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APPROXIMATE CONSERVATION LAWS AND SYMMETRY OPERATORS FOR FRACTIONAL DIFFERENTIAL HARRY-DYM EQUATION WITH A SMALL PERTURBATION PARAMETER

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Abstract. The approximate Lie group analysis of differential equations is applied in order to find symmetry operators of time-fractional Harry-Dym equation. First the method of finding symmetries is extended to approximate fractional differential equations and the corresponding reduced form of the equation are derived. The Riemann-Liouville and Caputo definitions are used in this case. Then, the perturbed conservation laws are computed with the modified version of Noether's theorem based on the formal Lagrangian.

1. Introduction

Lie symmetries of differential equations are so powerful tools for study structure of a given system of differential equations specially the exact solutions. Several problems in physics, enginnering, economics, etc., are described by differential equations, thus, the importance of this field shows itself more here [4, 5, 7, 11, 13, 15–17, 20, 30, 37, 39–42, 47]. Nowadays this theory is extended on fractional differential equations (FDEs).

Fractional partial differential equations (FPDEs) are widely used to describe various physical effects and many complex phenomena and the other various field such as: electrochemistry, quantitative biology, engineering, mechanics and etc. [28, 50]. Also the use of fractional differentiation for the mathematical modeling of real world has been widespread at the recent years. For example the optical soliton perturbation with fractional temporal evolution is one of the viable means to address a growing problem in telecommunication industry, namely the Internet bottleneck. For another

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example the use of fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties and etc. [31]. Consequently, several excellent and illustrative articles are written about this method [14, 32, 38, 43, 48]. Also we have some other valuable papers contain diverse methods to study fractional differential equations [12, 25, 27, 29, 44]

Conservation laws are the consequences of fundamental properties of nature. It is well known that they have a close connection with symmetries. For integer-order differential equations with Lagrangians, conservation laws can be found by corresponding variational symmetries using Noether's theorem; which makes a correspondence between a variational symmetry and a conservation law of a system (see, e.g., [21, 40]). As a limitation of Noether's theorem is what if the system has no any Lagrangian. So, for differential equations without Lagrangians their conservation laws can be constructed using several approaches, such as the concept of non-linear self-adjointness as the modification of Noether's theorem [1, 6, 18, 19, 22, 52], the direct construction method [2, 3], the method of A-operators [8].

However, conservation laws can be constructed not only for integer-order differential equations, but also for fractional differential equations (FDEs). Such equations contain differential and integral operators of non-integer order [26, 49], and they have received significant interest over the last few decades (see books [26, 51] and references therein). At present, FDEs are successfully used in science and engineering as mathematical models of systems and phenomena with memory and spatial non-locality [9, 45]. A problem of constructing conservation laws for FDEs is investigated by many researchers (see, e.g., [10] and references therein). In [35] the concept of non-linear self-adjointness was adopted for FDEs without Lagrangians, and in [36] the explicit algorithm for constructing conservation laws for a wide class of such equations using their Lie point symmetrie was developed.

The main purpose of the paper is to calculate the fractional symmetries of the perturbed equation

$$D_t^{\alpha} u = 2\left(\frac{1}{\sqrt{u}}\right)_{xxx} + \epsilon u_t, \quad \alpha \in (0,1). \tag{1.1}$$

The paper is outlined as follows. Some general and standard definitions and concepts of fractional derivatives including the basic results of the symmetry analysis of perturbed fractional differential equations are given in Section 2. Section 3 is devoted for the computation of symmetries and reduced equations constructed by the invariants of the operators. Finally, the method of finding conservation laws based on the modified Noether's theorem is extended to perturbed fractional differential equations in order to give the conservation laws of the Eq. (1.1).

2. Definitions for Approximating Fractional Differential Equations

For simplicity, we restrict our attention to the case of FDE with fractional derivatives with respect to only one independent variable x^1:

$$F(x, u, u_1, \dots, u_k, {}_aD_{x^1}^{\alpha_0}u, \dots, {}_aD_x^{\alpha_m}u, {}_{x^1}D_b^{\beta_0}u, \dots, {}_{x^1}D_b^{\beta_l}u) = 0, \tag{2.1}$$

where $k, l, m \in \mathbb{N}$, $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_m$, $0 < \beta_0 < \beta_1 < \cdots < \beta_l$, ${}_aD_{x_1}^{\alpha_i}u$ and ${}_{x_1}D_b^{\beta_j}u$ are the left-sided and the right-sided fractional derivatives, respectively [26]. In Eq. (2.1) $u = u(x)$ is a function of independent variables $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$, $x_1 \in (a, b)$, and

$$u_{(s)} \equiv \left\{u_{i_1\cdots i_s}\right\} = \left\{\frac{\partial^s u(x)}{\partial x^{i_1}\cdots\partial x^{i_s}}\right\}, \quad i_1, \dots, i_s = 1, \dots, n, s = 1, \dots, k, \tag{2.2}$$

are the so-called successive derivatives of differential variable u with respect to the independent variables ξ (see, e.g., [20]). If the orders of fractional derivatives in Eq. (2.1) are all nearly integers, then this equation can be approximated by a differential equation with small parameters which can be extracted from these orders. In a particular case when orders of all fractional derivatives in Eq. (2.1) have the same small deviation from the nearest integer number, this equation can be approximated by a differential equation with a single small parameter. In this case Eq. (2.1) can be written as

$$F\left(x, u, u_1, \dots, u_k, {}_aD_{x^1}^{\alpha}u, \dots, {}_aD_{x^1}^{\alpha+m}u, {}_{x^1}D_b^{\beta}u, \dots, {}_{x^1}D_b^{\beta+l}u\right) = 0, \tag{2.3}$$

where $\alpha \in (0, 1)$, $k, m \in \mathbb{N}$.

We assume that the left- and right-sided Riemann-Liouville fractional derivatives

$$({}_\alpha D_{x^1}^{\alpha+j}u)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x^1}\right)^{j+1}\int_\alpha^{x^1}\frac{u(\xi, x^2, \dots, x^n)}{(x^1-\xi)^\alpha}d\xi, \tag{2.4}$$

$$({}_{x^1}D_b^{\alpha+j}u)(x) = \frac{(-1)^{j+1}}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x^1}\right)^{j+1}\int_{x^1}^{b}\frac{u(\xi, x^2, \dots, x^n)}{(x^1-\xi)^\alpha}d\xi, \tag{2.5}$$

or the Caputo fractional derivatives

$$({}_\alpha^c D_{x^1}^{\alpha+j}u)(x) = \frac{1}{\Gamma(1-\alpha)}\int_\alpha^{x^1}\frac{\partial^{j+1}u(\xi, x^2, \dots, x^n)}{\partial\xi^{j+1}(x^1-\xi)^\alpha}d\xi, \tag{2.6}$$

$$({}_\alpha^c D_b^{\alpha+j}u)(x) = \frac{(-1)^{j+1}}{\Gamma(1-\alpha)}\int_{x^1}^{b}\frac{\partial^{j+1}u(\xi, x^2, \dots, x^n)}{\partial\xi^{j+1}(x^1-\xi)^\alpha}d\xi, \tag{2.7}$$

can be used in Eq. (2.3) as ${}_\alpha D_{x^1}^{\alpha+j}$ and ${}_{x^1}D_b^{\alpha+j}$, $\alpha \in (0, 1)$, $j = 0, 1, \dots, m$.

Let in Eq. (2.3) $\alpha = \epsilon$ or $\alpha = 1-\epsilon$, where ϵ is a small parameter: $0 \prec \epsilon \preceq 1$ function $u(x)$ is such that the left-sided Riemann-Liouville fractional derivatives ${}_\alpha D_{x^1}^{\alpha+\epsilon}u$, $j = 0, 1, \dots,$ and ${}_{x^1}D_b^{\alpha-\epsilon}u$, $j = 1, 2, \dots,$ exist and at each point $x^1 \in (a, b)$ they can be expanded into the series

$${}_\alpha D_{x_1}^{j\pm\epsilon} = \sum\binom{j\pm\epsilon}{s}\frac{(x^1-\alpha)^{s-j\pm\epsilon}}{\Gamma(1+s-j\pm\epsilon)}\cdot\frac{\partial^s u(\xi, x^2, \dots, x^n)}{\partial\xi^s},$$

where $\binom{j\pm\epsilon}{s} = \frac{\Gamma(1+j\pm\epsilon)}{\Gamma(1+s-j\pm\epsilon)s!}$ is a binomial coefficient. Then the following expansion is valid

$$
{}_{\alpha}D_{x_1}^{j\pm\epsilon}\frac{\partial^j u}{\partial(x^1)^j} \pm \epsilon \left\{ \begin{array}{c} [\psi(j+1) - \ln(x^1-\alpha)]\dfrac{\partial^j u}{\partial(x^1)^j} \\ -\sum_{s=1, s\neq j}^{+\infty} \dfrac{(-1)^{s-j}}{(s-j)} \cdot \dfrac{j!}{s!}(x^1-\alpha)^{s-j}\dfrac{\partial^s u}{\partial(x^1)^s} \end{array} \right\} + \mathcal{O}(\epsilon).
$$

Here $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function, and j is an integer number so that $j \pm \epsilon > 0$.

Similar expansion can be obtained for the right-sided Riemann-Liouville fractional derivative (2.4) and (2.5). In [37] it was proved that

$$
{}_{x^1}D_{b}^{j\pm\epsilon}\frac{\partial^j u}{\partial(x^1)^j} \pm \epsilon \left\{ \begin{array}{c} [\psi(j+1) - \ln(x^1-\alpha)]\dfrac{\partial^j u}{\partial(x^1)^j} \\ -\sum_{s=1, s\neq j}^{+\infty} \dfrac{(-1)^{s-j}}{(s-j)} \cdot \dfrac{j!}{s!}(x^1-\alpha)^{s-j}\dfrac{\partial^s u}{\partial(x^1)^s} \end{array} \right\} + \mathcal{O}(\epsilon).
$$

The expansion of Caputo's fractional derivatives is obtained similarly [26].

Definition 2.1. Relations between Caputo and Riemann-Liouville fractional derivatives are established by [49],

$$
\begin{aligned}
{}_{\alpha}^{C}D_{x^1}^{\alpha+j}u &= {}_{\alpha}D_{x^1}^{\alpha+j}u - \sum_{s=0}^{j} \frac{(x^1-a)^{s-j-\alpha}}{\Gamma(1+s-j-\alpha)} \cdot \frac{\partial^s u}{\partial(x^1)^s}\bigg|_{x^1=a}, \\
{}_{x^1}^{C}D_{\alpha}^{\alpha+j}u &= {}_{\alpha}D_{b}^{\alpha+j}u - \sum_{s=0}^{j} \frac{(b-x^1)^{s-j-\alpha}}{\Gamma(1+s-j-\alpha)} \cdot \frac{\partial^s u}{\partial(x^1)^s}\bigg|_{x^1=b}.
\end{aligned}
$$

For the left-sided Caputo fractional derivative (2.6) and (2.7) it can be proved that

$$
{}_{\alpha}^{C}D_{x^1}^{j\pm\epsilon}u = {}_{\alpha}D_{x^1}^{j\pm\epsilon}u \pm \sum_{s=0}^{j-1}(-1)^{s-j}(j-s-1)!(x^1-\alpha)^{s-j}\frac{\partial^s u}{\partial(x^1)^s}\bigg|_{x^1=\alpha} + g(x+\alpha), \tag{2.8}
$$

where

$$
g(x+\alpha) = \begin{cases} -[1+\epsilon(\psi(j+1)-\ln(x^1-\alpha)]\dfrac{\partial^j u}{\partial(x^1)^j}\bigg|_{x^1=\alpha}, & \text{for } {}_{\alpha}^{C}D_{x^1}^{j+\epsilon}u, \\ 0, & \text{for } {}_{\alpha}^{C}D_{x^1}^{j+\epsilon}u. \end{cases}
$$

Expansions for the right-sided Caputo derivatives (2.7) can be obtained from Eq. (2.8) by changing the point $x^1 = a$ to $x^1 = b$ and by changing the expression $(x^1 - a)$ to $(b - x^1)$.

If $\alpha = \epsilon$ or $\alpha = 1 - \epsilon$ then, Eq. (2.3) can be approximated by the perturbed integer-order differential equation

$$
F_{(0)}(x, u_1, \dots u_l) + \epsilon F_{(1)}(x, u_1, \dots, u_l, D_{x^1}^{l+1}u, D_{x^1}^{l+2}u, \dots) \simeq 0. \tag{2.9}
$$

Here $l = \max\{k, m\}$ for $\alpha = 1 - \epsilon$ and $l = \max\{k, m-1\}$ for $\alpha = \epsilon$.

Definition 2.2. The generator of an approximate Lie point transformation group [32, 34]

$$\begin{aligned}\tilde{x}^i \simeq & f^i(x,u,\alpha,\epsilon) \equiv f^i_{(0)}(x,u,\alpha) + \epsilon f^i_{(1)}(x,u,\alpha), \quad i=1,2,\dots,n,\\ \tilde{u} \simeq & g(x,u,\alpha,\epsilon) \equiv g_{(0)}(x,u,\alpha) + \epsilon g_{(1)}(x,u,\alpha),\end{aligned}$$

satisfying the conditions $\tilde{x}^i|_{\alpha=0} \simeq x^i$ and $\tilde{u}|_{\alpha=0} \simeq u$, is a first-order differential operator

$$V \simeq V_0 + \epsilon V_1 \equiv \left(\xi_0^i(x,u) + \epsilon \xi_1^i(x,u)\right)\frac{\partial}{\partial x^i} + (\eta_0^i(x,u) + \epsilon \eta_1^i(x,u))\frac{\partial}{\partial u}, \tag{2.10}$$

where

$$\begin{aligned}\xi_0^i(x,u) &= \frac{\partial f_0^i(x,u,\alpha)}{\partial \alpha}\Big|_{\alpha=0}, \quad \xi_1^i(x,u) = \frac{\partial f_1^i(x,u,\alpha)}{\partial \alpha}\Big|_{\alpha=0},\\ \eta_0(x,u) &= \frac{\partial g_0(x,u,\alpha)}{\partial \alpha}|_{\alpha=0}, \quad \eta_1(x,u) = \frac{\partial g_1(x,u,\alpha)}{\partial \alpha}\Big|_{\alpha=0}, \quad i=1,2,\dots,n.\end{aligned}$$

Definition 2.3. For (2.9) the operator (2.10) satisfying the equation [33],

$$V(f_0 + \epsilon f_1)|_{(2,9)} \simeq 0,$$

is called an approximate symmetry of (2.9).

3. Symmetry Analysis on Perturbed Equation

Constructing approximate symmetries first, you need to apply the extension ${}_\alpha D_{x^1}^{j\pm\epsilon}$ for equation by setting $\alpha = 1-\epsilon$. The approximation of Eq. (1.1) in the form

$$\begin{aligned}f_0 + \epsilon f_1 =& u_t + 2\left(\frac{1}{\sqrt{u}}\right)_{xxx}\\ &+ \epsilon\left[(\ln t + \gamma - 1)u_t + \frac{u}{t} + \sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D_t^{n+1}u\right] \approx 0,\end{aligned} \tag{3.1}$$

is achieved.

It follows from Eq. (3.1) that $\epsilon u_t \approx 0$. Therefore, $\epsilon D_t^{n+1} u \cong 0$ for $n \geq 0$ and the infinite series in (3.1) is also approximately equal to zero. Thus, (3.1) takes a very simple form

$$u_t + 2\left(\frac{1}{\sqrt{u}}\right)_{xxx} + \epsilon\frac{u}{t} \approx 0. \tag{3.2}$$

To begin the process we need to get symmetries of perturbed equation (3.1). These operators are found via a systematic computations. Thus, the Lie algebra G of the symmetries is spanned with the following geometric vector fields:

$$\begin{aligned}&V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad V_4 = t\frac{\partial}{\partial t} + \frac{2u}{3}\frac{\partial}{\partial u},\\ &V_5 = \frac{x^2}{2}\frac{\partial}{\partial x} - 2xu\frac{\partial}{\partial u}, \quad V_6 = \epsilon\frac{\partial}{\partial t}, \quad V_7 = \epsilon\frac{\partial}{\partial x}.\end{aligned}$$

4. Reduction

The process of reduction for perturbed FDE is as same as the others. In this section some reduced form of the Eq. (3.1) are given.

Case 1. For the symmetry V_1, the corresponding characteristic equation is

$$\frac{dt}{1}=\frac{dx}{0}=\frac{du}{0}.$$

Integration the group trajectories provides the following similarity variable and function

$$u=g(r),\quad r=x,$$

for reduction. Thus, the reduced equation

$$\frac{-4g'''g^2+18g'g''g-15g'^3}{4g^{7/2}}+\epsilon\left[(\ln t+\gamma-1)+\frac{g}{t}+\sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D_t^{n+1}u\right]\approx 0,$$

is obtained.

Case 2. Symmetry V_2, yields the characteristic equation

$$\frac{dt}{0}=\frac{dx}{1}=\frac{du}{0}.$$

Integration this system gives the following similarity variable and function

$$u=g(r),\quad r=t.$$

So, the reduced equation

$$g'+\epsilon\left[(\ln t+\gamma-1)g'+\frac{g}{t}+\sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D_t^{n+1}u\right]\approx 0,$$

is constructed.

Case 3. The operator V_3, has the characteristic equation

$$\frac{dt}{0}=\frac{dx}{x}=\frac{du}{-2u}.$$

The solution of this system gives the similarity variable and function

$$u=g(r)x^{-2},\quad r=t.$$

So, the reduced equation is

$$g'x^{-2}+\epsilon\left[(\ln t+\gamma-1)g'x^{-2}+\frac{g}{tx^2}+\sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D_t^{n+1}u\right]\approx 0.$$

Case 4. Finally for the symmetry V_4, we conclude

$$\frac{dt}{0}=\frac{dx}{\frac{1}{2}x^2}=\frac{du}{-2xu},$$

as the system of charachteristics. Integration provides the following similarity variable and function

$$u=g(r)x^{-4},\quad r=x.$$

These variables reduced the Eq. (2.2) to

$$g'x^{-4}+\epsilon\left[(\ln t+\gamma-1)g'x^{-4}+\frac{gx^{-4}}{t}+\sum_{n=1}^{+\infty}\frac{(-1)^n t^n}{n(n+1)!}D_t^{n+1}u\right]\approx 0.$$

5. Approximate Conservation Laws

There are several methods in order to find conservation laws of a given system of differential equations such as Noether's theorem, Boyer's generalization of Noether's theorem, direct method, homotopy operator method, Ibragimov's method etc. All cases have some advantages together with limitations. An illustrative comparison between these methods with so many examples is coming in [46]. In this section approximate conservation laws will be constructed by using approximate symmetries in a systematic method. This method is based on the extension of concept of non-linear self-adjointness by Ibragimov [19, 23] which presents a formal Lagrangian of perturbed Eq. (2.9). After introducing the adjoint equation and examining non-linear selfadjointness condition, conservation laws will be obtained in definite relationships.

5.1. Basic definitions for constructing conservation laws.

Definition 5.1. The approximate formal Lagrangian is defined by [24],

$$\mathcal{L}\simeq\mathcal{L}_0+\epsilon\mathcal{L}_1\equiv vF_0+v\epsilon F_1, \tag{5.1}$$

where v is a new dependent variable.

Then, the approximately adjoint equation can be written as

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial u}\approx&F^*_{(0)}(x,u,v,u_1,v_1,\dots,v_l,u_l)\\&+\epsilon F^*_{(1)}(x,u,v,u_1,v_1,\dots,v_i,u_l,u_l,D_{x^1}^{l+1}u,D_{x^1}^{l+2}u,D_{x^1}^{l+2}v,\dots)\simeq 0,\end{aligned}$$

where

$$\frac{\partial}{\partial u}=\frac{\partial}{\partial u}+\sum_{s=1}^{+\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}},$$

is the variational derivative. Here D_i denotes the operator of total differentiation with respect to x^i defined by

$$D_i=\frac{\partial}{\partial x^i}+u_i\frac{\partial}{\partial u}+v_i\frac{\partial}{\partial v}+\sum_{s=1}^{+\infty}\left(u_{i_1\cdots i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}+v_{i_1\cdots i_s}\frac{\partial}{\partial v_{i_1\cdots i_s}}\right).$$

Any approximate symmetry (2.10) of Eq. (2.9) leads to a conservation law

$$D_i(C^i)=0,\quad C^i=C_0^i+\epsilon C_1^i,$$

where

$$C_0^i=Q_0\left[\frac{\partial\mathcal{L}_0}{\partial u_i}+\sum_{s=1}^{l-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial\mathcal{L}_0}{\partial u_{i_1\cdots i_s}}\right]+\sum_{r=1}^{l-1}D_{k_1}\cdots D_{k_r}(Q_0)$$

$$\times \left[\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r}} + \sum_{s=1}^{l-r-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r i_1\cdots i_s}}\right],$$

$$\begin{aligned} C_1^i = {} & Q_1\left[\frac{\partial \mathcal{L}_0}{\partial u_i} + \sum_{s=1}^{l-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_0}{\partial u_{i_1\cdots i_s}}\right] + \sum_{r=1}^{l-1} D_{k_1}\cdots D_{k_r}(Q_1) \\ & \times \left[\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r}} + \sum_{s=1}^{l-r-1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_0}{\partial u_{ik_1\cdots k_r i_1\cdots i_s}}\right] \\ & + Q_0\left[\frac{\partial \mathcal{L}_1}{\partial u_i} + \sum_{s=1}^{+\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_1}{\partial u_{i_1\cdots i_s}}\right] + \sum_{r=1}^{+\infty} D_{k_1}\cdots D_{k_r}(Q_1) \\ & \times \left[\frac{\partial \mathcal{L}_1}{\partial u_{ik_1\cdots k_r}} + \sum_{s=1}^{+\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial \mathcal{L}_1}{\partial u_{ik_1\cdots k_r i_1\cdots i_s}}\right], \end{aligned}$$

where

$$Q_0 = \eta_0 - \xi_0^i u_i, \quad Q_1 = \eta_1 - \xi_1^i u_i,$$

and $\mathcal{L}_0, \mathcal{L}_1$ are defined in (5.1) upon the substitution

$$v \approx \varphi_0(x,u) + \epsilon\varphi_1(x,u) \neq 0.$$

5.2. **Approximate conservation laws for Eq.** (1.1)**.** Symmetries $V_1, \dots, V_5$ are stable since they do not vanish at $\epsilon = 0$. These symmetries can be used for constructing stable approximate conservation laws of Eq. (3.2). Approximate symmetry V_6 is the essential symmetry and can gives the essential approximate conservation laws. Symmetry V_7 is inessential and should be omitted.

The equation approximately adjoint to Eq. (3.2) can be obtained from Eq. (3.1) using the formal Lagrangian (5.1) written for Eq. (3.2). Direct calculations give

$$\begin{aligned} F_0^* + \epsilon F_1^* \equiv {} & v_t - \frac{15v_x^3}{4u^{7/2}} + \frac{9v_x v_{xx}}{2u^{5/2}} - \frac{v_{xxx}}{u^{3/2}} \\ & + \epsilon\left[(\ln t + \gamma - 1)v_t + \sum_{s=1}^{+\infty}\sum_{r=0}^{s}\binom{s-1}{r-1}\frac{t^r}{r(r+1)!}D_t^{r+1}v\right] \approx 0. \end{aligned}$$

It can be proved that Eq. (3.2) is approximately non-linearly self-adjoint. Indeed, substituting Eqs. (3.2) and (5.1) into

$$F_0^*|_{v(x,t,u)\approx\varphi(0)(x,t,u)+\epsilon\varphi(1)(x,t,u)} + \epsilon F_1^*|_{v(x,t,u)\approx\varphi(0)(x,t,u)} \approx \lambda_0 F_0 + \epsilon(\lambda_0 F_1 + \lambda_1 F_0),$$

and taking into account the representation $v(x,t,u) \approx \varphi(0)(x,t,u) + \epsilon\varphi(1)(x,t,u)$, we obtain the approximate equation for $\varphi(0)$ and $\varphi(1)$. Splitting this equation by ϵ yields

$$\begin{aligned} v_t = {} & \varphi_{(0)u}u_t + \epsilon\varphi_{(1)u}u_t + \varphi_{(0)t} + \epsilon\varphi_{(1)t}, \\ v_x = {} & \varphi_{(0)u}u_x + \epsilon\varphi_{(1)u}u_x + \varphi_{(0)x} + \epsilon\varphi_{(1)x}, \\ v_{xx} = {} & \varphi_{(0)ux}u_x + \varphi_{(0)u}u_{xx} + \varphi_{(0)xx} + \epsilon\varphi_{(1)ux}u_x + \epsilon\varphi_{(1)u}u_{xx} + \epsilon\varphi_{(1)xx}, \\ v_{xxx} = {} & \varphi_{(0)uxx}u_x + 2\varphi_{(0)ux}u_{xx} + \varphi_{(0)u}u_{xxx} + \varphi_{(0)xxx} + \epsilon\varphi_{(1)uxx}u_x + 2\epsilon\varphi_{(1)uxx}u_{xx}, \end{aligned}$$

$$+ \epsilon\varphi_{(1)u}u_{xxx} + \epsilon\varphi_{(1)xxx}.$$

Consequently, we have,

$$\begin{aligned}
&\varphi_{(0)u}u_x + \varphi_{(0)t} + \epsilon(\varphi_{(1)u}u_t + \varphi_{(1)t})\\
&- \frac{15(\varphi_{(0)u}u_x + \varphi_{(0)x} + \epsilon\varphi_{(1)u}u_x + \epsilon\varphi_{(1)x})^3}{8u^{7/2}}\\
&+ \frac{9(\varphi_{(0)u}u_x + \varphi_{(0)x} + \epsilon\varphi_{(1)u}u_x + \epsilon\varphi_{(1)x})(\varphi_{(0)xx}u_x + \varphi_{(0)u}u_{xx} + \varphi_{(0)xx} + \epsilon\varphi_{(1)ux}u_x}{4u^{5/2}}\\
&+ \frac{\epsilon\varphi_{(1)u}u_{xx} + \epsilon\varphi_{(1)xx})}{4u^{5/2}} - \varphi_{(0)uxx}u_x + 2\varphi_{(0)ux}u_{xx} + \varphi_{(0)u}u_{xxx} + \varphi_{(0)xxx}\\
&+ \epsilon\varphi_{(1)uxx}u_x + 2\epsilon\varphi_{(1)ux}u_{xx} + \epsilon\varphi_{(1)u}u_{xxx} + \frac{\epsilon\varphi_{(1)xxx}}{2u^{3/2}}\\
=&\lambda_0\left(u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\right) + \epsilon\Bigg\{\lambda_1u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\\
&+ \lambda_0\left[(\ln t + \gamma - 1)u_t + \frac{u}{t} + \sum_{n=1}^{+\infty}\frac{(-1)^nt^n}{n(n+1)!}D_t^{n+1}u\right]\Bigg\}.
\end{aligned}$$

Then,

$$v(x,t,u) \equiv \frac{\lambda_0}{u^{3/2}} + \epsilon\left(\lambda_1u^{-3/2} + \lambda_0\frac{x}{t} + C_1\right)$$

and

$$\begin{aligned}
\mathcal{L} \approx &\frac{\lambda_0}{u^{3/2}}\left(u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\right) + \epsilon\Bigg\{\left(\lambda_1u^{-3/2} + \lambda_0\frac{x}{t} + C_1\right)\\
&\times\left(u_t - \frac{15u_x^3}{4u^{7/2}} + \frac{9u_xu_{xx}}{2u^{5/2}} - \frac{u_{xxx}}{u^{3/2}}\right) + \frac{\lambda_0}{u^{3/2}}\Bigg[(\ln t + \gamma - 1)u_t\\
&+ \frac{u}{t} + \sum_{n=1}^{+\infty}\frac{(-1)^nt^n}{n(n+1)!}D_t^{n+1}u\Bigg]\Bigg\}.
\end{aligned}$$

Thus, the conserved vectors for Eq. (1.1) are constructed by the formula

$$\begin{aligned}
C_0^x =&Q_0\left(\frac{45\lambda_0u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right) + D_{xx}(Q_0)\left(\frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right),\\
C_1^x =&Q_1\left(\frac{45\lambda_0u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right) + D_{xx}(Q_1)\left(\frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right)\\
&+ Q_0\left[\frac{-45\lambda_1u_x^2}{64u^{21/2}} - \frac{9\lambda_1u_{xx}}{u^{5/2}} - \lambda_1\left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{u}t} + \frac{x}{t}\right)\right]\\
&+ D_{xx}(Q_0)\left(\lambda_1u^{-3/2} + \lambda_0\frac{x}{t} + 1\right),
\end{aligned}$$

$$
\begin{aligned}
C_0^t =& Q_0 \frac{\lambda_0}{u^{3/2}},\\
C_1^t =& \frac{\lambda_0}{u^{3/2}} \Bigg\{ Q_1 + Q_0 \left[\lambda_1 u^{-3/2} + \lambda_0 \frac{x}{t} + 1 + \ln t + \gamma + \sum_{i=1}^{+\infty} D_t^i(Q_0) \frac{t^i}{i(i+1)!} \right] \\
& + \sum_{s=1}^{+\infty} (-1)^s D_t^s(Q0) \left[\sum_{k=s}^{+\infty} \frac{t^k}{k(k+1)!} \right] \Bigg\},
\end{aligned}
$$

where

$$
C^x = C_0^x + \epsilon C_1^x, \quad C^t = C_0^t + \epsilon C_1^t.
$$

Now, the conservation laws of the Eq. (1.1) with respect to the symmetry operators are listed below.

(I) For V_1 we have $Q_0 = -u_t$ and $Q_1 = 0$, so component of approximate conservation laws is

$$
\begin{aligned}
C^x =& -u_t \left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5} \right) \\
& - u_t \left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1 \left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{u}t} + \frac{x}{t} \right) \right],\\
C^t =& -u_t \frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}} \Bigg\{ - u_t \Bigg[\lambda_1 u^{-3/2} + \lambda_0 \frac{x}{t} + 1 + \ln t + \gamma \\
& + \sum_{i=1}^{+\infty} D_t^i(-u_t) \frac{t^i}{i(i+1)!} \Bigg] + \sum_{s=1}^{+\infty} (-1)^s D_t^s(-u_t) \sum_{k=s}^{+\infty} \frac{t^k}{k(k+1)!} \Bigg\}.
\end{aligned}
$$

(II) Symmetry V_2, $Q_0 = -u_x$ and $Q_1 = 0$ yield

$$
\begin{aligned}
C^x =& -u_x \left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5} \right) \\
& - u_x \left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1 \left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{u}t} + \frac{x}{t} \right) \right],\\
C^t =& -u_x \frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}} \Bigg\{ - u_x \Bigg[\lambda_1 u^{-3/2} + \lambda_0 \frac{x}{t} + 1 + \ln t + \gamma \\
& + \sum_{i=1}^{+\infty} D_t^i(-u_x) \frac{t^i}{i(i+1)!} \Bigg] + \sum_{s=1}^{+\infty} (-1)^s D_t^s(-u_x) \sum_{k=s}^{+\infty} \frac{t^k}{k(k+1)!} \Bigg\}.
\end{aligned}
$$

(III) For V_3, the charachterisctics $Q_0 = -2u - xu_x$ and $Q_1 = 0$ are derived. In this case we have

$$
C^x = (-2u - xu_x) \left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5} \right)
$$

$$
\begin{aligned}
&- (2u - xu_x)\left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1\left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{ut}} + \frac{x}{t}\right)\right],\\
C^t &= (-2u - xu_x)\frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}}\Bigg\{\Bigg[(-2u - xu_x)(\lambda_1 u^{-3/2} + \lambda_0\frac{x}{t} + 1\\
&+ \ln t + \gamma + \sum_{i=1}^{+\infty} D_t^i(-2u - xu_x)\frac{t^i}{i(i+1)!}\Bigg]\\
&+ \sum_{s=1}^{+\infty}(-1)^s D_t^s(-2u - xu_x)\sum_{k=s}^{+\infty}\frac{t^k}{k(k+1)!}\Bigg\}.
\end{aligned}
$$

(IV) The components of conservation laws for V_4 with respect to $Q_0 = \frac{2}{3}u - tu_t$ and $Q_1 = 0$ are

$$
\begin{aligned}
C^x &= \left(\frac{2}{3}u - tu_t\right)\left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right)\\
&- \left(\frac{2}{3}u - tu_t\right)\left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1\left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{ut}} + \frac{x}{t}\right)\right],\\
C^t &= \left(\frac{2}{3}u - tu_t\right)\frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}}\Bigg\{\left(\frac{2}{3}u - tu_t\right)\Bigg[\lambda_1 u^{-3/2} + \lambda_0\frac{x}{t} + 1\\
&+ \ln t + \gamma + \sum_{i=1}^{+\infty} D_t^i\left(\frac{2}{3}u - tu_t\right)\frac{t^i}{i(i+1)!}\Bigg] + \sum_{s=1}^{+\infty}(-1)^s D_t^s\left(\frac{2}{3}u - tu_t\right)\\
&\times \sum_{k=s}^{+\infty}\frac{t^k}{k(k+1)!}\Bigg\}.
\end{aligned}
$$

(V) Now consider V_5. This operator has the characteristics $Q_0 = -2xu - \frac{x^2}{2}u_x$ and $Q_1 = 0$. So, the conserved vectors are

$$
\begin{aligned}
C^x &= \left(-2xu - \frac{x^2}{2}u_x\right)\left(\frac{45\lambda_0 u_x^2}{64u^{12}} + \frac{9u_x^2\lambda_0}{2u^5} - \frac{3u_x + u_{xx}}{u^4} + \frac{4u_x}{u^5}\right)\\
&- \left(-2xu - \frac{x^2}{2}u_x\right)\left[\frac{-45\lambda_1 u_x^2}{64u^{21/2}} - \frac{9\lambda_1 u_{xx}}{u^{5/2}} - \lambda_1\left(\frac{1}{u^{3/2}} + \frac{1}{\sqrt{ut}} + \frac{x}{t}\right)\right],\\
C^t &= \left(-2xu - \frac{x^2}{2}u_x\right)\frac{\lambda_0}{u^{3/2}} + \frac{\lambda_0}{u^{3/2}}\Bigg\{\left(-2xu - \frac{x^2}{2}u_x\right)\Bigg[\lambda_1 u^{-3/2} + \lambda_0\frac{x}{t} + 1\\
&+ \ln t + \gamma + \sum_{i=1}^{+\infty} D_t^i\left(-2xu - \frac{x^2}{2}u_x\right)\frac{t^i}{i(i+1)!}\Bigg]
\end{aligned}
$$

$$+\sum_{s=1}^{+\infty}(-1)^s D_t^s\left(-2xu-\frac{x^2}{2}u_x\right)\sum_{k=s}^{+\infty}\frac{t^k}{k(k+1)!}\Bigg\}.$$

(VI) The approximate symmetry V_6 with $Q_0=0$ and $Q_1=-u_t$ yields

$$C^x=-\epsilon\left[u_t\left(\frac{45\lambda_0 u_x^2}{64u^{12}}+\frac{9u_x^2\lambda_0}{2u^5}-\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)+u_{xxt}\left(\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)\right],$$

$$C_0^t=-\epsilon u_t\frac{\lambda_0}{u^{3/2}}.$$

(VII) Finally for V_7 ,$Q_0=0$ and $Q_1=-u_x$ we have

$$C^x=-\epsilon\left[u_x\left(\frac{45\lambda_0 u_x^2}{64u^{12}}+\frac{9u_x^2\lambda_0}{2u^5}-\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)+u_{xxt}\left(\frac{3u_x+u_{xx}}{u^4}+\frac{4u_x}{u^5}\right)\right],$$

$$C_0^t=-\epsilon u_x\frac{\lambda_0}{u^{3/2}}.$$

6. Concluding Remark

Recently, trending in FDEs led to proliferation of studies that it became an increasing interest in mathematics. Most studies in the field of FDEs have only focused of approximate analysis with some standard analytic methods. Symmetry analysis of differential equations is fast becoming a key instrument in all equations arising from a natural phenomenon. A key aspect of Lie group theory of differential equations is that this study provides an exciting opportunity to advance our knowledge about the exact solutions and geometric structures of the given system. Over the past years there has been an increasing interest in this field because the method could be implemented in all types of system of differential equations. This method is based on finding some differential operators (vector fields) called symmetries in order to find the exact solutions of differential equations. These operators are the largest local group of transformations acting on the independent and dependent variables of the system with the property that they transform solutions of the system to other solutions. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system [33].

Also there are several methods for finding conservation laws of a system of differential equations; such as Noether's method, Boyer's generalization of Noether's method, direct method, homotopy operator method, Ibragimov's method (modified version of Noether's theorem), etc. All five of these methods have some limitations in their use.

As it is mentioned in the paper the modified Noether's theorem is used for finding the conservation laws of Eq. (1.1). We note that there are several limitations to Noether's theorem. It is restricted to variational systems. Consequently, to be applicable to a given system as written, the given system must be of even order, have the same number of dependent variables as the number of equations in the system and have no dissipation. There is also the difficulty of finding symmetries admitted by the action functional. Moreover, the use of Noether's theorem to find conservation laws is coordinate-dependent. On comparing Ibragimov's method with Noether's theorem, we find that this method is similar to Noether's theorem which requires a Lagrangian to exist, however, Lagrangians exists only for very special types of differential equations within the construct of Noether's theorem. Ibragimov has attempted to overcome this difficulty by defining an adjoint equation for non-linear differential equations and constructing a Lagrangian for an arbitrary (linear and non-linear) equation considered together with its adjoint equation. With this reason Ibragimov's method is applied in this paper.

As mentioned in the paper, Lukashchuk discussed the basis of extended Lie group theory on approximate FDEs in [34]. The main target of this paper was to give a comprehensive analysis of the Lie group theory of perturbed fractional Harry-Dym equation. The work is done due to the developed method by Lukashchuk. This method is implemented by replacing the small perturbed parameter with a fractional expression of a FDE. For the next step we found symmetries and conservation laws of the equation. This is followed with the help of approximated Lie theory of FDEs.

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