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STABILITY OF MULTI-ADDITIVE FUNCTIONAL EQUATIONS ON RESTRICTED DOMAINS

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ABSTRACT. In the current investigation, we extend some stability results of Bae et al. (2022) to the case of n variables. In other words, we establish the Hyers-Ulam stability of multi-additive functional equations on restricted domains. We also study the asymptotic behavior of these equations. In addition, we extend a stability result for bilinear equations (as a special case) to multi-additive functional equations from a linear space to a 2-Banach space. A hyperstability result of multi-additive functional equations is presented as well.

1. INTRODUCTION AND PRELIMINARIES

A mathematical equation has stability when an approximate solution occurs near to the correct solution of the equation. The problem concerning the stability of equations is inspired through the fundamental problem of homomorphisms on groups posed by Ulam [23]. A foremost positive partial solution was presented by Hyers [14] as follows. Let $\mathbb{V}$ and $\mathbb{W}$ be Banach spaces. Assume that $\Phi : \mathbb{V} \to \mathbb{W}$ with

$$\|\Phi(u+v) - \Phi(u) - \Phi(v)\| \leq \delta,$$

for all $u, v \in \mathbb{V}$ and for some $\delta \geq 0$. Then, there exists an additive mapping $\mathcal{A} : \mathbb{V} \to \mathbb{W}$ which is uniquely determined such that $\|\Phi(v) - \mathcal{A}(v)\| \leq \delta$ for all $v \in \mathbb{V}$. The generalized version of Hyers' result has been obtained by Aoki [2] and Rassias [21] to approximate linear transformation in Banach spaces when the Cauchy difference bounded above by the sum of powers of norms. Considering different upper bounds, there are many interesting and motivating stability results on various functional equations pertaining to fascinating results are available in many articles and books.

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Assume that $(S,+)$ is a commutative semigroup and F is a linear space. A mapping $\Gamma : S^n \to F$ is said to be *multi-additive* if it satisfies $A(x+y) = A(x) + A(y)$ in all components, i.e.,

$$\begin{aligned}&\Gamma(\vartheta_1,\ldots,\vartheta_{i-1},\vartheta_i+\vartheta_i',\vartheta_{i+1},\ldots,\vartheta_n)\\ =&\Gamma(\vartheta_1,\ldots,\vartheta_{i-1},\vartheta_i,\vartheta_{i+1},\ldots,\vartheta_n)+\Gamma(\vartheta_1,\ldots,\vartheta_{i-1},\vartheta_i',\vartheta_{i+1},\ldots,\vartheta_n),\end{aligned}$$

for all $i \in \{1,\ldots,n\}$. Ciepliński [8] represented a system of additive functional equations defining a multi-additive mapping as one unified equation.

Theorem 1.1. ([8, Theorem 2]) *A mapping* $\Gamma : S^n \to F$ *is multi-additive if and only if*

$$\Gamma\left(\vartheta_1^{[n]}+\vartheta_2^{[n]}\right)=\sum_{i_1,\ldots,i_n\in\{1,2\}}\Gamma(\vartheta_{i_1 1},\ldots,\vartheta_{i_n n}), \tag{1.1}$$

where $\vartheta_i^{[n]} = (\vartheta_{i1},\ldots,\vartheta_{in}) \in S^n$ *with* $i \in \{1,2\}$.

Equation (1.1) is called the *multi-additive functional equation*. Regarding to (1.1), Ciepliński presented the next Găvruţa stability result.

Theorem 1.2. *Assume that* $(S,+)$ *is a commutative semigroup with an identity element and* $\mathbb{W}$ *is a Banach space. Suppose* $\phi : S^{2n} \to \mathbb{W}$ *is a mapping such that*

$$\widetilde{\phi}(s_{11},s_{12},\ldots,s_{n1},s_{n2}):=\sum_{j=0}^{+\infty}\frac{1}{2^{n(j+1)}}\phi(2^js_{11},2^js_{12},\ldots,2^js_{n1},2^js_{n2}),$$

for all $(s_{11},s_{12},\ldots,s_{n1},s_{n2}) \in S^{2n}$. *If* $f : S^n \to \mathbb{W}$ *is a mapping satisfying*

$$\left\|f\left(s_{11}+s_{12},\ldots,s_{n1}+s_{n2}\right)-\sum_{j_1,\ldots,j_n\in\{1,2\}}f(s_{j_1 1},\ldots,s_{j_n n})\right\|\le\phi(s_{11},s_{12},\ldots,s_{n1},s_{n2}),$$

for all $(s_{11},s_{12},\ldots,s_{n1},s_{n2}) \in S^{2n}$, *then there exists a unique multi-additive mapping* $\mathcal{A} : S^n \to \mathbb{W}$ *for which*

$$\|f(s_{11},\ldots,s_{n1})-\mathcal{A}(s_{11},\ldots,s_{n1})\|\le\widetilde{\phi}(s_{11},s_{11},\ldots,s_{n1},s_{n1}),$$

for all $(s_{11}\ldots,s_{n1}) \in S^n$. *Furthermore,* $\mathcal{A}$ *is given by*

$$\mathcal{A}(s_{11},\ldots,s_{n1})=\lim_{j\to+\infty}\frac{1}{2^{nj}}f(2^js_{11},\ldots,2^js_{n1}),$$

for all $(s_{11}\ldots,s_{n1}) \in S^n$.

Note that the solutions, structure and stability results of multi-additive mappings (in certain types of Banach spaces) were studied as follows: the stability and hyperstability of a multi-additive functional equation [4], multi-Jensen and multi-Euler-Lagrange additive mappings [6], the Ulam stability of a generalized for multi-additive functional equation [10], approximate multi-additive mappings in 2-Banach spaces [9], the set-valued multi-additive functional equations [15], approximate multi-Jensen and multi-Euler-Lagrange additive mappings in n-Banach spaces [24]; for more basic properties of multi-additive mappings, we refer to [16].

Regarding to the Ulam-Hyers stability of functional equations or inequalities satisfied on restricted domains or fulfills under restricted conditions, we remind that many authors have been worked on this topic. For example, Rassias et al. [20] proved the Ulam stability of Jensen and Jensen type mappings on restricted domains. Furthermore, many researchers have studied and proved the stability results of several interesting functional equations on restricted domains; one can refer to [1] (for some generalized functional equations on 2-Banach spaces with restricted domains), [7] (Measure zero stability problem of a quadratic functional equation), [18] (Hyers-Ulam stability of Davison functional equation on restricted domains) and references therein; see also [11].

In 2022, Bae et al. [3] studied the Hyers-Ulam stability of bi-linear functional equations on some restricted unbounded domains, and investigated some asymptotic behaviors of 2-linear functions. Motivated by the mentioned paper, in the current work, we establish the Hyers stability of multi-additive functional equations in restricted domains and then obtain some asymptotic behaviors of such functional equations. In continuation, we extend a stability result of [3] to multi-additive functional equations when the domain is a linear space. Indeed, we show that a multi-additive functional equation can be hyperstable when the range of the corresponding mapping is a 2-Banach space.

2. Stability of Multi-additive Functional Equations

In this section, we assume that $\mathbb{V}$ is a normed space and $\mathbb{W}$ is a Banach space. For simplicity, throughout the text, for any $l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, and for any $v = (v_1, \ldots, v_n) \in \mathbb{V}^n$, we define the scalar multiple $lv := (lv_1, \ldots, lv_n)$. Moreover, for each $i \in \{1, 2\}$, we denote $v_i^{[n]} := (v_{i1}, \ldots, v_{in}) \in \mathbb{V}^n$ and any element of V^n is specified by a power $[n]$.

To simplify the calculations involved in our results, we define the difference operator for a mapping $f : \mathbb{V}^n \to \mathbb{W}$ by

$$\mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) := f\left(v_1^{[n]} + v_2^{[n]}\right) - \sum_{i_1,\ldots,i_n \in \{1,2\}} f(v_{i_1 1}, \ldots, v_{i_n n}),$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$.

The following theorem concerns the Hyers-Ulam stability of multi-additive functional equations and follows directly from Theorem 1.2.

Theorem 2.1. *Assume that $(S, +)$ is a commutative semigroup with an identity element and $\mathbb{W}$ is a Banach space. Suppose that $f : S^n \to \mathbb{W}$ is a mapping satisfies*

$$\left\|\mathcal{D}f\left(s_1^{[n]}, s_2^{[n]}\right)\right\| \leq \varepsilon,$$

for all $s_1^{[n]}, s_2^{[n]} \in S^n$, where $\varepsilon \geq 0$. Then, there exists a unique multi-additive mapping $\mathcal{A} : S^n \to \mathbb{W}$ such that

$$\left\| f\left(s^{[n]}\right) - \mathcal{A}\left(s^{[n]}\right) \right\| \leq \frac{\varepsilon}{2^n - 1},$$

for all $s^{[n]} \in S^n$. Furthermore, $\mathcal{A}$ is given by

$$\mathcal{A}\left(s^{[n]}\right) = \lim_{j \to +\infty} \frac{1}{2^{nj}} f\left(2^j s^{[n]}\right),$$

for all $s^{[n]} \in S^n$.

We consider here a version of Theorem 2.1 in the context of a restricted domain. In other words, we show that the zero property of Theorem 6 from [5] is redundant and can be removed in the following result.

Theorem 2.2. *Given $\varepsilon \geq 0$ and $d > 0$. Suppose that $f : \mathbb{V}^n \to \mathbb{W}$ is a mapping fulfilling*

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) \right\| \leq \varepsilon, \tag{2.1}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$ with $\sum_{i=1}^{2} \sum_{j=1}^{n} \|v_{ij}\| \geq d$. Then, there exists a unique multi-additive mapping $\mathcal{A} : \mathbb{V}^n \to \mathbb{W}$ such that

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}\left(v^{[n]}\right) \right\| \leq \frac{2^{n+1}}{2^n - 1}\varepsilon,$$

for all $v^{[n]} \in \mathbb{V}^n$.

Proof. Putting $v_1^{[n]} = v_2^{[n]} = v^{[n]} := (u_1, \ldots, u_n)$ in (2.1), we obtain

$$\left\| f\left(2v^{[n]}\right) - 2^n f\left(v^{[n]}\right) \right\| \leq \varepsilon,$$

for all $v^{[n]} \in \mathbb{V}^n$ with $\sum_{j=1}^{n} \|u_j\| \geq d$. This leads, through a standard iterative approach, to the inequality

$$\left\| \frac{f\left(2^{m+1} v^{[n]}\right)}{2^{n(m+1)}} - \frac{f\left(2^l v^{[n]}\right)}{2^{nl}} \right\| \leq \varepsilon \sum_{k=l}^{m} \frac{1}{2^{(k+1)n}}, \quad \sum_{j=1}^{n} \|u_j\| \geq d, \tag{2.2}$$

for all $m > l \geq 0$. This inequality implies that the sequence $\left\{ \frac{f\left(2^m v^{[n]}\right)}{2^{nm}} \right\}_m$ is Cauchy in $\mathbb{W}$ for all $v^{[n]} \in \mathbb{V}^n$ with $\sum_{j=1}^{n} \|u_j\| \geq d$. It is straightforward to confirm that this sequence is indeed Cauchy for all $v^{[n]} \in \mathbb{V}^n$. Since $\mathbb{W}$ is complete, there exists a mapping $\mathcal{A} : \mathbb{V}^n \to \mathbb{W}$ defined by

$$\mathcal{A}\left(v^{[n]}\right) = \lim_{m \to +\infty} \frac{f\left(2^m v^{[n]}\right)}{2^{nm}}, \quad v^{[n]} \in \mathbb{V}^n.$$

Based on assumption (2.1) and the definition of $\mathcal{A}$, we deduce that $\mathcal{DA}\left(v_1^{[n]}, v_2^{[n]}\right) = 0$, for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n \setminus \{\mathbf{0}_n\}$, where $\mathbf{0}_n = (\overbrace{0, \dots, 0}^{n-\text{times}})$. Moreover, we have

$$\mathcal{A}\left(\mathbf{0}_n\right) = \lim_{m\to+\infty} \frac{f\left(\mathbf{0}_n\right)}{2^{nm}} = 0.$$

Thus

$$\mathcal{DA}\left(v_1^{[n]}, v_2^{[n]}\right) = 0, \quad v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n.$$

Consequently, the mapping $\mathcal{A}$ satisfies (1.1), and according to Theorem 1.1, it must be multi-additive. Furthermore, by setting $l = 0$ and allowing $m \to +\infty$ in (2.2), we obtain the following estimate

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}\left(v^{[n]}\right)\right\| \le \frac{\varepsilon}{2^n - 1}, \qquad \sum_{j=1}^{n} \|u_j\| \ge d. \tag{2.3}$$

Assume that $v_1^{[n]} \in \mathbb{V}^n$. We choose $v_2^{[n]} \in \mathbb{V}^n$ such that $\|v_{2j}\| \ge d + \|v_{1j}\|$ for each $j \in \{1, \dots, n\}$. With this choice, it is straightforward to verify that $\sum_{j=1}^n \|v_{1j} + v_{2j}\| \ge d$. Additionally, we have

$$\sum_{\substack{i_1,\dots,i_n\in\{1,2\}\\(i_1,\dots,i_n)\neq(1,\dots,1)}} \|v_{i_1 1}\| + \dots + \|v_{i_n n}\| \ge d.$$

From these inequalities with (2.1) and (2.3), it follows that

$$\left\| f\left(v_1^{[n]} + v_2^{[n]}\right) - \sum_{i_1,\dots,i_n\in\{1,2\}} f(v_{i_1 1}, \dots, v_{i_n n})\right\| \le \varepsilon,$$

$$\left\|\mathcal{A}\left(v_1^{[n]} + v_2^{[n]}\right) - f\left(v_1^{[n]} + v_2^{[n]}\right)\right\| \le \frac{\varepsilon}{2^n - 1},$$

$$\left\|\mathcal{A}\left(v_2^{[n]}\right) - f\left(v_2^{[n]}\right)\right\| \le \frac{\varepsilon}{2^n - 1},$$

$$\left\| \sum_{\substack{i_1,\dots,i_n\in\{1,2\}\\(i_1,\dots,i_n)\neq(1,\dots,1)}} \left[f\left(v_{i_1 1}, \dots, v_{i_n n}\right) - \mathcal{A}\left(v_{i_1 1}, \dots, v_{i_n n}\right)\right]\right\| \le \frac{(2^n - 1)\varepsilon}{2^n - 1}.$$

By summing the four inequalities above and applying the fact that $\mathcal{DA}\left(v_1^{[n]}, v_2^{[n]}\right) = 0$, we deduce that

$$\left\|\mathcal{A}\left(v_1^{[n]}\right) - f\left(v_1^{[n]}\right)\right\| \le \frac{2^{n+1}}{2^n - 1}\varepsilon,$$

for all $v_1^{[n]} \in \mathbb{V}^n$. This completes the proof. □

Remark 2.1. It is worth mentioning that the conclusion of Theorem 2.2 remains valid even if the mapping f satisfies condition (2.1) only for those $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$ where

$$\max\left\{\|v_{ij}\| : i = 1, 2,\ j = 1, \dots, n\right\} \ge d.$$

Let us now define the set

$$\Omega := \left\{ \left(v_1^{[n]}, v_2^{[n]}\right) \in \mathbb{V}^{2n} : \|v_{ij}\| < d, \text{ for all } i = 1, 2 \text{ and } j = 1, \dots, n \right\},$$

where $d > 0$ is a fixed constant. Clearly, we have the inclusion

$$\left\{ \left(v_1^{[n]}, v_2^{[n]}\right) \in \mathbb{V}^{2n} : \sum_{i=1}^{2} \sum_{j=1}^{n} \|v_{ij}\| \geq 2nd \right\} \subset \mathbb{V}^{2n} \setminus \Omega.$$

This observation naturally leads us to the following corollary.

Corollary 2.1. *Suppose that* $f : \mathbb{V}^n \to \mathbb{W}$ *satisfies* (2.1) *for all* $\left(v_1^{[n]}, v_2^{[n]}\right) \in \mathbb{V}^{2n} \setminus \Omega$. *Then, there exists a unique multi-additive mapping* $\mathcal{A} : \mathbb{V}^n \to \mathbb{W}$ *such that*

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}\left(v^{[n]}\right) \right\| \leq \frac{2^{n+1}}{2^n - 1} \varepsilon,$$

for all $v^{[n]} \in \mathbb{V}^n$.

An immediate consequence of Theorem 2.2 is the following description of the asymptotic behavior of the mapping f.

Corollary 2.2. *A mapping* $f : \mathbb{V}^n \to \mathbb{W}$ *is multi-additive if and only if*

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) \right\| \to 0 \tag{2.4}$$

as $\sum_{i=1}^{2} \sum_{j=1}^{n} \|v_{ij}\| \to +\infty$.

Proof. Suppose that the mapping f satisfies condition (2.4). Then, for each $m \in \mathbb{N}$, there exists a constant $d_m > 0$ such that $\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) \right\| \leq \frac{1}{m}$ for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$ with $\sum_{i=1}^{2} \sum_{j=1}^{n} \|v_{ij}\| \geq d_m$. From Theorem 2.2, it follows that there exists a unique multi-additive mapping $\mathcal{A}_m : \mathbb{V}^n \to \mathbb{W}$ satisfying

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}_m\left(v^{[n]}\right) \right\| \leq \frac{2^{n+1}}{(2^n - 1)m} \leq \frac{2^{n+1}}{2^n - 1}, \tag{2.5}$$

for all $v^{[n]} \in \mathbb{V}^n$. In particular, for $m = 1$, we also obtain

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}_1\left(v^{[n]}\right) \right\| \leq \frac{2^{n+1}}{2^n - 1},$$

for all $v^{[n]} \in \mathbb{V}^n$. From (2.5) with the uniqueness of $\mathcal{A}_1$, we conclude that $\mathcal{A}_m = \mathcal{A}_1$ for all $m \in \mathbb{N}$. Therefore, (2.5) implies

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}_1\left(v^{[n]}\right) \right\| \leq \frac{2^{n+1}}{(2^n - 1)m},$$

for all $m \in \mathbb{N}$. Taking the limit as $m \to +\infty$, we get $f = \mathcal{A}_1$, i.e., the mapping f is indeed multi-additive. The reverse implication is straightforward. □

Theorem 2.3. *Let* $\mathbf{w}$ *be a fixed element in* $\mathbb{W}$. *For a mapping* $f : \mathbb{V}^n \to \mathbb{W}$ *the following assertions are equivalent:*

(i) $\lim_{\sum_{i=1}^{2}\sum_{j=1}^{n}\|v_{ij}\|\to+\infty} \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) = \mathbf{w}$;

(ii) $\mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) = \mathbf{w}$ *for all* $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$.

Proof. (i)⇒(ii) Define a mapping $\Psi : \mathbb{V}^n \to \mathbb{W}$ by setting

$$\Psi\left(v^{[n]}\right) = f\left(v^{[n]}\right) + \frac{1}{2^n-1}\mathbf{w}, \quad v^{[n]} \in \mathbb{V}^n.$$

Then, for any $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$, we have

$$\mathcal{D}\Psi\left(v_1^{[n]}, v_2^{[n]}\right) = \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right) - \mathbf{w}.$$

From the assumption in (i), it follows that

$$\lim_{\sum_{i=1}^{2}\sum_{j=1}^{n}\|v_{ij}\|\to+\infty} \mathcal{D}\Psi\left(v_1^{[n]}, v_2^{[n]}\right) = 0.$$

Applying Corollary 2.2, we conclude that $\mathcal{D}\Psi\left(v_1^{[n]}, v_2^{[n]}\right) = 0$ for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$. This establishes the claim (ii). The implication (ii)⇒(i) is trivial. □

The following theorem is a special case of the pointwise stability result for multi-additive mappings established by Ciepliński [8, Theorem 1].

Theorem 2.4. *Assume that* $(S, +)$ *is a commutative semigroup and* $\mathbb{W}$ *is a Banach space. Suppose that* $f : S^n \to \mathbb{W}$ *satisfies*

$$\begin{aligned}&\Big\| f(s_1, \ldots, s_{j-1}, s_j + s_j', s_{j+1}, \ldots, s_n) \\ &- f(s_1, \ldots, s_{j-1}, s_j, s_{j+1}, \ldots, s_n) - f(s_1, \ldots, s_{j-1}, s_j', s_{j+1}, \ldots, s_n)\Big\| \le \varepsilon,\end{aligned} \tag{2.6}$$

for all $j \in \{1, \ldots, n\}$, *all* $(s_1, \ldots, s_{j-1}, s_j, s_j', s_{j+1}, \ldots, s_n) \in S^{n+1}$ *and some* $\varepsilon \ge 0$. *Then, for each* $j \in \{1, \ldots, n\}$ *there exists a multi-additive mapping* $\mathcal{A}_j : S^n \to \mathbb{W}$ *such that* $\|f(s_1, \ldots, s_n) - \mathcal{A}_j(s_1, \ldots, s_n)\| \le \varepsilon$, *for all* $(s_1, \ldots, s_n) \in S^n$. *Moreover, for any* $j \in \{1, \ldots, n\}$, *the mapping* $\mathcal{A}_j$ *is given through*

$$\mathcal{A}_j(s_1, \ldots, s_n) = \lim_{m\to+\infty} \frac{1}{2^m} f(s_1, \ldots, s_{j-1}, s_j, 2^m s_j, s_{j+1}, \ldots, s_n),$$

for all $(s_1, \ldots, s_n) \in S^n$.

By making use of Theorem 2.4, we derive the Hyers-Ulam stability result within a restricted domain. In fact, we remove the zero property from [5, Theorem 5] and obtain a more exact approximation.

Theorem 2.5. *Given* $\varepsilon \ge 0$ *and* $d > 0$. *Suppose that* $f : \mathbb{V}^n \longrightarrow \mathbb{W}$ *is a function satisfying* (2.6) *for all* $j \in \{1, \ldots, n\}$ *and all* $(v_1, \ldots, v_{j-1}, v_j, v_j', v_{j+1}, \ldots, v_n) \in \mathbb{V}^{n+1}$ *with* $\|v_j'\| + \sum_{j=1}^{n}\|v_j\| \ge d$, *then for any* $j \in \{1, \ldots, n\}$ *there exists a multi-additive mapping* $\mathcal{A}_j : \mathbb{V}^n \to \mathbb{W}$ *such that*

$$\|f(v_1, \ldots, v_n) - \mathcal{A}_j(v_1, \ldots, v_n)\| \le 5\varepsilon,$$

for all $(v_1, \dots, v_n) \in \mathbb{V}^n$.

Proof. Let $j \in \{1, \dots, n\}$ be fixed. Assume that $\|v'_j\| + \sum_{i=1}^n \|v_i\| < d$. If

$$\|v'_j\| + \sum_{i=1}^{n} \|v_i\| = 0,$$

we choose $u \in \mathbb{V}$ such that $\|u\| = d$. Otherwise, take $u = \left(1 + \frac{d}{\|u_0\|}\right) u_0$, where $u_0 \in \mathbb{V} \setminus \{0\}$ with $\|u_0\| >$max$\{\|v_j\|, \|v'_j\|\}$. Obviously, $\|u\| > d$ and

$$\sum_{k\in\{1,\dots,n\}\setminus\{j\}} \|v_k\| + \|v_j + u\| + \|v'_j - u\| \geq d.$$

Using (2.6), we get

$$\begin{aligned}
&\|f(v_1, \dots, v_{j-1}, 0, v_{j+1}, \dots, v_n)\| \\
=&\left\| f(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n) - \sum_{w\in\{u,0\}} f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \right\| \leq \varepsilon.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&f(v_1, \dots, v_{j-1}, v_j + v'_j, v_{j+1}, \dots, v_n) - \sum_{w\in\{v_j,v'_j\}} f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \\
=&f(v_1, \dots, v_{j-1}, v_j + u + v'_j - u, v_{j+1}, \dots, v_n) \\
&- \sum_{w\in\{v_j+u,v'_j-u\}} f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \\
&+ f(v_1, \dots, v_{j-1}, v_j + u, v_{j+1}, \dots, v_n) - \sum_{w\in\{v_j,u\}} f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \\
&+ f(v_1, \dots, v_{j-1}, v'_j - u, v_{j+1}, \dots, v_n) - \sum_{w\in\{v_j,-u\}} f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \\
&+ \sum_{w\in\{u,-u\}} f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) + f(v_1, \dots, v_{j-1}, 0, v_{j+1}, \dots, v_n) \\
&- f(v_1, \dots, v_{j-1}, 0, v_{j+1}, \dots, v_n).
\end{aligned}$$

It follows from (2.6) and the above relations that

$$\begin{aligned}
&\Big\| f(v_1, \dots, v_{j-1}, v_j + v'_j, v_{j+1}, \dots, v_n) \\
&- f(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_n) - f(v_1, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_n) \Big\| \leq 5\varepsilon,
\end{aligned}$$

for all $(v_1, \dots, v_{j-1}, v_j, v'_j, v_{j+1}, \dots, v_n) \in \mathbb{V}^{n+1}$. Now, Theorem 2.4 implies that there exists a multi-additive mapping $\mathcal{A}_j : \mathbb{V}^n \to \mathbb{W}$ such that

$$\|f(v_1, \dots, v_n) - \mathcal{A}_j(v_1, \dots, v_n)\| \leq 5\varepsilon,$$

for all $(v_1, \dots, v_n) \in \mathbb{V}^n$. □

In the next theorem, we show that a better bound for the obtained approximation in Theorem 2.5 can be improved through a different proof.

Theorem 2.6. *Under the assumptions of Theorem* 2.5*, for any* $j \in \{1, \dots, n\}$*, there exists a multi-additive mapping* $\mathcal{A}_j : \mathbb{V}^n \to \mathbb{W}$ *such that*

$$\|f(v_1, \dots, v_n) - \mathcal{A}_j(v_1, \dots, v_n)\| \leq 3\varepsilon, \tag{2.7}$$

for all $(v_1, \dots, v_n) \in \mathbb{V}^n$*.*

Proof. Fix an index $j \in \{1, \dots, n\}$. Putting $v_j = v_j' = w$ in (2.6), we arrive at

$$\|f(v_1, \dots, v_{j-1}, 2w, v_{j+1}, \dots, v_n) - 2f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n)\| \leq \varepsilon,$$

for all $(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \in V^n$ satisfying $\|w\| \geq d$. From this, it follows that

$$\left\| \frac{f(v_1, \dots, v_{j-1}, 2^m w, v_{j+1}, \dots, v_n)}{2^m} - \frac{f(v_1, \dots, v_{j-1}, 2^k w, v_{j+1}, \dots, v_n)}{2^k} \right\| \leq \sum_{i=k}^{m-1} \frac{\varepsilon}{2^{i+1}}, \tag{2.8}$$

for all $(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \in V^n$ with $\|w\| \geq d$ and integers $m > k \geq 0$. From (2.8), we deduce that the sequence

$$\left\{ \frac{f(v_1, \dots, v_{j-1}, 2^m w, v_{j+1}, \dots, v_n)}{2^m} \right\}_m$$

is Cauchy, and therefore convergent for all $(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \in V^n$. We then consider the mapping $\mathcal{A}_j : \mathbb{V}^n \to \mathbb{W}$ defined via

$$\mathcal{A}_j(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) := \lim_{m \to +\infty} \frac{f(v_1, \dots, v_{j-1}, 2^m w, v_{j+1}, \dots, v_n)}{2^m}.$$

Clearly, $\mathcal{A}_j(\mathbf{0}_n) = 0$. From (2.6), it follows that $\mathcal{A}_j$ is multi-additive. Moreover, by (2.8) we have

$$\|\mathcal{A}_j(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) - f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n)\| \leq \varepsilon, \tag{2.9}$$

for all $(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \in V^n$ with $\|w\| \geq d$. Let $u \in V$ and choose $w \in V$ such that $\min\{\|w\|, \|u + w\|\} \geq d$. Applying (2.6) together with (2.9) gives

$$\begin{aligned} &\Big\| f(v_1, \dots, v_{j-1}, u + w, v_{j+1}, \dots, v_n) \\ &- f(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n) - f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) \Big\| \leq \varepsilon, \\ &\|\mathcal{A}_j(v_1, \dots, v_{j-1}, u + w, v_{j+1}, \dots, v_n) - f(v_1, \dots, v_{j-1}, u + w, v_{j+1}, \dots, v_n)\| \leq \varepsilon, \\ &\|f(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n) - \mathcal{A}_j(v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_n)\| \leq \varepsilon. \end{aligned}$$

By summing these inequalities and using the multi-additivity of $\mathcal{A}_j$, we obtain

$$\|\mathcal{A}_j(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n) - f(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n)\| \leq 3\varepsilon,$$

for all $(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n) \in V^n$, which establishes (2.7). □

3. Stability Results in 2-Banach Spaces

We begin by briefly reviewing some historical background on 2-normed spaces. The concept of a 2-normed space was first introduced and explored by Gähler [12, 13].

Let V be an at least two-dimensional real linear space over $\mathbb{R}$. Let $\|\cdot,\cdot\| : V \times V \to \mathbb{R}$ be a function satisfying the following conditions for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$:

(i) $\|u, v\| = 0$ if and only if u and v are linearly dependent;
(ii) $\|u, v\| = \|v, u\|$;
(iii) $\|\alpha u, v\| = |\alpha|\|u, v\|$;
(iv) $\|u, v + w\| \leq \|u, v\| + \|u, w\|$.

The function $\|\cdot,\cdot\|$ is then referred to as a 2-*norm* on V, and the pair $(V, \|\cdot,\cdot\|)$ is known as a 2-*normed space*. Obviously, it follows from (ii)-(iv) that $\|u, v\| \geq 0$ for each $u, v \in V$.

Recall that a sequence $\{u_m\}_m$ in a 2-normed space $(V, \|\cdot,\cdot\|)$ is said to be 2-*Cauchy* if there exist two linearly independent vectors $v, w \in V$ such that

$$\lim_{k,j\to+\infty} \|u_k - u_j, v\| = \lim_{k,j\to+\infty} \|u_k - u_j, w\| = 0.$$

Furthermore, the sequence $\{u_m\}_m$ is said to be 2-*convergent* if there exists an element $v \in V$ such that $\lim_{m\to+\infty} \|u_m - v, w\| = 0$ for all $w \in V$. This element v is referred to as the *limit* of the sequence $\{u_m\}_m$, and is denoted by $\lim_{m\to+\infty} u_m$. A 2-normed space $(V, \|\cdot,\cdot\|)$ is said to be a 2-*Banach space* if every 2-Cauchy sequence in V converges in the 2-norm sense.

Lemma 3.1 ([19])**.** *Let $(V, \|\cdot,\cdot\|)$ be a 2-normed space. Then, the following assertions hold.*

(i) $|\|u, v\| - \|w, v\|| \leq \|u - w, v\|$ *for all* $u, v, w \in V$.
(ii) *If* $\|u, v\| = 0$ *for all* $v \in V$*, then* $u = 0$.
(iii) *For any 2-convergent sequence* $\{u_m\}_m$ *in* $(V, \|\cdot,\cdot\|)$*, we have*

$$\lim_{m\to+\infty} \|u_m, v\| = \left\| \lim_{m\to+\infty} u_m, v \right\|,$$

for all $v \in V$.

Lemma 3.2. ([17]) *Suppose that $(V, \|\cdot,\cdot\|)$ is a 2-normed space and $u, v \in V$ are linearly independent elements. If $x \in V$ and $\|x, u\| = \|x, v\| = 0$, then $x = 0$.*

Remark 3.1. Let $(W, \|\cdot,\cdot\|)$ be a 2-normed space, and suppose that $\{w_1, w_2\}$ is a linearly independent subset of W. According to Theorem 8 in [22], the function $\|\cdot\| : W \to [0, +\infty)$ defined by $\|w\| = \|w, w_1\| + \|w, w_2\|$, is a norm on W.

Here, we investigate the hyperstability of multi-additive mappings defined from a Cartesian power of a linear space into a 2-Banach space. It is worth noting that this result extends Theorem 6 in [3], which was established for linear spaces.

Theorem 3.1. *Let $\delta, \varepsilon \in [0, +\infty)$ and consider a linear space V and a 2-normed space W. Let $\{w_1, w_2\}$ be a fixed linearly independent subset of W. Assume that the mapping $f : V^n \to W$ satisfies the inequality*

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), w \right\| \leq \begin{cases} \delta + \varepsilon \|w\|^p, & p < 1, \\ \varepsilon \|w\|^p, & p > 1, \end{cases} \tag{3.1}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $w \in W \setminus \{0\}$, where $\| \cdot \|$ denotes the norm introduced in Remark 3.1. Then, f is a multi-additive mapping.

Proof. Assume that $p < 1$. The case $p > 1$ can be handled in a similar manner. By replacing w with mw in (3.1), we arrive at

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), w \right\| \leq \frac{\delta}{m} + m^{p-1} \varepsilon \|w\|,$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $m \in \mathbb{N}$. Letting $m \to +\infty$ yields

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), w \right\| = 0,$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $w \in \{w_1, w_2\}$. Therefore, by invoking Lemma 3.2, we conclude that f is multi-additive. $\square$

A class of the Hyers stability of multi-additive functional equations is given as follows.

Theorem 3.2. *Let $\delta, \varepsilon \in [0, +\infty)$ and consider a linear space V and a 2-Banach space W. Let $\{w_1, w_2\}$ be a fixed linearly independent subset of W. Assume that the mapping $f : V^n \to W$ satisfies the inequality*

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), w \right\| \leq \delta + \varepsilon \|w\|, \tag{3.2}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $w \in W$, where $\| \cdot \|$ denotes the norm introduced in Remark 3.1. Then, there exists a unique multi-additive mapping $\mathcal{A} : V^n \to W$ such that

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}\left(v^{[n]}\right), w \right\| \leq \frac{\varepsilon \|w\|}{2^n - 1}, \tag{3.3}$$

for all $v^{[n]} \in V^n$ and $w \in W$.

Proof. Substituting w with mw in inequality (3.2), we obtain

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), w \right\| \leq \frac{\delta}{m} + \varepsilon \|w\|,$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n, w \in W$ and $m \in \mathbb{N}$. Taking the limit as $m \to +\infty$, we get

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), w \right\| \leq \varepsilon \|w\|, \tag{3.4}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $w \in W$. Putting $v_1^{[n]} = v_2^{[n]} := v^{[n]}$ in (3.4), we obtain

$$\left\| f\left(2v^{[n]}\right) - 2^n f\left(v^{[n]}\right), w \right\| \leq \varepsilon \|w\|,$$

for all $v^{[n]} \in V^n$ and all $w \in W$. Thus, through a straightforward calculation, one arrives at the inequality

$$\left\| \frac{f\left(2^m v^{[n]}\right)}{2^{nm}} - \frac{f\left(2^l v^{[n]}\right)}{2^{nl}}, w \right\| \leq \sum_{q=l}^{m-1} \frac{\varepsilon \|w\|}{2^{(q+1)n}}, \tag{3.5}$$

which holds for all $v^{[n]} \in V^n$, $w \in W$, and integers $m > l \geq 0$. This shows that the sequence $\left\{ \frac{f\left(2^m v^{[n]}\right)}{2^{nm}} \right\}_m$ is 2-Cauchy in W, and hence it converges in the 2-norm for each $v \in V^n$. Based on this, define the mapping $\mathcal{A} : V^n \to W$ by

$$\mathcal{A}\left(v^{[n]}\right) := \lim_{m \to +\infty} \frac{f\left(2^m v^{[n]}\right)}{2^{nm}}, \quad v^{[n]} \in V^n.$$

By setting $l = 0$ and taking the limit as $m \to +\infty$ in inequality (3.5), and then applying Lemma 3.1, we arrive at the desired inequality (3.3). Now, replacing $\left(v_1^{[n]}, v_2^{[n]}\right)$ by $\left(2^m v_1^{[n]}, 2^m v_2^{[n]}\right)$ in (3.4) and dividing both sides of the resultant by 2^{nm}, we get

$$\left\| \frac{1}{2^{nm}} \mathcal{D}f\left(2^{nm} v_1^{[n]}, 2^{nm} v_2^{[n]}\right), w \right\| \leq \frac{\varepsilon \|w\|}{2^{mn}},$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n, w \in W$, and $m \in \mathbb{N}$. Taking the limit as $m \to +\infty$ in the above inequality and applying Lemma 3.1, we conclude that $\mathcal{D}\mathcal{A}\left(v_1^{[n]}, v_2^{[n]}\right) = 0$ for all $v_1^{[n]}, v_2^{[n]} \in V^n$. Hence, $\mathcal{A}$ is a multi-additive mapping.

To establish the uniqueness of $\mathcal{A}$, assume that $\mathfrak{A} : V^n \to W$ is another multi-additive mapping satisfying

$$\left\| f\left(v^{[n]}\right) - \mathfrak{A}\left(v^{[n]}\right), w \right\| \leq \frac{\varepsilon \|w\|}{2^n - 1},$$

for all $v^{[n]} \in V^n$ and $w \in W$. Since $\mathfrak{A}\left(2^m v^{[n]}\right) = 2^{mn} \mathfrak{A}\left(v^{[n]}\right)$ for all $v^{[n]} \in V^n$ and all $m \in \mathbb{N}$, it follows that

$$\mathfrak{A}\left(v^{[n]}\right) = \lim_{m \to +\infty} \frac{f\left(2^m v^{[n]}\right)}{2^{nm}}, \quad v^{[n]} \in V^n.$$

Hence, $\mathcal{A} = \mathfrak{A}$. □

In the incoming result, we establish the Rassias stability of multi-additive functional equations when both domain and range of a mapping are 2-normed space.

Theorem 3.3. *Let $r \in (0, +\infty)$, with $r \neq n$, and suppose V is a 2-normed space, W is a 2-Banach space, and $g : V \to W$ is a surjective mapping. Assume that a mapping $f : V^n \to W$ satisfies the inequality*

$$\left\| \mathcal{D}f\left(v_1^{[n]}, v_2^{[n]}\right), g(u) \right\| \leq \delta \sum_{i=1}^{2} \sum_{j=1}^{n} \|v_{ij}, u\|^r, \tag{3.6}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ *and* $u \in V$*, where* $\delta > 0$ *is a constant. Then, there exists a unique multi-additive mapping* $\mathcal{A} : V^n \to W$ *such that*

$$\left\| f\left(v^{[n]}\right) - \mathcal{A}\left(v^{[n]}\right), g(u) \right\| \leq \frac{2\delta}{|2^n - 2^r|} \sum_{j=1}^{n} \|v_j, u\|^r, \tag{3.7}$$

for all $v^{[n]} = (v_1, \ldots, v_n) \in V^n$ *and* $u \in V$*.*

Proof. Without loss of generality, we may assume that $r < n$; the argument for $r > n$ follows in a similar manner. Let us choose $v_1^{[n]} = v_2^{[n]} := v^{[n]} = (v_1, \ldots, v_n)$ in (3.6). This yields

$$\left\| f\left(2v^{[n]}\right) - 2^n f\left(v^{[n]}\right), g(u) \right\| \leq 2\delta \sum_{j=1}^{n} \|v_j, u\|^r,$$

for all $v^{[n]} \in V^n$ and $u \in V$. This inequality leads to

$$\left\| \frac{f\left(2^m v^{[n]}\right)}{2^{nm}} - \frac{f\left(2^l v^{[n]}\right)}{2^{nl}}, g(u) \right\| \leq \frac{2\delta}{2^n} \sum_{q=l}^{m-1} \left(\frac{2^r}{2^n}\right)^q \sum_{j=1}^{n} \|v_j, u\|^r, \tag{3.8}$$

for all $v^{[n]} \in V^n$, $u \in V$ and $m > l \geq 0$. Hence, the sequence $\left\{ \frac{f\left(2^m v^{[n]}\right)}{2^{nm}} \right\}$ forms a 2-Cauchy sequence in W. Since W is 2-complete, this sequence converges (in the 2-norm sense) for every $v^{[n]} \in V^n$. Define the mapping $\mathcal{A} : V^n \to W$ through

$$\mathcal{A}\left(v^{[n]}\right) := \lim_{m \to +\infty} \frac{f\left(2^m v^{[n]}\right)}{2^{nm}}, \quad v^{[n]} \in V^n.$$

Putting $l = 0$ in (3.8) and taking the limit as $m \to +\infty$, and then applying Lemma 3.1, we arrive at the inequality stated in (3.7). Now, replacing $\left(v_1^{[n]}, v_2^{[n]}\right)$ with $\left(2^m v_1^{[n]}, 2^m v_2^{[n]}\right)$ in (3.6) and then dividing both sides by 2^{nm}, we obtain

$$\frac{1}{2^{nm}} \left\| \mathcal{D}\left(2^m v_1^{[n]}, 2^m v_2^{[n]}\right), g(u) \right\| \leq \delta \left(\frac{2^r}{2^n}\right)^m \sum_{i=1}^{2} \sum_{j=1}^{n} \|v_{ij}, u\|^r,$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$, $u \in V$ and $m \in \mathbb{N}$. Letting $m \to +\infty$ in (3.8), we arrive at

$$\left\| \mathcal{D}\mathcal{A}\left(v_1^{[n]}, v_2^{[n]}\right), g(u) \right\| = 0,$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$ and $u \in V$. Since g is surjective, it follows that $\mathcal{D}\mathcal{A}\left(v_1^{[n]}, v_2^{[n]}\right) = 0$ for all $v_1^{[n]}, v_2^{[n]} \in V^n$. Therefore, $\mathcal{A}$ is a multi-additive mapping. From (3.7), the uniqueness of $\mathcal{A}$ can be readily deduced. □

4. Conclusions

In this article, we have extended and generalized some stability results of Bae et al. [On asymptotic behavior of a 2-Linear functional equation, Math. **10** (2022), 1685] to the case of n variables. We have also established the Hyers-Ulam stability of multi-additive functional equations on restricted domains. In [5], the first author proposed a question as follows: Can we remove the zero property from the approximate multi-additive mappings and obtain better approximations? The answer is affirmative. In fact, we have removed the zero property in Theorem 2.2 and Theorem 2.5 and have found the more exact approximations and less errors. In continuation, the authors have studied the asymptotic behavior of multi-additive mappings. Furthermore, we have generalized a stability result for bilinear equations (as a special case) to multi-additive functional equations from a linear space to a 2-Banach space. Finally, we have presented a hyperstability result of multi-additive functional equations.

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