

STEINER HARARY INDEX

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ABSTRACT. The *Harary index* $H(G)$ of a connected graphs G is defined as $H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}$ where $d_G(u, v)$ is the distance between vertices u and v of G . The Steiner distance in a graph, introduced by Chartrand et al. in 1989, is a natural generalization of the concept of classical graph distance. For a connected graph G of order at least 2 and $S \subseteq V(G)$, the *Steiner distance* $d_G(S)$ of the vertices of S is the minimum size of all connected subgraphs whose vertex set contain S . Recently, Furtula, Gutman, and Katanić introduced the concept of Steiner Harary index and give its chemical applications. The *k-center Steiner Harary index* $SH_k(G)$ of G is defined by $SH_k(G) = \sum_{S \subseteq V(G), |S|=k} \frac{1}{d_G(S)}$. Expressions for SH_k for some special graphs are obtained. We also give sharp upper and lower bounds of SH_k of a connected graph, and establish some of its properties in the case of trees.

1. INTRODUCTION

All graphs in this paper are undirected, finite, and simple. We refer to [3] for graph theoretical notation and terminology not described here. For a graph G , let $V(G)$, $E(G)$, and $e(G)$ denote the set of vertices, the set of edges, and the size of G , respectively. Distance is one of the basic concepts of graph theory [4]. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u, v) = d_G(u, v)$ between u and v is the length of a shortest path connecting u and v . If v is a vertex of a connected graph G , then the *eccentricity* $\varepsilon(v)$ of v is defined by $\varepsilon(v) = \max\{d(u, v) \mid u \in V(G)\}$. Furthermore, the *radius* $\text{rad}(G)$ and *diameter* $\text{diam}(G)$ of G are defined by $\text{rad}(G) = \min\{\varepsilon(v) \mid v \in V(G)\}$ and $\text{diam}(G) = \max\{\varepsilon(v) \mid v \in V(G)\}$. These latter two concepts are related by the inequalities $\text{rad}(G) \leq \text{diam}(G) \leq 2 \text{rad}(G)$. Goddard and Oellermann gave a survey paper on this subject [13].

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The *Wiener index* $W(G)$ of G is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v).$$

The first investigations of this distance-based graph invariant were done by Harold Wiener in 1947, who realized that there exist correlations between the boiling points of paraffins and their molecular structure, see [26–28]. Mathematicians study the Wiener index since the 1970s [10].

The Wiener index obtained wide attention and numerous results have been worked out, see the surveys [9, 15, 16, 30]. The *Harary index* $H(G)$ of G is defined by

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u, v)}.$$

For more details on the Harary index, we refer to [2, 17, 20, 29].

The Steiner distance of a graph, introduced by Chartrand et al. in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an *S -Steiner tree* or a *Steiner tree connecting S* (or simply, an *S -tree*) is a subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G . Then the *Steiner distance* $d_G(S)$ among the vertices of S (or simply the distance of S) is the minimum size of connected subgraphs whose vertex set contain S . Note that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d_G(S)$, then H is a tree. Clearly, $d_G(S) = \min\{|E(T)|, S \subseteq V(T)\}$, where T is subtree of G . Furthermore, if $S = \{u, v\}$, then $d_G(S) = d_G(u, v)$ is nothing new, but the classical distance between u and v .

Let n and k be integers such that $2 \leq k \leq n$. The *Steiner k -eccentricity* $\varepsilon_k(v)$ of a vertex v of G is defined by $\varepsilon_k(v) = \max\{d_G(S) \mid S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. The *Steiner k -radius* of G is $s \text{ rad}_k(G) = \min\{\varepsilon_k(v) \mid v \in V(G)\}$, while the *Steiner k -diameter* of G is $s \text{ diam}_k(G) = \max\{\varepsilon_k(v) \mid v \in V(G)\}$. Note that for every connected graph G , $\varepsilon_2(v) = \varepsilon(v)$ for all vertices v of G , $s \text{ rad}_2(G) = \text{rad}(G)$ and $s \text{ diam}_2(G) = \text{diam}(G)$. For more details on Steiner distance, we refer to [1, 5, 6, 8, 13, 25].

The following observation is easily seen.

Proposition 1.1. *Let k be an integer such that $2 \leq k \leq n$. If H is a spanning subgraph of G , then $s \text{ diam}_k(G) \leq s \text{ diam}_k(H)$.*

Li et al. [18] generalized the concept of Wiener index by Steiner distance. The *Steiner Wiener k -index* or *k -center Steiner Wiener index* $SW_k(G)$ of G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d_G(S).$$

For $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider SW_k for $2 \leq k \leq n - 1$, but the above definition

implies $SW_1(G) = 0$ and $SW_n(G) = n - 1$. We refer to [18, 19, 21–24] for more details on Steiner Wiener index.

Furtula et al. [11] introduced the concept of Steiner Harary index. The *Steiner Harary k -index* or *k -center Steiner Harary index* $SH_k(G)$ of G is defined as

$$SH_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \frac{1}{d_G(S)}.$$

For $k = 2$, the above defined Steiner Harary index coincides with the ordinary Harary index. It is usual to consider SH_k for $2 \leq k \leq n - 1$, but the above definition implies $SH_1(G) = 0$ and $SH_n(G) = \frac{1}{n-1}$.

In Section 2, we obtain the exact values of the Steiner Harary k -index of the path, complete graph, and complete bipartite graph. In Section 3, we obtain sharp lower and upper bounds for SH_k for connected graphs and for trees. In Section 4 we establish some relations for SH_k of trees. Our basic idea is from [18, 19].

2. RESULTS FOR SOME SPECIAL GRAPHS

Beginning this section, we note that the special case for $k = 2$ of all formulas derived here for the Steiner Harary index, thus pertaining to the ordinary Harary index, are well known and mentioned many times in the earlier literature.

Proposition 2.1. *Let K_n be the complete graph of order n , and let k be an integer such that $2 \leq k \leq n$. Then $SH_k(K_n) = \frac{1}{k-1} \binom{n}{k}$.*

Proof. For any $S \subseteq V(K_n)$ and $|S| = k$, without loss of generality, we let $S = \{u_1, u_2, \dots, u_k\}$. Since K_n is the complete graph of order n , it follows that the tree T induced by the edges in $\{u_1u_2, u_1u_3, \dots, u_1u_k\}$ is an S -Steiner tree, and hence $d_{K_n}(S) \leq k - 1$. Since $|S| = k$, it follows that $d_{K_n}(S) \geq k - 1$. Therefore, $d_{K_n}(S) = k - 1$. From the arbitrariness of S and the symmetry of K_n , we have

$$SH_k(K_n) = \sum_{\substack{S \subseteq V(K_n) \\ |S|=k}} \frac{1}{d_{K_n}(S)} = \frac{1}{k-1} \binom{n}{k},$$

as desired. □

Proposition 2.2. *Let $K_{a,b}$ be the complete bipartite graph of order $a + b$ ($1 \leq a \leq b$), and let k be an integer such that $2 \leq k \leq a + b$. Then*

$$SH_k(K_{a,b}) = \begin{cases} \frac{1}{k-1} \binom{a+b}{k} - \frac{1}{k(k-1)} \binom{a}{k} - \frac{1}{k(k-1)} \binom{b}{k}, & \text{if } 1 \leq k \leq a; \\ \frac{1}{k-1} \binom{a+b}{k} - \frac{1}{k(k-1)} \binom{b}{k}, & \text{if } a < k \leq b; \\ \frac{1}{k-1} \binom{a+b}{k}, & \text{if } b < k \leq a + b. \end{cases}$$

Proof. Let $G = K_{a,b}$, and let $U = \{u_1, u_2, \dots, u_a\}$ and $W = \{w_1, w_2, \dots, w_b\}$ be the two parts of $G = K_{a,b}$.

First, we consider the case $1 \leq k \leq a$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap U = \emptyset$, or $S \cap W = \emptyset$, or $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$. If $S \cap U = \emptyset$, then $S \subseteq W$. Without loss of generality, let $S = \{w_1, w_2, \dots, w_k\}$. Then the tree T induced by the edges in $\{u_1w_1, u_1w_2, \dots, u_1w_k\}$ is an S -Steiner tree, and hence $d_G(S) \leq k$. Since $G = K_{a,b}$ is a complete bipartite graph, it follows that any tree connecting S must use at least k edges, and hence $d_G(S) \geq k$. Therefore, $d_G(S) = k$. If $S \cap W = \emptyset$, then $S \subseteq U$. Without loss of generality, let $S = \{u_1, u_2, \dots, u_k\}$. Then the tree T induced by the edges in $\{w_1u_1, w_1u_2, \dots, w_1u_k\}$ is a Steiner tree connecting S , and hence $d_G(S) \leq k$. Since $G = K_{a,b}$ is a complete bipartite graph, it follows that any tree connecting S must use at least k edges, and hence $d_G(S) \geq k$. Therefore, $d_G(S) = k$. Suppose $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$. Without loss of generality, let $S = \{u_1, u_2, \dots, u_x, w_1, w_2, \dots, w_{k-x}\}$. Then the tree T induced by the edges in $\{u_1w_1, w_1u_2, w_1u_3, \dots, w_1u_x, u_1w_2, u_1w_3, \dots, u_1w_{k-x}\}$ is an S -Steiner tree, and hence $d_G(S) \leq k - 1$. Since $|S| = k$, it follows that any tree connecting S must use at least $k - 1$ edges, and hence $d_G(S) = k - 1$. Thus,

$$\begin{aligned} SH_k(G) &= \sum_{\substack{S \subseteq V(G) \\ S \cap U = \emptyset}} \frac{1}{d_G(S)} + \sum_{\substack{S \subseteq V(G) \\ S \cap U = \emptyset}} \frac{1}{d_G(S)} + \sum_{\substack{S \subseteq V(G) \\ S \cap U \neq \emptyset, S \cap W \neq \emptyset}} \frac{1}{d_G(S)} \\ &= \frac{1}{k} \binom{a}{k} + \frac{1}{k} \binom{b}{k} + \frac{1}{k-1} \left[\sum_{x=1}^a \binom{a}{x} \binom{b}{k-x} \right] \\ &= \frac{1}{k} \binom{a}{k} + \frac{1}{k} \binom{b}{k} + \frac{1}{k-1} \left[\binom{a+b}{k} - \binom{b}{k} - \binom{a}{k} \right] \\ &= \frac{1}{k-1} \binom{a+b}{k} - \frac{1}{k(k-1)} \binom{a}{k} - \frac{1}{k(k-1)} \binom{b}{k}. \end{aligned}$$

Next, we consider the case $a < k \leq b$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap U = \emptyset$ or $S \cap U \neq \emptyset$. If $S \cap U = \emptyset$, then $S \subseteq W$ and $d_G(S) = k$. Suppose $S \cap U \neq \emptyset$. Then $d_G(S) = k - 1$, and hence

$$\begin{aligned} SH_k(G) &= \sum_{\substack{S \subseteq V(G) \\ S \cap U = \emptyset}} \frac{1}{d_G(S)} + \sum_{\substack{S \subseteq V(G) \\ S \cap U \neq \emptyset}} \frac{1}{d_G(S)} \\ &= \frac{1}{k} \binom{b}{k} + \frac{1}{k-1} \left[\sum_{x=1}^a \binom{a}{x} \binom{b}{k-x} \right] \\ &= \frac{1}{k} \binom{b}{k} + \frac{1}{k-1} \left[\sum_{x=1}^{\infty} \binom{a}{x} \binom{b}{k-x} \right] \\ &= \frac{1}{k} \binom{b}{k} + \frac{1}{k-1} \left[\binom{a+b}{k} - \binom{b}{k} \right] \\ &= \frac{1}{k-1} \binom{a+b}{k} - \frac{1}{k(k-1)} \binom{b}{k}. \end{aligned}$$

In this end, we consider the remaining case $b < k \leq a + b$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap U \neq \emptyset$ and $S \cap U \neq \emptyset$. Then $d_G(S) = k - 1$, and hence

$$SH_k(G) = \sum_{\substack{S \subseteq V(G) \\ S \cap U = \emptyset}} \frac{1}{d_G(S)} = \frac{1}{k-1} \binom{a+b}{k}.$$

The proof is now complete. \square

From the above proposition, we can derive the following corollary.

Corollary 2.1. *Let S_n be the star of order n ($n \geq 3$), and let k be an integer such that $2 \leq k \leq n$. Then*

$$SH_k(S_n) = \frac{kn - n + k}{k^2(k-1)} \binom{n-1}{k-1}.$$

Proof. From Proposition 2.2, we have that $SH_k(S_n) = SH_k(K_{1,n-1}) = \binom{n}{n-1} \frac{1}{n-1} = \frac{1}{n-1}$ for $k = n$ and $SH_k(S_n) = SH_k(K_{1,n-1}) = \frac{1}{k} \binom{n-1}{k} + \frac{1}{k-1} \binom{n-1}{k-1}$ for $2 \leq k \leq n-1$. We conclude that

$$SH_k(S_n) = \frac{1}{k} \binom{n-1}{k} + \frac{1}{k-1} \binom{n-1}{k-1} = \frac{kn - n + k}{k^2(k-1)} \binom{n-1}{k-1}.$$

\square

Proposition 2.3. *Let P_n be the path of order n ($n \geq 3$), and let k be an integer such that $2 \leq k \leq n$. Then*

$$SH_k(P_n) = n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1}.$$

Proof. Let $V(P_n) = \{u_1, u_2, \dots, u_n\}$. Choose $S \subseteq V(P_n)$ and $|S| = k$. Without loss of generality, let $S = \{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$ where $i_1 \leq i_2 \leq \dots \leq i_k$. Clearly, $k-1 \leq d(S) \leq n-1$. Observe that $d(S) = d_{P_n}(u_{i_1}, u_{i_k})$. Then $k-1 \leq d_{P_n}(u_{i_1}, u_{i_k}) \leq n-1$. Let $d_{P_n}(u_{i_1}, u_{i_k}) = t$. Thus, $k-1 \leq t \leq n-1$ and $1 \leq i_1 \leq n-t$. Therefore, we have $(n-t)$ ways to choose u_{i_1} . Once the vertex u_{i_1} is chosen, then $u_{i_k} = u_{i_1+t}$ is determined. Since $d_{P_n}(u_{i_1}, u_{i_k}) = t$, we have $\binom{t-1}{k-2}$ ways to choose $u_{i_2}, u_{i_3}, \dots, u_{i_{k-1}}$. So there are $(n-t) \binom{t-1}{k-2}$ ways to determine S . Thus,

$$\begin{aligned} SH_k(P_n) &= \sum_{k-1 \leq t \leq n-1} (n-t) \frac{1}{t} \binom{t-1}{k-2} \\ &= n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \sum_{k-1 \leq t \leq n-1} \binom{t-1}{k-2} \\ &= n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1}, \end{aligned}$$

as desired. \square

3. LOWER AND UPPER BOUNDS FOR GENERAL GRAPHS

The following proposition is immediate.

Proposition 3.1. *Let G be a connected graph of order n , $e \in E(G)$, and let k be an integer such that $2 \leq k \leq n$. Furthermore, let F be the graph with vertex set $V(F) = V(G)$ and edge set $E(G) \setminus e$. Then*

$$SH_k(F) \leq SH_k(G).$$

This straightforwardly leads to the following theorem.

Proposition 3.2. *Let G be a connected graph of order n , and T a spanning tree of G . Let k be an integer such that $2 \leq k \leq n$. Then*

$$SH_k(T) \leq SH_k(G)$$

with equality if and only if G is a tree.

A lower bound for the Steiner Harary index of an arbitrary tree is given by the next theorem.

Theorem 3.1. *Let T be a tree of order n , and let k be an integer such that $2 \leq k \leq n$. Then*

$$n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1} \leq SH_k(T) \leq \frac{kn - n + k}{k^2(k-1)} \binom{n-1}{k-1}.$$

Moreover, among all trees of order n , the star S_n maximizes the Steiner Harary k -index whereas the path P_n minimizes the Steiner Harary k -index.

Proof. The validity of the second inequality is verified by induction on n . For $n = k$, we have $d_T(S) = k - 1 = n - 1$ for $S \subseteq V(T)$, where T is a tree of order n . Then $SH_k(T) = SH_n(T) = \frac{1}{n-1} = \frac{1}{d_T(S)} = \frac{n^2 - n + n}{n^2(n-1)} \binom{n-1}{n-1}$, as desired. Assume now that the second inequality holds for all trees of order n . Let T be a tree on $n + 1$ vertices and v its pendent vertex. Let u be the vertex adjacent to v in T . Furthermore, let T' be the subtree of T induced by $V(T) \setminus v$. Then the inequality holds for T' . By the induction hypothesis,

$$SH_k(T') \leq \frac{kn - n + k}{k^2(k-1)} \binom{n-1}{k-1}.$$

If $v \in S$ but $u \notin S$, then $d_T(S) \geq k$ for $S \subseteq V(T)$, $|S| = k$. If both $v \in S$ and $u \in S$, then $d_T(S) \geq k - 1$ for $S \subseteq V(T)$, $|S| = k$. Therefore, we have

$$\begin{aligned} SH_k(T) &= SH_k(T') + \sum_{\substack{S \subseteq V(T), |S|=k \\ v \in S, u \notin S}} \frac{1}{d_T(S)} + \sum_{\substack{S \subseteq V(T), |S|=k \\ u, v \in S}} \frac{1}{d_T(S)} \\ &\leq \frac{kn - n + k}{k^2(k-1)} \binom{n-1}{k-1} + \frac{1}{k} \binom{n-1}{k-1} + \frac{1}{k-1} \binom{n-1}{k-2} \end{aligned}$$

$$= \frac{(k-1)(n+1) + k}{k^2(k-1)} \binom{n}{k-1}.$$

The first inequality is also verified by induction on n . For $n = k$, we have $d_T(S) = k - 1 = n - 1$ for $S \subseteq V(T)$, where T is a tree of order n . Then $SH_k(T) = SH_n(T) = \frac{1}{d_T(S)} = \frac{1}{n-1} = n \cdot \frac{1}{n-1} \binom{n-2}{n-2} - \binom{n-1}{n-1}$, as desired. Assume now that the second inequality holds for all trees of order n . Let T be a tree on $n+1$ vertices and v its pendent vertex. Furthermore, let T' be the subtree of T induced by $V(T) \setminus v$. Then the inequality holds for T' and we obtain

$$SH_k(T) = SH_k(T') + \sum_{\substack{S \subseteq V(T), |S|=k \\ v \in S}} \frac{1}{d_T(S)}.$$

By the induction hypothesis,

$$SH_k(T') \geq n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1}.$$

Let $V(T') = \{v_1, v_2, \dots, v_n\}$. For the vertex $v \in V(T)$, we can find a subtree T_1 such that $v \in V(T_1)$ and $d(V(T_1)) = k - 1$. Without loss of generality, let $V(T_1) = \{v, v_1, v_2, \dots, v_{k-1}\}$. Clearly, there is only one subset of $V(T_1)$ containing v and hence

$$\sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_1, v_2, \dots, v_{k-1} \in S}} \frac{1}{d_T(S)} = \frac{1}{d(V(T_1))} = \frac{1}{k-1}.$$

Pick up a vertex from $V(T) \setminus \{v, v_1, v_2, \dots, v_{k-1}\} = \{v_k, v_{k+1}, \dots, v_n\}$, say v_k , such that v_k is adjacent to one element of $\{v_1, v_2, \dots, v_{k-1}\}$. Clearly, the tree T_2 induced by the edges in $E(T_1) \cup \{v_k v_j\}$ is of order $k+1$, where $1 \leq j \leq k-1$ and $v_k v_j \in E(T)$. Then $d_T(V(T_2)) = d_T(\{v, v_1, v_2, \dots, v_{k-1}, v_k\}) = k$. It is clear that there are at most $\binom{k-1}{k-2}$ subsets of $V(T_2)$ containing both v and v_k . For each such subset $S \subseteq V(T_2)$ with $|S| = k$, $d_T(S) \leq d_T(V(T_2)) \leq k$. Thus, we have

$$\sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_k \in S}} \frac{1}{d_T(S)} \geq \binom{k-1}{k-2} \frac{1}{d_T(V(T_2))} \geq \frac{1}{k} \binom{k-1}{k-2}.$$

Pick up a vertex from $V(T) \setminus \{v, v_1, v_2, \dots, v_{k-1}, v_k\} = \{v_{k+1}, v_{k+2}, \dots, v_n\}$, say v_{k+1} , such that v_{k+1} is adjacent to one element of $\{v_1, v_2, \dots, v_k\}$. Clearly, the tree T_3 induced by the edges in $E(T_2) \cup \{v_{k+1} v_j\}$ is of order $k+2$, where $1 \leq j \leq k$ and $v_{k+1} v_j \in E(T)$. Then $d_T(V(T_3)) = d_T(\{v, v_1, v_2, \dots, v_{k+1}\}) = k+1$. Obviously, we can find $\binom{k}{k-2}$ subsets of $V(T_3)$ containing both v and v_{k+1} . Thus,

$$\sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_{k+1} \in S}} \frac{1}{d_T(S)} \geq \binom{k}{k-2} \frac{1}{d_T(V(T_3))} \geq \frac{1}{k+1} \binom{k}{k-2}.$$

Continue the above procedure, we get

$$\begin{aligned}
SH_k(T) &= SH_k(T') + \sum_{\substack{S \subseteq V(T), |S|=k \\ v \in S}} \frac{1}{d_T(S)} \\
&\geq SH_k(T') + \sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_1, v_2, \dots, v_{k-1} \in S}} \frac{1}{d_T(S)} + \sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_k \in S}} \frac{1}{d_T(S)} \\
&\quad + \sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_{k+1} \in S}} \frac{1}{d_T(S)} + \cdots + \sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_n \in S}} \frac{1}{d_T(S)} \\
&\geq SH_k(T') + \frac{1}{k-1} \binom{k-2}{k-2} + \frac{1}{k} \binom{k}{k-2} + \cdots + \frac{1}{n} \binom{n-1}{k-2} \\
&= n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1} + \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} + \frac{1}{n} \binom{n-1}{k-2} \\
&= (n+1) \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1} + \frac{1}{n} \binom{n-1}{k-2} \\
&= (n+1) \sum_{k-1 \leq t \leq n} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-2} - \binom{n-1}{k-1} \\
&= (n+1) \sum_{k-1 \leq t \leq n} \frac{1}{t} \binom{t-1}{k-2} - \binom{n}{k-1}. \quad \square
\end{aligned}$$

We recall that Theorem 3.1 provides a generalization of the much older results known for the Harary index, i.e., it yields this previous result by setting $k = 2$.

Theorem 3.2. *Let G be a connected graph of order n , and let k be an integer such that $2 \leq k \leq n$. Then*

$$n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1} \leq SH_k(G) \leq \frac{1}{k-1} \binom{n}{k}.$$

Moreover, the lower bound is sharp.

Proof. From Proposition 2.1, we have $SH_k(K_n) = \frac{1}{k-1} \binom{n}{k}$. According to Proposition 3.1, each subgraph G of K_n with $E(G) \subseteq E(K_n)$ has Steiner Harary index less than the Steiner Harary of K_n . Since each graph of order n is a subgraph of the complete graph, the inequality holds. From Proposition 3.2, we know that $SH_k(T) \leq SH_k(G)$, where T is a spanning tree of G . Combining this with Theorem 3.1, we have $SH_k(G) \geq SH_k(T) \geq SH_k(P_n) = n \sum_{k-1 \leq t \leq n-1} \frac{1}{t} \binom{t-1}{k-2} - \binom{n-1}{k-1}$, as desired. \square

4. ON STEINER HARARY INDICES OF TREES

For $k = n, n-1$, we have the following results for trees.

Theorem 4.1. *Let T be a tree of order n , possessing p pendent vertices. Then*

$$(4.1) \quad SH_{n-1}(T) = \frac{n^2 - 2n - p}{(n-1)(n-2)},$$

irrespective of any other structural detail of T .

Proof. Since $k = n - 1$, the respective subsets S contain all except one vertices of T . If the vertex missing from S is pendent, then the vertices contained in S form a tree of order $n - 1$. Therefore $d_G(S) = n - 2$. There are p such subsets, contributing to SH_{n-1} by $p \times \frac{1}{n-2}$.

If the vertex of T , not present in S , is non-pendent, then the vertices contained in S cannot form a tree, and the respective Steiner tree must contain all the n vertices of T . Therefore, $d_G(S) = n - 1$. There are $n - p$ such subsets, contributing to SH_{n-1} by $(n - p) \times \frac{1}{n-1}$.

Thus, $SH_{n-1}(T) = \frac{p}{n-2} + \frac{n-p}{n-1}$, which straightforwardly leads to (4.1). \square

Theorem 4.2. *Let T be a tree of order n , possessing p pendent vertices. Let q be the number of vertices of degree 2 in T such that each of them is adjacent to a pendant vertex. Then*

$$(4.2) \quad SH_{n-2}(T) = \left[\binom{p}{2} + q \right] \frac{1}{n-3} + \binom{n-p}{2} \frac{1}{n-1} + \frac{pn - p^2 - q}{n-2}.$$

Proof. For any $S \subseteq V(G)$ and $|S| = n - 2$, we let $\bar{S} = \{u, v\}$. If $d_T(u) = d_T(v) = 1$, then $d_T(S) = n - 3$ and this case contributes SH_{n-2} by

$$\sum_{\substack{u, v \in \bar{S} \\ d_T(u)=d_T(v)=1}} \frac{1}{d_T(S)} = \binom{p}{2} \frac{1}{n-3}.$$

If $d_T(u) \geq 2$ and $d_T(v) \geq 2$, then $d_T(S) = n - 1$ and this case contributes SH_{n-2} by

$$\sum_{\substack{u, v \in \bar{S} \\ d_T(u) \geq 2, d_T(v) \geq 2}} \frac{1}{d_T(S)} = \binom{n-p}{2} \frac{1}{n-1}.$$

Suppose that $d_T(u) = 1$ and $d_T(v) \geq 2$. If $d_T(u) = 1$, $d_T(v) = 2$ and $uv \in E(T)$, then $d_T(S) = n - 3$. If $d_T(u) = 1$, $d_T(v) \geq 3$ and $uv \in E(T)$, then $d_T(S) = n - 2$. If $d_T(u) = 1$, $d_T(v) \geq 2$ and $uv \notin E(T)$, then $d_T(S) = n - 2$. Therefore, this case contributes SH_{n-2} by

$$\begin{aligned} SH_{n-2}(T) &= \sum_{\substack{u, v \in \bar{S} \\ d_T(u)=1, d_T(v) \geq 2}} \frac{1}{d_T(S)} \\ &= \sum_{\substack{u, v \in \bar{S}, uv \in E(T) \\ d_T(u)=1, d_T(v)=2}} \frac{1}{d_T(S)} + \sum_{\substack{u, v \in \bar{S}, uv \in E(T) \\ d_T(u)=1, d_T(v) \geq 3}} \frac{1}{d_T(S)} + \sum_{\substack{u, v \in \bar{S}, uv \notin E(T) \\ d_T(u)=1, d_T(v) \geq 2}} \frac{1}{d_T(S)} \end{aligned}$$

$$= \frac{q}{n-3} + \frac{p-q}{n-2} + \frac{p(n-p-1)}{n-2}.$$

From the above argument, we have

$$\begin{aligned} SH_{n-2}(T) &= \binom{p}{2} \frac{1}{n-3} + \binom{n-p}{2} \frac{1}{n-1} + \frac{q}{n-3} + \frac{p-q}{n-2} + \frac{p(n-p-1)}{n-2} \\ &= \left[\binom{p}{2} + q \right] \frac{1}{n-3} + \binom{n-p}{2} \frac{1}{n-1} + \frac{pn-p^2-q}{n-2}. \quad \square \end{aligned}$$

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