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ON BERNSTEIN-TYPE INEQUALITIES FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract. In this paper, we shall use a parameter β and obtain some Bernstein-type inequalities for rational functions with prescribed poles which generalize the results of Qasim and Liman and Li, Mohapatra and Rodriguez and others.

1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex polynomials of degree at most n. If $P \in \mathbb{P}_n$, then concerning the estimate of $|P'(z)|$ on $|z| = 1$, we have

$$|P'(z)| \leq n \sup_{|z|=1} |P(z)|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein [2], who proved it in 1912. Later, in 1969 (see [10]), Malik improved the above inequality (1.1) and established that if $P \in \mathbb{P}_n$, then for $|z| = 1$, we have

$$|P'(z)| + |Q'(z)| \leq n \sup_{|z|=1} |P(z)|, \tag{1.2}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

It is worth mentioning that equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin, so it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials

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$P(z)$ having no zeros in $|z|<1$, then (1.1) can be replaced by

$$\sup_{|z|=1}|P'(z)| \le \frac{n}{2}\sup_{|z|=1}|P(z)|, \tag{1.3}$$

whereas if $P(z)$ has no zeros in $|z|>1$, then

$$\sup_{|z|=1}|P'(z)| \ge \frac{n}{2}\sup_{|z|=1}|P(z)|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [9], whereas inequality (1.4) is due to Turán [12]. Li, Mohapatra and Rodriguez [14] gave a new perspective to the above inequalities and extended them to rational functions with prescribed poles. Essentially, in the inequalities referred to, they replaced the polynomial $P(z)$ by a rational function $r(z)$ with prescribed poles $a_1, a_2, \dots, a_n$ and z^n by a Blaschke product $B(z)$. Before proceeding towards their results, let us introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, \dots, n$, let

$$W(z) := \prod_{j=1}^{n}(z-a_j)$$

and let

$$B(z) := \prod_{j=1}^{n}\left(\frac{1-\overline{a}_j z}{z-a_j}\right), \quad \mathcal{R}_n := \mathcal{R}_n(a_1, a_2, \dots, a_n) = \left\{\frac{P(z)}{W(z)} : P \in \mathbb{P}_n\right\}.$$

Then $\mathcal{R}_n$ is the set of rational functions with poles $a_1, a_2, \dots, a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. For $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, the conjugate transpose r^* of r is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$. The rational function $r \in \mathcal{R}_n$ is called self-inversive if $r^*(z) = \lambda r(z)$ for some λ with $|\lambda| = 1$.

As an extension of (1.2) to rational functions, Li, Mohapatra and Rodriguez [14, Theorem 2] showed that if $r \in \mathcal{R}_n$, then

$$|r'(z)| + |(r^*(z))'| \le |B'(z)|\sup_{|z|=1}|r(z)|, \quad \text{for } |z|=1. \tag{1.5}$$

Equality holds in (1.5) for $r(z) = \alpha B(z)$ with $|\alpha| = 1$.

For $r \in \mathcal{R}_n$ to be self-inversive, Li, Mohapatra and Rodriguez [14, Corollary 4] proved that

$$|r'(z)| \le \frac{|B'(z)|}{2}\sup_{|z|=1}|r(z)|. \tag{1.6}$$

In the same paper, Li, Mohapatra and Rodriguez [14] showed that inequality (1.6) also holds for rational functions $r \in \mathcal{R}_n$ having no zeros in $|z|<1$ with prescribed poles. The latest development of further results along this line can be found in the monographs and papers [3–5, 7, 8, 11].

More recently, Qasim and Liman [6] proved several results by considering a specialized class of rational functions $r(t(z))$, defined by

$$(r \circ t)(z) = r(t(z)) := \frac{P(t(z))}{W(t(z))},$$

where $t(z)$ is a polynomial of degree m and $r \in \mathcal{R}_n$, so that $r(t(z)) \in \mathcal{R}_{mn}$, and

$$W(t(z)) = \prod_{j=1}^{mn} \Big(z - a_j\Big).$$

Also the Blaschke product is given by

$$B(z) = \frac{\Big(W(t(z))\Big)^*}{W(t(z))} = \frac{z^{mn}\overline{W(t(\frac{1}{\bar{z}}))}}{W(t(z))} = \prod_{j=1}^{mn} \Big(\frac{1 - \overline{a}_j z}{z - a_j}\Big).$$

Assume that the mn poles of $r(t(z))$ are denoted by a_j, $j = 1, 2, \ldots, mn$, and $|a_j| > 1$. They proved the following Bernstein-type inequality for rational functions $r(t(z)) \in \mathcal{R}_{mn}$ with restricted zeros.

Theorem 1.1. *If $r(t(z)) \in \mathcal{R}_{mn}$ and all the mn zeros of $r(t(z))$ lie in $|z| \geq 1$, then for $|z| = 1$*

$$|r'(t(z))| \leq \frac{|B'(z)|}{2m\mu} \sup_{|z|=1} |r(t(z))|, \tag{1.7}$$

where $t(z)$ has all its zeros in $|z| \leq 1$ and $\mu = \inf_{|z|=1} |t(z)|$.

2. Lemmas

For the proofs of our theorems we need the following lemmas.

Lemma 2.1. *If $r \in \mathcal{R}_n$ has n zeros all lie in $|z| \leq 1$, then*

$$|r'(z)| \geq \frac{1}{2}|B'(z)||r(z)|, \quad \text{for } |z| = 1.$$

The above lemma is due to Li, Mohapatra and Rodriguez [14].

Lemma 2.2. *Let A and B be any two complex numbers, then*

(i) if $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$;

(ii) conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$, then $|A| \geq |B|$.

The above lemma is due to Li [13].

Lemma 2.3. *If $r(t(z))$, $s(t(z)) \in \mathcal{R}_{mn}$ and all the mn zeros of $s(t(z))$ lie in $|z| \leq 1$ and $|r(t(z))| \leq |s(t(z))|$ for $|z| = 1$. Then for every $\beta \in \mathbb{C}$, with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq \left|B(z)s'(t(z))t'(z) + \frac{\beta}{2}B'(z)s(t(z))\right|. \tag{2.1}$$

The result is sharp and equality holds in (2.1) for $r(t(z)) = \alpha s(t(z))$, with $|\alpha| = 1$.

Proof. The proof of this lemma is identical to the proof of Theorem 3.2 of Li [13], but for the sake of completeness we give the brief outlines of its proof. First assume that no zero of $s(t(z))$ are on the unit circle $|z| = 1$ and therefore, all the mn zeros of $s(t(z))$ are in $|z| < 1$. By Rouche's theorem, the rational function $\lambda r(t(z)) + s(t(z))$ has all its zeros in $|z| < 1$ for $|\lambda| < 1$ and has no poles in $|z| \leq 1$. On applying Lemma 2.1 to $\lambda r(t(z)) + s(t(z))$, we get on $|z| = 1$

$$2|B(z)|\left|\lambda(r(t(z)))' + (s(t(z)))'\right| \geq |B'(z)|\left|\lambda r(t(z)) + s(t(z))\right|. \tag{2.2}$$

Now, note that $B'(z) \neq 0$ (e.g. see formula (14) in [14]). So, the right hand side of (2.2) is non zero. Thus, by using (i) of Lemma 2.2, we have for all $\beta \in \mathbb{C}$, with $|\beta| < 1$,

$$2B(z)\Big(\lambda r'(t(z))t'(z) + s'(t(z))t'(z)\Big) \neq -\beta B'(z)\Big(\lambda r(t(z)) + s(t(z))\Big),$$

for $|z| = 1$. Equivalently, for $|z| = 1$,

$$\lambda\Big(2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z))\Big) \neq -\Big(2B(z)s'(t(z))t'(z) + \beta B'(z)s(t(z))\Big),$$

for $|\lambda| < 1$ and $|\beta| < 1$. Using (ii) of Lemma 2.2, we have

$$|2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z))| \leq |2B(z)s'(t(z))t'(z) + \beta B'(z)s(t(z))| \tag{2.3}$$

for $|z| = 1$ and $|\beta| < 1$. Now, using the continuity in zeros and β, we can obtain the (2.3), when some zeros of $s(t(z))$ lie on the unit circle $|z| = 1$ and $|\beta| \leq 1$. $\square$

Applying Lemma 2.3 to the rational function $r(t(z))$ and $B(z)\sup_{|z|=1}|r(t(z))|$, we get the following.

Lemma 2.4. *If $r(t(z)) \in \mathcal{R}_{mn}$, then for all $\beta \in \mathbb{C}$, with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq |B(z)|\left|1 + \frac{\beta}{2}\right|\sup_{|z|=1}|r(t(z))|.$$

Lemma 2.5. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq 1$, then*

$$\inf_{|z|=1}|P'(z)| \geq n \inf_{|z|=1}|P(z)|. \tag{2.4}$$

The result is best possible and equality in (2.4) holds for polynomials, having all zeros at the origin.

The above lemma is due to Aziz and Dawood [1].

3. Main Results

In this note, we shall use a parameter β and obtain generalizations of (1.5), (1.6) and (1.7). We shall always assume that all the poles of $r(t(z)) \in \mathcal{R}_{mn}$ lie in $|z| > 1$.

Theorem 3.1. *If $r(t(z)) \in \mathcal{R}_{mn}$ and $|z| = 1$, then for every β, with $|\beta| \leq 1$,*

$$\begin{aligned}&\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| + \left|B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2}B'(z)(r(t(z)))^*\right| \\ &\leq |B'(z)|\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \sup_{|z|=1} |r(t(z))|.\end{aligned} \tag{3.1}$$

Proof. Let $M := \sup_{|z|=1} |r(t(z))|$. Therefore, for every $\lambda \in \mathbb{C}$, with $|\lambda| > 1$, $|r(t(z))| < |\lambda M B(z)|$ for $|z| = 1$.

By Rouche's theorem, all the mn zeros of $G(z) = r(t(z)) + \lambda M B(z)$ lie in $|z| < 1$. If $H(z) = B(z)\overline{G(\frac{1}{\bar{z}})}$, then $|H(z)| = |G(z)|$ for $|z| = 1$ and hence, for any γ, with $|\gamma| < 1$, the rational function $\gamma H(z) + G(z)$ has all mn zeros in $|z| < 1$. By applying Lemma 2.1 to $\gamma H(z) + G(z)$, we have

$$2\Big|B(z)\Big(\gamma H'(z) + G'(z)\Big)\Big| \geq |B'(z)|\Big|\gamma H(z) + G(z)\Big|, \quad \text{for } |z| = 1. \tag{3.2}$$

Since $B'(z) \neq 0$ therefore, the right hand side of (3.2) is non zero. Thus, by using (i) of Lemma 2.2, we have for all $\beta \in \mathbb{C}$, with $|\beta| < 1$,

$$2B(z)\Big(\gamma H'(z) + G'(z)\Big) \neq -\beta B'(z)\Big(\gamma H(z) + G(z)\Big), \quad \text{for } |z| = 1.$$

Equivalently, for $|z| = 1$,

$$-\gamma\Big(2B(z)H'(z) + \beta B'(z)H(z)\Big) \neq -\Big(2B(z)G'(z) + \beta B'(z)G(z)\Big), \tag{3.3}$$

for $|\gamma| < 1$, $|\beta| < 1$. Using (ii) of Lemma 2.2 in (3.3), we have

$$\Big|2B(z)G'(z) + \beta B'(z)G(z)\Big| \leq \Big|2B(z)H'(z) + \beta B'(z)H(z)\Big|, \tag{3.4}$$

for $|z| = 1$, $|\beta| < 1$. Now, using $G(z) = r(t(z)) + \lambda M B(z)$ and since

$$H(z) = B(z)\overline{G\left(\frac{1}{\bar{z}}\right)} = B(z)\left(\overline{r\left(t\left(\frac{1}{\bar{z}}\right)\right)} + \bar{\lambda}M\overline{B\left(\frac{1}{\bar{z}}\right)}\right) = (r(t(z)))^* + \bar{\lambda}M,$$

for $|z| = 1$ in (3.4), we get, for $|\beta| < 1$ and $|z| = 1$,

$$\begin{aligned}&\Big|2B(z)\Big[(r(t(z)))^*\Big]' + \beta B'(z)(r(t(z)))^* + \bar{\lambda}\beta M B'(z)\Big| \\ &\leq \Big|2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z)) + \lambda B(z)B'(z)\Big(2 + \beta\Big)M\Big|.\end{aligned} \tag{3.5}$$

By choosing a suitable argument of λ and applying Lemma 2.4 on the right hand side of (3.5), we get, for $|z|=1$ and $|\beta|<1$,

$$\begin{aligned}&\Big|2B(z)\Big[(r(t(z)))^*\Big]' + \beta B'(z)(r(t(z)))^*\Big| - |\lambda|\Big|\beta B'(z)\Big|M\\ &\le|\lambda|\Big|B(z)B'(z)\Big(2+\beta\Big)\Big|M - \Big|2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z))\Big|.\end{aligned} \tag{3.6}$$

Note that $|B(z)|=1$ for $|z|=1$. Making $|\lambda|\to 1$ and using continuity for $|\beta|=1$ in (3.6), we get (3.1) and this proves the desired result. $\square$

For $t(z)=z$, Theorem 3.1 reduces to the following result.

Corollary 3.1. *If $r\in\mathcal{R}_n$ and $|z|=1$, then for every β, with $|\beta|\le 1$,*

$$\begin{aligned}&\left|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\right| + \left|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\right|\\ &\le|B'(z)|\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\sup_{|z|=1}|r(z)|.\end{aligned} \tag{3.7}$$

Remark 3.1. For $\beta=0$, (3.7) reduces to (1.5).

Theorem 3.2. *If $r(t(z))\in\mathcal{R}_{mn}$ is self-inversive and $|z|=1$, then for every β with $|\beta|\le 1$, we have*

$$\left|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))\right| \le \frac{|B'(z)|}{2}\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\sup_{|z|=1}|r(t(z))|. \tag{3.8}$$

Proof. Since $r(t(z))$ is self-inversive, therefore, we have $(r(t(z)))^*=\lambda r(t(z))$ with $|\lambda|=1$. Hence, for all $\beta\in\mathbb{C}$,

$$\left|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))\right| = \left|B(z)\Big[(r(t(z)))^*\Big]'+\frac{\beta}{2}B'(z)(r(t(z)))^*\right|. \tag{3.9}$$

Combining Theorem 3.1 and (3.9), we have for every β, with $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}2\left|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))\right| =&\left|B'(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))\right|\\ &+\left|B(z)\Big[(r(t(z)))^*\Big]'+\frac{\beta}{2}B'(z)(r(t(z)))^*\right|\\ \le&|B'(z)|\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\sup_{|z|=1}|r(t(z))|,\end{aligned}$$

which proves Theorem 3.2 completely. $\square$

Remark 3.2. If we take $\beta=0$ in inequality (3.8) and make use of the Lemma 2.5, after supposing that $t(z)$ has all its zeros in $|z|\le 1$, we get the following result.

Corollary 3.2. *If $r(t(z)) \in \mathcal{R}_{mn}$ is self-inversive, where $t(z)$ has all its zeros in $|z| \leq 1$, then for $|z| = 1$,*

$$|r'(t(z))| \leq \frac{|B'(z)|}{2m\mu} \sup_{|z|=1} |r(t(z))|, \tag{3.10}$$

where $\mu = \inf_{|z|=1} |t(z)|$.

Remark 3.3. For $t(z) = z$, (3.10) reduces to (1.6).

We end this section by proving the following interesting generalization of (1.7).

Theorem 3.3. *Suppose $r(t(z)) \in \mathcal{R}_{mn}$ and all the mn zeros of $r(t(z))$ lie in $|z| \geq 1$. Then for every β, with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left| B(z)r'(t(z))t'(z) + \frac{\beta}{2} B'(z)r(t(z)) \right| \leq \frac{|B'(z)|}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \sup_{|z|=1} |r(t(z))|. \tag{3.11}$$

Proof. Since $r(t(z)) \in \mathcal{R}_{mn}$ has all its mn zeros in $|z| \geq 1$ and $(r(t(z)))^* = B(z)\overline{r(t(\frac{1}{\bar{z}}))}$, therefore, all the zeros of $(r(t(z)))^*$ lie in $|z| \leq 1$. Also, $|r(t(z))| = |(r(t(z)))^*|$ for $|z| = 1$. Hence, by Lemma 2.3, it follows for every β, with $|\beta| \leq 1$ and $|z| = 1$,

$$\left| B(z)r'(t(z))t'(z) + \frac{\beta}{2} B'(z)r(t(z)) \right| \leq \left| B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2} B'(z)(r(t(z)))^* \right|. \tag{3.12}$$

Combining Theorem 3.1 and (3.12), we have for every β, with $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} 2\left| B(z)r'(t(z))t'(z) + \frac{\beta}{2} B'(z)r(t(z)) \right| &\leq \left| B'(z)r'(t(z))t'(z) + \frac{\beta}{2} B'(z)r(t(z)) \right| \\ &\quad + \left| B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2} B'(z)(r(t(z)))^* \right| \\ &\leq |B'(z)| \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \sup_{|z|=1} |r(t(z))|, \end{aligned}$$

which is equivalent to (3.11) and this completes the proof of Theorem 3.3. □

Remark 3.4. If we take $\beta = 0$ in (3.11) and assume that $t(z)$ has all its zeros in $|z| \leq 1$, we get (1.7) by virtue of Lemma 2.5.

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