2.9}$$

where $A_1 \in (\max\{(1-s)/q, 0\}, 1/q)$ *and* $A_2 \in (\max\{(1-s)/p, 0\}, 1/p)$.

Proof. Applying Lemma 2.2 and Lemma 2.4 we get

$$\omega_m(pA_2) \leq \Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty} K(m,x)\left(\frac{x}{m}\right)^{-\alpha p A_2}(dx)^\alpha.$$

Further, using homogeneity of function K and substituting $u = x/m$, we have

$$\begin{aligned}\omega_m(pA_2) &\leq \Gamma(1+\alpha) m^{\alpha(1-s)}\frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty} K(1,u)u^{-\alpha p A_2}(du)^\alpha\\ &= \Gamma(1+\alpha) m^{\alpha(1-s)} k(pA_2),\end{aligned}$$

which implies (2.8), where we used the definition of $k(\beta)$ in equation (2.7). Similarly, we obtain (2.9). ☐

The main results are stated below.

Theorem A. *Let* $1/p + 1/q = 1$, $p > 1$, *and let* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be non-negative real sequences. If* $K(x,y)$, A_1, A_2 *are defined as in Lemma* 2.5, *then the following inequalities hold and are equivalent*

$$\begin{aligned} I := \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n) a_m^\alpha b_n^\alpha \leq & L\left(\sum_{m=1}^{+\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}}\\ &\times \left(\sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}}\end{aligned} \tag{2.10}$$

and

$$\begin{aligned} J := &\left(\sum_{n=1}^{+\infty} n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^\alpha\right)^p\right)^{\frac{1}{p}}\\ \leq & L\left(\sum_{m=1}^{+\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}},\end{aligned} \tag{2.11}$$

where $L = \Gamma(1+\alpha) k(pA_2)^{1/p} k(2-s-qA_1)^{1/q}$.

Proof. By using the local fractional Hölder's inequality (2.1), we have

$$\begin{aligned}
I &= \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\, a_m^{\alpha}\frac{m^{\alpha A_1}}{n^{\alpha A_2}}\; b_n^{\alpha}\frac{n^{\alpha A_2}}{m^{\alpha A_1}}\\
&\le \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\frac{m^{\alpha p A_1}}{n^{\alpha p A_2}} a_m^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\frac{n^{\alpha q A_2}}{m^{\alpha q A_1}} b_n^{\alpha q}\right)^{\frac{1}{q}}\\
&= \left(\sum_{m=1}^{+\infty}\left(\sum_{n=1}^{+\infty} K(m,n)\left(\frac{m}{n}\right)^{\alpha p A_2}\right) m^{\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}}\\
&= \left(\sum_{n=1}^{+\infty}\left(\sum_{m=1}^{+\infty} K(m,n)\left(\frac{n}{m}\right)^{\alpha q A_1}\right) n^{\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, from Lemma 2.5, we get the inequality (2.10).

We suppose that the inequality (2.10) is valid. To obtain (2.11), we set

$$b_n^{\alpha} := n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^{\alpha}\right)^{p-1}.$$

It follows that

$$J^p = \sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}.$$

By using the inequality (2.10), we have

$$\begin{aligned}
&\sum_{n=1}^{+\infty} n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^{\alpha}\right)^{p} = J^p = I\\
\le& L\left(\sum_{m=1}^{+\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}},
\end{aligned}$$

which implies the inequality (2.11) holds. By using the two dimensional Hölder's inequality in Lemma 2.1, we have

$$\begin{aligned}
I =& \sum_{n=1}^{+\infty}\left(n^{\alpha(s-1)\frac{1}{q}+\alpha(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^{\alpha}\right)\right) n^{\alpha(1-s)\frac{1}{q}+\alpha(A_2-A_1)} b_n^{\alpha}\\
\le& J\left(\sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}}.
\end{aligned}$$

From (2.11) and the above inequality, we have (2.10). Therefore, the inequalities (2.11) and (2.10) are equivalent. □

Now, we consider some special choises of the parameters A_1 and A_2. More precisely, let the parameters A_1 and A_2 satisfy condition

$$pA_2 + qA_1 = 2 - s. \tag{2.12}$$

Then, the constant L from Theorem A becomes

$$L^* = \Gamma(1+\alpha)k(pA_2). \tag{2.13}$$

Further, the inequalities (2.10) and (2.11) take form

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)a_m^\alpha b_n^\alpha \leq L^*\left(\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1}a_m^{\alpha p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{+\infty} n^{-\alpha+\alpha pqA_2}b_n^{\alpha q}\right)^{\frac{1}{q}} \tag{2.14}$$

and

$$\left(\sum_{n=1}^{+\infty} n^{\alpha(p-1)(1-pqA_2)}\left(\sum_{m=1}^{+\infty} K(m,n)a_m^\alpha\right)^p\right)^{\frac{1}{p}} \leq L^*\left(\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1}a_m^{\alpha p}\right)^{\frac{1}{p}}. \tag{2.15}$$

In the following theorem we show that, if the parameters A_1 and A_2 satisfy condition (2.12), then one obtains the best possible constant.

Theorem B. *Let s, A_1, A_2 and $K(x,y)$ be defined as in Theorem* A. *If the parameters A_1 and A_2 satisfy condition $pA_2+qA_1 = 2-s$, then the constant $L^* = \Gamma(1+\alpha)k(pA_2)$ in inequalities* (2.14) *and* (2.15) *is the best possible.*

Proof. For this purpose, set $\widetilde{a}_m^\alpha = m^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}$ and $\widetilde{b}_n^\alpha = n^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}$ where $0<\varepsilon<\frac{1-pA_2}{q}$. Now, let us suppose that the inequality (2.14) is valid. By using Lemma 2.2, we have

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)\varepsilon^\alpha} &= \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} u^{-\alpha-\alpha\varepsilon}(du)^\alpha \leq \frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} m^{-\alpha-\alpha\varepsilon}\\ &= \frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1}\widetilde{a}_m^{\alpha p}\\ &\leq \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{+\infty} u^{-\alpha-\alpha\varepsilon}(du)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} u^{-\alpha-\alpha\varepsilon}(du)^\alpha.\end{aligned}$$

Hence, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1}\widetilde{a}_m^{\alpha p} \leq \frac{1}{\varepsilon^\alpha\Gamma(1+\alpha)} + O(1), \tag{2.16}$$

and similarly

$$\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} n^{-\alpha+\alpha pqA_2}\widetilde{b}_n^{\alpha q} \leq \frac{1}{\varepsilon^\alpha\Gamma(1+\alpha)} + O(1). \tag{2.17}$$

Suppose that the constant L^*, defined by (2.13), is not the best possible in inequality (2.14). That implies that there exits constant M, smaller than L^*, such that the inequality (2.14) is still valid if we replace L^* with M. Hence, if we insert relations (2.16) and (2.17) in inequality (2.14), with the constant M instead of L^*, we have

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\widetilde{a}_m^\alpha\widetilde{b}_n^\alpha \leq \frac{1}{\varepsilon^\alpha}(M+o(1)). \tag{2.18}$$

Further, we estimate the left-hand side of inequality (2.14). Namely, if we insert the above defined sequences $(\widetilde{a}_m^\alpha)_{m\in\mathbb{N}}$ and $(\widetilde{b}_n^\alpha)_{n\in\mathbb{N}}$ in the left-hand side of (2.14), we easily obtain the inequality

(2.19)
$$\begin{aligned} J_\varepsilon :=&\frac{1}{\Gamma^2(1+\alpha)}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}K(m,n)\widetilde{a}_m^\alpha\widetilde{b}_n^\alpha \\ \geq&\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty}x^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty}K(x,y)y^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(dy)^\alpha\right)(dx)^\alpha, \end{aligned}$$

where we used Lemma 2.2. By using the substitution $u = y/x$, we obtain

$$J_\varepsilon \geq \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty}x^{-\alpha-\alpha\varepsilon}\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{x}}^{+\infty}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha\right)(dx)^\alpha. \tag{2.20}$$

Further, since the kernel K is strictly decreasing in both variables, it follows that $K(1,0) > K(1,t)$, for $t > 0$, so we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{x}}^{+\infty}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha \\ >&\frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha-\frac{K(1,0)}{\Gamma(1+\alpha)}\int_0^{\frac{1}{x}}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha \\ =&k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{K(1,0)}{\Gamma(1+\alpha)(1-pA_2-\frac{\varepsilon}{q})^\alpha}x^{\alpha pA_2+\frac{\alpha\varepsilon}{q}-\alpha} \end{aligned}$$

and, consequently,

$$J_\varepsilon \geq \frac{1}{\varepsilon^\alpha}\cdot\frac{k\left(pA_2+\frac{\varepsilon}{q}\right)}{\Gamma(1+\alpha)}+\frac{K(1,0)}{\Gamma^2(1+\alpha)}\cdot\frac{1}{(1-pA_2-\frac{\varepsilon}{q})^\alpha(pA_2-\frac{\varepsilon}{p}-1)^\alpha}. \tag{2.21}$$

Now, the relations (2.19), (2.20) and (2.21) yield the estimate for the left-hand side of inequality (2.14):

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}K(m,n)\widetilde{a}_m^\alpha\widetilde{b}_n^\alpha > \frac{1}{\varepsilon^\alpha}(L^*+o(1)). \tag{2.22}$$

Finally, by comparing (2.18) and (2.22), and by letting $\varepsilon \longrightarrow 0^+$, we get that $L^* \leq M$, which contradicts with the assumption that the constant M is smaller than L^*.

The equivalence of inequalities (2.14) and (2.15) means that the constant L^* is the best possible in the inequality (2.15). The proof is now completed. ☐

As corollaries of Theorem B we have the following results. We processed with the kernel $K_1(x,y) = (x+y)^{-\alpha s}$, $s > 0$. By using local fractional calculus, we have

$$\frac{\partial^\alpha}{\partial x^\alpha}\cdot\frac{1}{(m+x)^{\alpha s}} = -\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\cdot\frac{1}{(m+x)^{\alpha(s+1)}} < 0, \quad x > 0,$$

and similarly

$$\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \cdot \frac{1}{(m+x)^{\alpha s}} = \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s-1)\alpha)} \cdot \frac{1}{(m+x)^{\alpha(s+2)}} > 0, \quad x > 0.$$

Now, by applying Lemma 2.4 we obtain

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} K_1(x,y) x^{-\alpha r} < 0 \quad \text{and} \quad \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} K_1(x,y) x^{-\alpha r} > 0$$

and

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}} K_1(x,y) y^{-\alpha r} < 0 \quad \text{and} \quad \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} K_1(x,y) y^{-\alpha r} > 0,$$

for $r > 0$.

In what follows we suppose that

$$A_1 = \frac{2-s}{2q}, \quad A_2 = \frac{2-s}{2p}. \tag{2.23}$$

Then, based on equation (2.23), the constant L^* from Theorem B becomes

$$\begin{aligned} L^* =& \Gamma(1+\alpha) k(pA_2) = \Gamma(1+\alpha) k\left(1 - \frac{s}{2}\right) \\ =& \Gamma(1+\alpha) \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} \frac{u^{-\alpha-\frac{\alpha s}{2}}}{(1+u)^{\alpha s}} (du)^{\alpha} = \Gamma(1+\alpha) B_{\alpha}\left(\frac{s}{2}, \frac{s}{2}\right). \end{aligned}$$

Now, from Theorem B, we get the following result.

Corollary 2.1. *Let* $1/p + 1/q = 1$, $p > 1$, $0 < s < 2$, *and* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be non-negative real sequences. Then the following inequalities hold and are equivalent*

$$\begin{aligned} \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{a_m^{\alpha} b_n^{\alpha}}{(m+n)^{\alpha s}} \leq & \Gamma(1+\alpha) B_{\alpha}\left(\frac{s}{2}, \frac{s}{2}\right) \\ & \times \left(\sum_{m=1}^{+\infty} m^{\alpha p(1-\frac{s}{2})-\alpha} a_m^{\alpha p} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{+\infty} n^{\alpha p(1-\frac{s}{2})-\alpha} b_n^{\alpha q} \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\left(\sum_{n=1}^{+\infty} n^{\frac{\alpha p s}{2}-\alpha} \left(\sum_{m=1}^{+\infty} \frac{a_m^{\alpha}}{(m+n)^{\alpha s}} \right)^{p} \right)^{\frac{1}{p}} \leq \Gamma(1+\alpha) B_{\alpha}\left(\frac{s}{2}, \frac{s}{2}\right) \left(\sum_{m=1}^{+\infty} m^{\alpha p(1-\frac{s}{2})-\alpha} a_m^{\alpha p} \right)^{\frac{1}{p}},$$

where the constant $\Gamma(1+\alpha)B_{\alpha}(s/2, s/2)$ *is the best possible.*

3. Conclusion

In this paper, we have firstly obtained a fractal Hölder's inequality and some related inequalities. According to the basic results, some new discrete local fractional Hilbert-type inequalities have been investigated. At the same time, some new fractional Hilbert-type inequalities with homogeneous kernels have been given.

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