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GEOMETRIC PROPERTIES AND COMPACT OPERATOR ON FRACTIONAL RIESZ DIFFERENCE SPACE

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Abstract. In this article we introduce the Riesz difference sequence space $r_p^q\left(\Delta^{B\alpha}\right)$ of fractional order α, defined by the composition of fractional backward difference operator $\Delta^{B\alpha}$ given by $(\Delta^{B\alpha}v)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}v_{k-i}$ and the Riesz matrix R^q. We give some topological properties, obtain the Schauder basis and determine the α-, β- and γ- duals and investigate certain geometric properties of the space $r_p^q\left(\Delta^{B\alpha}\right)$. Finally, we characterize certain classes of compact operators on the space $r_p^q\left(\Delta^{B\alpha}\right)$ using Hausdorff measure of non-compactness.

1. Introduction

Throughout this article we shall use the symbol l^0 to denote the space of all real valued sequences. Let V and W be two sequence spaces and let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of real entries. In the rest of the paper, for ambiguity we shall write $A = (a_{nk})$ in place of $A = (a_{nk})_{n,k=0}^{\infty}$. We write A_n to denote the sequences in the nth row of the matrix A. We say that the matrix A defines a matrix mapping from V to W if for every sequence $v = (v_k)$, the A-transform of v, i.e., $Av = \{(Av)_n\} \in W$, where

$$(Av)_n = \sum_k a_{nk}v_k, \quad n \in \mathbb{N}. \tag{1.1}$$

Define the sequence space V_A by

$$V_A = \left\{v = (v_k) \in l^0 : Av \in V\right\}. \tag{1.2}$$

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Then the sequence space V_A is called the domain of the matrix A in the space V. Also, we use the notation (V, W) to represent the class of all matrices A from V to W. Thus $A \in (V, W)$ if and only if the series on the right hand side of the equality (1.1) converges for each $n \in \mathbb{N}$ and $v \in V$ such that $Av \in W$ for all $v \in V$. Besides, we denote the unit sphere and the closed unit ball of a set V by $S(V)$ and $B(V)$, respectively.

Throughout this paper s will denote the conjugate of p, that is $s = \frac{p}{p-1}$ for $1 < p < \infty$ or $s = \infty$ for $p = 1$ or $s = 1$ for $p = \infty$.

Definition 1.1. Let x be a real number such that $x \notin \{0, -1, -2, \ldots\}$. Then the gamma function of x is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.3}$$

Clearly, $\Gamma(x+1) = x!$ for $x \in \mathbb{N}$. Also, $\Gamma(x+1) = x\Gamma(x)$ for any real number $x \notin \{0, -1, -2, \ldots\}$.

The domains $c_0(\Delta^F)$, $c(\Delta^F)$ and $\ell_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0, c and ℓ_∞ are introduced by Kızmaz [24]. Aftermore, the domain bv_p of the backward difference matrix Δ^B in the space ℓ_p have recently been investigated for $0 < p < 1$ by Altay and Başar [6], and for $1 \le p \le \infty$ by Başar and Altay [7]. Aftermore, several other authors [13, 15, 16, 18–21, 30, 31, 43] generalized the notion of difference operator Δ and studied difference sequence spaces of integer order. However, for a positive proper fraction α, Baliarsingh [10] (see also [9]) introduced generalized fractional forward and backward difference operators $\Delta^{F\alpha}$ and $\Delta^{B\alpha}$ defined by

$$(\Delta^{F\alpha} v)_k = \sum_i (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i} \quad \text{and} \quad (\Delta^{B\alpha} v)_k = \sum_i (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k-i},$$

respectively. We give a short survey concerned with sequence spaces defined by fractional difference operator. Baliarsingh [10] introduced the difference sequence spaces $V(\Gamma, \Delta^\alpha, u)$ of fractional order α for $V = \{\ell_\infty, c, c_0\}$, where $u = (u_n)$ is a sequence satisfying certain conditions. Baliarsingh and Dutta [9] studied the difference sequence spaces $V(\Gamma, \Delta^\alpha, p)$ for $V = \{\ell_\infty, c, c_0\}$. Moreover, Altay and Başar [4] and Altay et al. [5] introduced the Euler sequence spaces e_0^r, e_c^r and e_∞^r, respectively. In [3], Polat and Başar introduced the spaces $e_0^r(\Delta^{Bm})$, $e_c^r(\Delta^{Bm})$ and $e_\infty^r(\Delta^{Bm})$ consisting of all sequences whose m^{th} order differences are in the Euler spaces e_0^r, e_c^r and e_∞^r, respectively. Kadak and Baliarsingh [22] studied Euler difference sequence spaces of fractional order $e_p^r(\Delta^{B\alpha})$, $e_0^r(\Delta^{B\alpha})$, $e_c^r(\Delta^{B\alpha})$ and $e_\infty^r(\Delta^{B\alpha})$ by introducing the Euler mean difference operator $E^r(\Delta^{B\alpha})$. Extending these spaces Meng and Mei [29] introduced binomial difference sequence spaces $b_0^{r,s}(\Delta^{B\alpha})$, $b_c^{r,s}(\Delta^{B\alpha})$ and $b_\infty^{r,s}(\Delta^{B\alpha})$ of fractional order. Yaying et al. [40] also studied the compactness related results on these spaces. Yaying and Hazarika [41] also examined the sequence space $b_p^{r,s}(\Delta^{B\alpha})$. Furthermore, Yaying [42] also studied paranormed Riesz difference sequence spaces $r_\infty^q(\Delta^{B\alpha})$, $r_0^q(\Delta^{B\alpha})$ and $r_c^q(\Delta^{B\alpha})$ of fractional order. Nayak, Et and Baliarsingh [35] examined the sequence spaces $V(u, v, \Delta^{B\alpha}, p)$ derived by combining the weighted mean operator $G(u, v)$ and backward fractional difference operator $\Delta^{B\alpha}$. Özger [37] studied geometric properties and Hausdorff measure of non-compactness related results of certain sequence spaces defined by the fractional difference operators. More recently Baliarsingh and Kadak [11] investigated certain class of mappings and Hausdorff measure of non-compactness of certain generalised Euler difference sequence spaces of fractional order. Further, one may also refer [12] for a more generalized fractional difference operators.

Definition 1.2. Let (q_k) be a sequence of positive numbers and define $Q_n = \sum_{k=0}^{n} q_k$, $n \in \mathbb{N}$. Then the Riesz mean matrix $R^q = (r^q_{nk})$ is defined as

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Malkowsky [25] introduced the sequence spaces r^q_∞, r^q_c and r^q_0 as the set of all sequences whose R^q-transforms are in the spaces ℓ_∞, c and c_0, respectively. Altay and Başar [1] studied the sequence space $r^q(p)$ as

$$r^q(p) = \left\{ v = (v_k) \in l^0 : \sum_{n \in \mathbb{N}} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k v_k \right|^{p_k} < \infty \right\},$$

where $p = (p_k)$ is a bounded sequence of positive real numbers. Altay and Başar [2] also studied the sequence spaces $r^q_\infty(p)$, $r^q_0(p)$ and $r^q_c(p)$ defined by

$$r^q_\infty(p) = \left\{ v = (v_k) \in l^0 : \sup_{n \in \mathbb{N}} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k v_k \right|^{p_k} < \infty \right\},$$

$$r^q_0(p) = \left\{ v = (v_k) \in l^0 : \lim_{n \to \infty} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k v_k \right|^{p_k} = 0 \right\} \quad \text{and}$$

$$r^q_c(p) = \left\{ v = (v_k) \in l^0 : \lim_{n \to \infty} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k v_k - l \right|^{p_k} = 0, \text{ for some } l \in \mathbb{R} \right\}.$$

Since then several authors studied and examined Riesz sequence spaces. For more studies on Riesz sequence spaces, one may refer to [25,42] and the references mentioned therein.

2. Riesz Difference Operator of Fractional Order and Sequence Spaces

First we give the definitions of $R^q(\Delta^{B\alpha})$ and its inverse.

Definition 2.1 ([42]). The product matrix $R^q(\Delta^{B\alpha})$ of Riesz mean R^q and the backward difference operator $\Delta^{B\alpha}$ is defined as follows:

$$\left(R^q(\Delta^{B\alpha})\right)_{nk} = \begin{cases} \sum_{i=k}^{n} (-1)^{i-k} \frac{\Gamma(\alpha+1)}{(i-k)!\Gamma(\alpha-i+k+1)} \cdot \frac{q_i}{Q_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Definition 2.2. ([42, Lemma 2.1]). The inverse of the product matrix $R^q(\Delta^{B\alpha})$ is given by:

$$\left(R^q(\Delta^{B\alpha})\right)^{-1}_{nk} = \begin{cases} (-1)^{n-k}\sum_{j=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \cdot \frac{Q_k}{q_j}, & 0 \le k < n,\\ \frac{Q_n}{q_n}, & k = n,\\ 0, & k > n. \end{cases}$$

We define the $R^q(\Delta^{B\alpha})$-transform of a sequence $v = (v_k)$ as follows:

(2.1)

$$u_n = \left(R^q(\Delta^{B\alpha})v\right)_n = \sum_{k=0}^{n-1}\left[\sum_{j=k}^{n}(-1)^{j-k}\frac{\Gamma(\alpha+1)}{(j-k)!\Gamma(\alpha-j+k+1)}\cdot\frac{q_j}{Q_n}\right]v_k + \frac{q_n}{Q_n}v_n,$$

where $n \in \mathbb{N}$. Now we introduce the Riesz difference sequence space $r_p^q(\Delta^{B\alpha})$ of fractional order α as follows:

$$r_p^q(\Delta^{B\alpha}) = \left\{v = (v_n) \in l^0 : R^q(\Delta^{B\alpha})v \in \ell_p\right\}, \quad \text{where } 1 \le p \le \infty.$$

The above sequence space can be expressed in the notation of (1.2) as follows:

$$r_p^q(\Delta^{B\alpha}) = (\ell_p)_{R^q(\Delta^{B\alpha})}, \quad 1 \le p \le \infty.$$

The sequence space $r_p^q(\Delta^{B\alpha})$ may be reduced to the following classes of sequence spaces in the special cases of α.

1. If $\alpha = 0$, then the sequence space $r_p^q(\Delta^{B\alpha})$ reduces to $r_p^q = (\ell_p)_{R^q}$ for $1 \le p \le \infty$.
2. If $\alpha = 1$, then the sequence space $r_p^q(\Delta^{B\alpha})$ reduces to $r_p^q(\Delta^B)$, where $(\Delta^B v)_k = v_k - v_{k-1}$ for all $k \in \mathbb{N}$.
3. If $\alpha = m \in \mathbb{N}$, then the sequence space $r_p^q(\Delta^{B\alpha})$ reduces to $r_p^q(\Delta^{Bm})$, where $(\Delta^{Bm}v)_k = \sum_{j=0}^{m}(-1)^j\binom{m}{j}v_{m-j}$ for all $k \in \mathbb{N}$.

We begin with the following theorem.

Theorem 2.1. *The sequence space $r_p^q(\Delta^{B\alpha})$ is a BK-space normed by*

$$\|v\|_{r_p^q(\Delta^{B\alpha})} = \left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_p} = \left(\sum_k\left|\left(R^q(\Delta^{B\alpha})v\right)_k\right|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{2.2}$$

and

$$\|v\|_{r_\infty^q(\Delta^{B\alpha})} = \left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_\infty} = \sup_{k\in\mathbb{N}}\left|\left(R^q(\Delta^{B\alpha})v\right)_k\right|. \tag{2.3}$$

Proof. The proof is a routine verification and hence omitted. □

Theorem 2.2. *The Riesz difference space $r_p^q(\Delta^{B\alpha})$ is linearly isomorphic to ℓ_p, where $1 \le p \le \infty$.*

Proof. We prove the result for the space $r_p^q(\Delta^{B\alpha})$, $1 \le p < \infty$. Define the mapping $T : r_p^q(\Delta^{B\alpha}) \to \ell_p$ by $v \mapsto u = Tv = R^q(\Delta^{(\alpha)})v$. It is easy to see that T is linear and injective. Let $u = (u_k) \in \ell_p$ and define the sequence $v = (v_k)$ by

$$v_k = \sum_{j=0}^{k-1} \left[\sum_{i=j}^{j+1} (-1)^{k-j} \frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)} \cdot \frac{Q_j}{q_i} u_j \right] + \frac{Q_k}{q_k} u_k, \quad k \in \mathbb{N}. \tag{2.4}$$

Then

$$\begin{aligned}
\|v\|_{r_p^q(\Delta^{B\alpha})} &= \left\| R^q(\Delta^{B\alpha}) v \right\|_{\ell_p} = \left(\sum_k \left| \left(R^q(\Delta^{B\alpha}) v \right)_k \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_k \left| \sum_{j=0}^{k-1} \left(\sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\alpha+1)}{(i-j)!\Gamma(\alpha-i+j+1)} \cdot \frac{q_i}{Q_k} \right) v_j + \frac{q_k}{Q_k} v_k \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_k \left| \sum_{j=0}^{k} \delta_{kj} u_j \right|^p \right)^{\frac{1}{p}} = \left(\sum_k |u_k|^p \right)^{\frac{1}{p}} = \|u\|_{\ell_p} < \infty,
\end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, $v \in r_p^q(\Delta^{B\alpha})$. Consequently, T is surjective and norm preserving. Thus, $r_p^q(\Delta^{B\alpha}) \cong \ell_p$, $1 \le p < \infty$. Similarly, we can show that $r_\infty^q(\Delta^{B\alpha}) \cong \ell_\infty$. □

We now construct sequence of points in the space $r_p^q(\Delta^{B\alpha})$ which will form the Schauder basis for that space. First we recall the definition of Schauder basis for a normed space $(V, \|\cdot\|)$.

Definition 2.3. A sequence $v = (v_k)$ of a normed space $(V, \|\cdot\|)$ is called a Schauder basis of the space V if for every $\nu \in V$ there exists a unique sequence of scalars (c_k) such that

$$\lim_{n\to\infty} \left\| \nu - \sum_{k=0}^{n} c_k v_k \right\| = 0.$$

We know by Theorem 2.2 that the mapping $T : r_p^q(\Delta^{B\alpha}) \to \ell_p$ is an isomorphism. Hence it is evident that the inverse image of the usual basis $\{e^{(k)}\}_{k\in\mathbb{N}}$ of the space ℓ_p, $1 \le p < \infty$, forms the basis of the new space $r_p^q(\Delta^{B\alpha})$. This immediately gives us the following theorem.

Theorem 2.3. *Let $1 \le p < \infty$ and define the sequence $b^{(k)}(q) = (b_n^{(k)}(q))$ of the elements of the space $r_p^q(\Delta^{B\alpha})$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \sum_{i=j}^{j+1} (-1)^{k-j} \frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)} \cdot \frac{Q_j}{q_i}, & k < n, \\ \frac{Q_n}{q_n}, & k = n, \\ 0, & k > n. \end{cases} \tag{2.5}$$

Then the sequence $\{b^{(k)}(q)\}$ is basis for the space $r_p^q(\Delta^{B\alpha})$ and every $v \in r_p^q(\Delta^{B\alpha})$ has a unique representation of the form

$$v = \sum_k \lambda_k b^{(k)}(q), \tag{2.6}$$

where $\lambda_k = \Big(R^q(\Delta^{B\alpha})v\Big)_k$ for all $k \in \mathbb{N}$.

Corollary 2.1. *The sequence space $r_p^q(\Delta^{B\alpha})$ is separable for $1 \leq p < \infty$.*

3. α-, β- and γ-Duals

In this section we obtain the α-, β- and γ-duals of $r_p^q(\Delta^{B\alpha})$. We note that the notation α used for α-dual has different meaning to that of the operator $\Delta^{B\alpha}$. First we recall the definitions of α-, β- and γ-duals of the space $V \subset l^0$.

Definition 3.1. The α-, β- and γ-duals of the subset $V \subset l^0$ are defined by

$$\begin{aligned} V^\alpha &= \{t = (t_k) \in l^0 : tv = (t_k v_k) \in \ell_1 \text{ for all } v \in V\}, \\ V^\beta &= \{t = (t_k) \in l^0 : tv = (t_k v_k) \in cs \text{ for all } v \in V\}, \\ V^\gamma &= \{t = (t_k) \in l^0 : tv = (t_k v_k) \in bs \text{ for all } v \in V\}, \end{aligned}$$

respectively.

Now, we quote certain lemmas given by Stielglitz and Tietz [38] which are necessary to establish our results. Throughout $\mathcal{N}$ will denote the collection of all finite subsets of $\mathbb{N}$.

Lemma 3.1. *$A = (a_{nk}) \in (\ell_p, \ell_1)$ if and only if $\sup_{K \in \mathcal{N}} \sum_k \left| \sum_{n \in K} a_{nk} \right| < \infty,\ 1 < p \leq \infty$.*

Lemma 3.2. *$A = (a_{nk}) \in (\ell_p, c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} \text{ exists for all } k \in \mathbb{N}, \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^s < \infty,\ 1 < p < \infty. \tag{3.2}$$

Lemma 3.3. *$A = (a_{nk}) \in (\ell_p, \ell_\infty)$ if and only if (3.2) holds, with $1 < p \leq \infty$.*

Lemma 3.4. *$A = (a_{nk}) \in (\ell_1, \ell_1)$ if and only if $\sup_{k \in \mathbb{N}} \sum_n |a_{nk}| < \infty$.*

Lemma 3.5. *$A = (a_{nk}) \in (\ell_1, c)$ if and only if (3.1) holds and*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}| < \infty. \tag{3.3}$$

Lemma 3.6. *$A = (a_{nk}) \in (\ell_1, \ell_\infty)$ if and only if (3.2) holds.*

Theorem 3.1. *Define the sets $d_1(q)$ and $d_2(q)$ by*

$$d_1(q) = \left\{ t = (t_k) \in l^0 : \sup_{k \in \mathbb{N}} \sum_n |d_{nk}| < \infty \right\}$$

and

$$d_2(q) = \left\{ t = (t_k) \in l^0 : \sup_{K \in \mathbb{N}} \sum_k \left| \sum_{n \in K} d_{nk} \right|^q < \infty \right\},$$

where the matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum\limits_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\alpha+1)}{(n-j)!\Gamma(-\alpha-n+j+1)} \cdot \frac{Q_k}{q_k} t_n, & 0 \le k < n, \\ \frac{Q_n}{q_n} t_n, & k = n, \\ 0, & k > n. \end{cases}$$

Then $\left[r_1^q(\Delta^{B\alpha})\right]^\alpha = d_1(q)$ and $\left[r_p^q(\Delta^{B\alpha})\right]^\alpha = d_2(q)$ for $1 < p < \infty$.

Proof. Consider the sequence $t = (t_k) \in l^0$ and $v = (v_k)$ is as defined in (2.4), then we have

$$\begin{aligned} t_n v_n &= \sum_{j=0}^{n-1} \left[\sum_{i=j}^{j+1} (-1)^{n-j} \frac{\Gamma(-\alpha+1)}{(n-i)!\Gamma(-\alpha-n+i+1)} \cdot \frac{Q_j}{q_i} t_n u_j \right] + \frac{Q_n}{q_n} t_n u_n \\ &= (Du)_n, \quad \text{for each } n \in \mathbb{N}, \end{aligned} \tag{3.4}$$

Thus, we deduce from (3.4) that $tv = (t_k v_k) \in \ell_1$ whenever $v = (v_k) \in r_1^q(\Delta^{B\alpha})$ or $r_p^q(\Delta^{B\alpha})$ if and only if $Du \in \ell_1$ whenever $u = (u_k) \in \ell_1$ or ℓ_p. This yields us the fact that $t = (t_n) \in \left[r_1^q(\Delta^{B\alpha})\right]^\alpha$ or $\left[r_p^q(\Delta^{B\alpha})\right]^\alpha$ if and only if $D \in (\ell_1, \ell_1)$ or $D \in (\ell_p, \ell_1)$.

Thus, by using Lemma 3.1 and Lemma 3.4, we conclude that

$$\left[r_1^q(\Delta^{B\alpha})\right]^\alpha = d_1(q) \quad \text{and} \quad \left[r_p^q(\Delta^{B\alpha})\right]^\alpha = d_2(q).$$ □

Theorem 3.2. *Define the sets $d_3(q)$, $d_4(q)$ and $d_5(q)$ as follows:*

$$d_3(q) = \left\{ t = (t_k) \in l^0 : \sum_k \left| \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k \right|^q < \infty \right\},$$

$$d_4(q) = \left\{ t = (t_k) \in l^0 : \sup_{n,k} \left| \Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) Q_k \right| < \infty \right\} \quad \textit{and}$$

$$d_5(q) = \left\{ t = (t_k) \in l^0 : \left\{ \frac{Q_k}{q_k} t_k \right\} \in \ell_\infty \right\},$$

where

$$\Delta^{B\alpha} \left(\frac{t_k}{q_k} \right) = \frac{t_k}{q_k} + \sum_{j=k+1}^{n} (-1)^{j-k} t_j \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)q_i}. \tag{3.5}$$

Then $\left[r_1^q(\Delta^{B\alpha})\right]^\beta = d_4(q) \cap d_5(q)$ and $\left[r_p^q(\Delta^{B\alpha})\right]^\beta = d_3(q) \cap d_5(q)$.

Proof. We give the proof for the space $r_p^q(\Delta^{B\alpha})$, $1 < p < \infty$, to avoid repetition of the similar statements. Let $t = (t_k) \in l^0$ and $v = (v_k)$ is as defined in (2.4). Consider the following equation

$$\begin{aligned}\sum_{k=0}^{n} t_k v_k &= \sum_{k=0}^{n} t_k \left[\sum_{j=0}^{k-1}\left(\sum_{i=j}^{j+1}(-1)^{k-j}\frac{\Gamma(-\alpha+1)}{(k-i)!\Gamma(-\alpha-k+i+1)}\frac{Q_j}{q_i}u_j\right) + \frac{Q_k}{q_k}u_k\right]\\
&= \sum_{k=0}^{n-1} u_k Q_k \left[\frac{t_k}{q_k} + \sum_{j=k+1}^{n}(-1)^{j-k}t_j\sum_{i=k}^{k+1}\frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)q_i}\right] + \frac{Q_n}{q_n}t_n u_n \qquad (3.6)\\
&= \sum_{k=0}^{n-1} u_k Q_k \Delta^{B\alpha}\left(\frac{t_k}{q_k}\right) + \frac{Q_n}{q_n}t_n u_n = (Cu)_n, \quad \text{for each } n \in \mathbb{N},\end{aligned}$$

where $C = (c_{nk})$ is a matrix defined by

$$c_{nk} = \begin{cases}\Delta^{B\alpha}\left(\frac{t_k}{q_k}\right)Q_k, & 0 \le k < n,\\ \frac{Q_n}{q_n}t_n, & k = n,\\ 0, & k > n,\end{cases}$$

and $\Delta^{B\alpha}\left(\frac{t_k}{q_k}\right)$ is as defined in (3.5). Clearly the columns of the matrix C are convergent, since

$$\lim_{n\to\infty} c_{nk} = \Delta^{B\alpha}\left(\frac{t_k}{q_k}\right)Q_k.$$

Thus, we deduce from (3.6) that $tv = (t_k v_k) \in cs$ whenever $v = (v_k) \in r_p^q(\Delta^{B\alpha})$ if and only if $Cu \in c$ whenever $u = (u_k) \in \ell_p$. This yields the fact that $t = (t_k) \in \left[r_p^q(\Delta^{B\alpha})\right]^\beta$ if and only if $C \in (\ell_p, c)$. Thus by using Lemma 3.2 with (3.6), we get that

$$\sum_k \left|\Delta^{B\alpha}\left(\frac{t_k}{q_k}\right)Q_k\right|^q < \infty \quad \text{and} \quad \sup_k\left|\frac{Q_k}{q_k}t_k\right| < \infty.$$

Thus, $\left[r_p^q(\Delta^{B\alpha})\right]^\beta = d_3(q) \cap d_5(q)$. □

Theorem 3.3. *Let* $1 < p < \infty$*. Then* $\left[r_p^q(\Delta^{B\alpha})\right]^\gamma = d_3(q)$ *and* $\left[r_1^q(\Delta^{B\alpha})\right]^\gamma = d_4(q)$.

Proof. The proof is analogous to the previous theorem except that Lemma 3.3 in case of $r_p^q(\Delta^{B\alpha})$ and Lemma 3.6 in case of $r_1^q(\Delta^{B\alpha})$ are employed instead of the Lemma 3.2. □

4. Certain Geometric Properties of the Space $r_p^q(\Delta^{B\alpha})$

In this section, we investigate certain geometric properties of the space $r_p^q(\Delta^{B\alpha})$. We first recall certain notions and definitions which are necessary to establish our results.

Definition 4.1. A point $w \in S(V)$ is an extreme point if for every $u, v \in S(V)$ the equality $2w = u + v$ implies $u = v$. A Banach space V is said to be rotund if every point of $S(V)$ is an extreme point.

Definition 4.2. A Banach space V is said to have Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 4.3. Let $1 < p < \infty$. A Banach space is said to have the Banack-Saks type p if every weakly null sequence has a subsequence (x_k) such that for some $K > 0$

$$\|x_k\| \leq K n^{\frac{1}{p}}, \quad \text{for all } n = 1, 2, 3, \ldots$$

Definition 4.4. Let V be a real vector space. A functional $\sigma : V \to [0, \infty)$ is called a modular if

(a) $\sigma(v) = 0$ if and only if $v = \theta$;
(b) $\sigma(\lambda v) = \sigma(v)$ for scalars $|\lambda| = 1$;
(c) $\sigma(\lambda u + \delta v) \leq \sigma(u) + \sigma(v)$ for all $u, v \in V$ and $\lambda, \delta > 0$ with $\lambda + \mu = 1$.

The modular σ is called convex if $\sigma(\lambda u + \delta v) \leq \lambda\sigma(u) + \delta\sigma(v)$ for $u, v \in V$ and $\lambda, \delta > 0$ with $\lambda + \delta = 1$.

We define the operator σ_p, $1 \leq p < \infty$, on $r_p^q(\Delta^{B\alpha})$ by

$$\sigma_p(v) = \sum_n \left| R^q(\Delta^{B\alpha}) \right|^p. \tag{4.1}$$

It is clear that $\sigma_p(v)$ is a convex modular on $r_p^q(\Delta^{B\alpha})$. Now we equip the sequence space $r_p^q(\Delta^{B\alpha})$ with the Luxemborg norm defined by

$$\|v\| = \inf \left\{ \kappa > 0 : \sigma_p \left(\frac{v}{\kappa} \right) \leq 1 \right\}.$$

Now, we give certain basic properties of the modular σ_p.

Proposition 4.1. *The modular σ_p on $r_p^q(\Delta^{B\alpha})$ satisfies the following statements.*

(a) *If $0 < k < 1$, then $k^p \sigma_p\left(\frac{v}{k}\right) \leq \sigma_p(v)$ and $\sigma_p(kv) \leq k\sigma_p(v)$.*
(b) *If $k > 1$, then $\sigma_p(v) \leq k^p \sigma_p\left(\frac{v}{k}\right)$.*
(c) *If $k \geq 1$, then $\sigma_p(v) \leq k\sigma_p(v) \leq \sigma_p(kv)$.*

Proposition 4.2. *The following statements hold for $v \in r_p^q(\Delta^{B\alpha})$.*

(a) *If $\|v\| < 1$, then $\sigma_p(v) \leq \|v\|$.*
(b) *If $\|v\| > 1$, then $\sigma_p(v) \geq \|v\|$.*
(c) *$\|v\| = 1$ if and only if $\sigma_p(v) = 1$.*
(d) *$\|v\| < 1$ if and only if $\sigma_p(v) < 1$.*
(e) *$\|v\| > 1$ if and only if $\sigma_p(v) > 1$.*
(f) *If $0 < k < 1$, $\|v\| > k$, then $\sigma_p(v) > k^p$.*
(g) *If $k \geq 1$, $\|v\| < k$, then $\sigma_p(v) < k^p$.*

Proof. The results can be established analogously to [44, Proposition 17, p.7] (also see [23, Proposition 3], [36, Proposition 6]). Hence, we omit details. □

Proposition 4.3. *Let* (v_n) *be a sequence in* $r_p^q(\Delta^{B\alpha})$.

(a) *If* $\lim_{n\to\infty} \|x_n\| = 1$, *then* $\lim_{n\to\infty} \sigma_p(x_n) = 1$.
(b) *If* $\lim_{n\to\infty} \sigma_p(x_n) = 0$, *then* $\lim_{n\to\infty} \|x_n\| = 0$.

Proof. The proof is analogous to the proof of the [36, Theorem 10, page 4]. So we omit details. □

Theorem 4.1. *The sequence space* $r_p^q(\Delta^{B\alpha})$ *is a Banach space with respect to the Luxemborg norm.*

Proof. It is enough to show that every Cauchy sequence in $r_p^q(\Delta^{B\alpha})$ is convergent in Luxemborg norm. Let $v^{(n)} = (v_j^{(n)})$ be a Cauchy sequence in $r_p^q(\Delta^{B\alpha})$ and $\varepsilon \in (0,1)$. Then there exists a positive integer n_0 such that $\left\|v^{(n)} - v^{(m)}\right\| < \varepsilon$ for all $m, n \geq n_0$. Using Part (a) of Proposition 4.2, we obtain

$$\sigma_p(v^{(n)} - v^{(m)}) < \left\|v^{(n)} - v^{(m)}\right\| < \varepsilon, \tag{4.2}$$

for all $n, m \geq n_0$. This gives

$$\sum_k \left|\left(R^q(\Delta^{B\alpha})(v^{(n)} - v^{(m)})\right)_k\right|^p < \varepsilon. \tag{4.3}$$

Thus, for each fixed k and for all $n, m \geq n_0$

$$\left|\left(R^q(\Delta^{B\alpha})(v^{(n)} - v^{(m)})\right)_k\right| = \left|\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k - \left(R^q(\Delta^{B\alpha})v^{(m)}\right)_k\right| < \varepsilon.$$

Hence, the sequence $\left\{\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k\right\}$ is Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k \in \mathbb{R}$ such that $\left\{\left(R^q(\Delta^{B\alpha})v^{(n)}\right)_k\right\} \to \left(R^q(\Delta^{B\alpha})v\right)_k$ as $n \to \infty$. Therefore as $n \to \infty$, using (4.3), we have

$$\sum_k \left|\left(R^q(\Delta^{B\alpha})(v^{(n)} - v)\right)_k\right|^p < \varepsilon, \quad \text{for all } n \geq n_0.$$

It remains to show that (v_k) is an element of $r_p^q(\Delta^{B\alpha})$. Since $\left\{\left(R^q(\Delta^{B\alpha})v^{(m)}\right)_k\right\} \to \left(R^q(\Delta^{B\alpha})v\right)_k$ as $m \to \infty$ we have

$$\lim_{m\to\infty} \sigma_p(v^{(n)} - v^{(m)}) = \sigma_p(v^{(n)} - v).$$

Thus, by using the inequality (4.2), we get that $\sigma_p(v^{(n)} - v) < \left\|v^{(n)} - v\right\| < \varepsilon$ for all $n \geq n_0$. This implies that $v^{(n)} \to v$ as $n \to \infty$. Thus, we have $v = v^{(n)} - (v^{(n)} - v) \in r_p^q(\Delta^{B\alpha})$.

Hence, the space $r_p^q(\Delta^{B\alpha})$ is complete under the Luxemborg norm. □

Theorem 4.2. *The sequence space* $r_p^q(\Delta^{B\alpha})$ *equipped with the Luxemborg norm is rotund if and only if* $p > 1$.

Proof. Let the space $r_p^q(\Delta^{B\alpha})$ be rotund and take $p=1$. Now consider the following sequences for a proper fraction α

$$u=\left(1,\alpha-\frac{q_0}{q_1},\frac{\alpha(\alpha+1)}{2!}-\alpha\frac{q_0}{q_1},\frac{\alpha(\alpha+1)(\alpha+2)}{3!}-\frac{\alpha(\alpha+1)}{2!}\cdot\frac{q_0}{q_1},\dots\right)$$

and

$$v=\left(0,\frac{Q_1}{q_1},\alpha\frac{Q_1}{q_1}-\frac{Q_1}{q_2},\frac{\alpha(\alpha+1)}{2!}\cdot\frac{Q_1}{q_1}-\alpha Q_1\frac{q_1}{q_2},\dots\right).$$

Then $u\neq v$ and it can be clearly seen that

$$\sigma_p(u)=\sigma_p(v)=\sigma_p\left(\frac{u+v}{2}\right)=1.$$

Then by Part (c) of Proposition 4.2, $u,v,\frac{u+v}{2}\in S\left[r_p^q(\Delta^{B\alpha})\right]$ which contradicts the fact that $r_p^q(\Delta^{B\alpha})$ is not rotund. Hence, $p>1$.

Conversely, let $w\in S\left[r_p^q(\Delta^{B\alpha})\right]$ and $u,v\in S\left[r_p^q(\Delta^{B\alpha})\right]$, $1<p<\infty$, be such that $w=\frac{u+v}{2}$. By the convexity of σ_p and using the property (c) of Proposition 4.2, we have

$$1=\sigma_p(w)\leq\frac{1}{2}\left[\sigma_p(u)+\sigma_p(v)\right]\leq\frac{1}{2}+\frac{1}{2}=1.$$

This implies that $\sigma_p(u)=\sigma_p(v)=1$ and $\sigma_p(w)=\frac{\sigma_p(u)+\sigma_p(v)}{2}$.

Thus from the definition of σ_p and from the above discussion, we get

$$\sum_n\left|\left(R^q(\Delta^{B\alpha})w\right)_n\right|^p=\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})u\right)_n\right|^p+\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})v\right)_n\right|^p.$$

Again $w=\frac{u+v}{2}$, we have

$$\sum_n\left|\left(R^q(\Delta^{B\alpha})\left(\frac{u+v}{2}\right)\right)_n\right|^p=\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})u\right)_n\right|^p+\frac{1}{2}\sum_n\left|\left(R^q(\Delta^{B\alpha})v\right)_n\right|^p.$$

This implies that

$$\left|\left(R^q(\Delta^{B\alpha})\left(\frac{u+v}{2}\right)\right)_n\right|^p=\frac{1}{2}\left|\left(R^q(\Delta^{B\alpha})u\right)_n\right|^p+\frac{1}{2}\left|\left(R^q(\Delta^{B\alpha})v\right)_n\right|^p. \tag{4.4}$$

From (4.4), it follows immediately that $u=v$. Thus the space $r_p^q(\Delta^{B\alpha})$ is rotund. ☐

Theorem 4.3. *The sequence space $R^q(\Delta^{B\alpha})$ has the Kadec-Klee property.*

Proof. Let $v\in S\left[r_p^q(\Delta^{B\alpha})\right]$ and $(v^{(n)})\subset r_p^q(\Delta^{B\alpha})$ such that $\left\|v^{(n)}\right\|\to 1$ and $v^{(n)}\to v$ weakly. Using Part (a) of Proposition 4.3, we get

$$\sigma_p(v^{(n)})\to 1\quad\text{as}\quad n\to\infty. \tag{4.5}$$

Also $v\in S\left[r_p^q(\Delta^{B\alpha})\right]$ and using Part (c) of Proposition 4.2, we observe that

$$\sigma_p(v)=1. \tag{4.6}$$

Thus observing equations (4.5) and (4.6), we write

$$\sigma_p(v^{(n)})\to\sigma_p(v)\quad\text{as}\quad n\to\infty.$$

Since $v^{(n)} \to v$ weakly and the jth coordinate mapping $\pi_j : r_p^q(\Delta^{B\alpha}) \to \mathbb{R}$ defined by $\pi_j(v) = v_j$ is continuous imply that $v_k^{(n)} \to v_k$ as $n \to \infty$. Therefore, $v^{(n)} \to v$ as $n \to \infty$. This completes the proof. $\square$

Theorem 4.4. *The space $r_p^q(\Delta^{B\alpha})$, $1 < p < \infty$, has the Banach-Saks type p.*

Definition 4.5. The Gurarii's modulus of convexity for a normed linear space V is defined by

$$\beta_V(\varepsilon) = \inf\left\{1 - \inf_{0\le\alpha\le 1} \|\alpha v + (1-\alpha)u\| : v, u \in S(V), \|v - u\| = \varepsilon\right\},$$

where $0 < \varepsilon < 2$.

Theorem 4.5. *The Gurarii's modulus of convexity for the space $r_p^q(\Delta^{B\alpha})$, $1 \le p < \infty$, is*

$$\beta_{r_p^q(\Delta^{B\alpha})} \le 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}, \quad \textit{where } 0 \le \varepsilon \le 2.$$

Proof. Let $z \in r_p^q(\Delta^{B\alpha})$. Then

$$\|z\|_{r_p^q(\Delta^{B\alpha})} = \left\|R^q(\Delta^{B\alpha})z\right\|_{\ell_p} = \left(\sum_n \left|\left(R^q(\Delta^{B\alpha})z\right)_n\right|^p\right)^{\frac{1}{p}}.$$

Let $0 \le \varepsilon \le 2$ and we define the following two sequences:

$$u = \left(\left(\left[R^q(\Delta^{B\alpha})\right]^{-1}\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, \left[R^q(\Delta^{B\alpha})\right]^{-1}\left(\frac{\varepsilon}{2}\right), 0, 0, \dots\right)$$

and

$$v = \left(\left(\left[R^q(\Delta^{B\alpha})\right]^{-1}\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, \left[R^q(\Delta^{B\alpha})\right]^{-1}\left(\frac{-\varepsilon}{2}\right), 0, 0, \dots\right).$$

Then $\left\|R^q(\Delta^{B\alpha})u\right\|_{\ell_p} = \|u\|_{r_p^q(\Delta^{B\alpha})} = 1$ and $\left\|R^q(\Delta^{B\alpha})v\right\|_{\ell_p} = \|v\|_{r_p^q(\Delta^{B\alpha})} = 1$. That is $u, v \in S\left[r_p^q(\Delta^{B\alpha})\right]$ and $\left\|R^q(\Delta^{B\alpha})u - R^q(\Delta^{B\alpha})v\right\|_{\ell_p} = \|u - v\|_{r_p^q(\Delta^{B\alpha})} = \varepsilon$. Thus, for $0 \le \alpha \le 1$

$$\begin{aligned}\|\alpha u + (1-\alpha)v\|^p_{r_p^q(\Delta^{B\alpha})} &= \left\|\alpha R^q(\Delta^{B\alpha})u + (1-\alpha)R^q(\Delta^{B\alpha})v\right\|^p_{\ell_p}\\ &= 1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha - 1|\left(\frac{\varepsilon}{2}\right)^p.\end{aligned}$$

Then $\inf_{0\le\alpha\le1} \|\alpha u + (1-\alpha)v\|^p_{r_p^q(\Delta^{B\alpha})} = 1 - \left(\frac{\varepsilon}{2}\right)^p$. Therefore, for $p \ge 1$

$$\beta_{r_p^q(\Delta^{B\alpha})} \le 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

$\square$

Corollary 4.1. (a) *For $\varepsilon = 2$, $\beta_{r_p^q(\Delta^{B\alpha})} \le 1$. Hence, $r_p^q(\Delta^{B\alpha})$ is strictly convex.*

(b) *For $0 < \varepsilon < 2$, $0 < \beta_{r_p^q(\Delta^{B\alpha})} < 1$. Hence, $r_p^q(\Delta^{B\alpha})$ is uniformly convex.*

5. Hausdorff Measure of Non Compactness

In this section, we characterize certain classes of compact operators on the space $r_p^q(\Delta^{B\alpha})$ using Hausdorff measure of non-compactness. First we recall certain known definitions, results and notations that are essential for our investigation.

If V and W are Banach spaces then by $B(V,W)$, we denote the class of all bounded linear operators $L : V \to W$. $B(V,W)$ itself is a Banach space with the operator norm defined by $\|L\| = \sup_{v\in S(V)} \|L(v)\|$. We denote

$$\|a\|_V^* = \sup_{v\in S(V)} \left|\sum_k a_k v_k\right|, \tag{5.1}$$

for $a \in l^0$, provided that the series on the right hand side is finite which is the case whenever V is a BK space and $a \in V^\beta$ [39]. Also L is said to be compact if $D(V) = V$ for the domain of V and for every bounded sequence (v_n) in V, the sequence $(L(v_n))$ has a convergent subsequence in W. We denote the class of all such operators by $C(V,W)$.

The Hausdorff measure of noncompactness of a bounded set Q in a metric space V is defined by

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^n S(v_i, r_i),\ v_i \in V,\ r_i < \varepsilon,\ i = 1,2,\dots,n,\ n \in \mathbb{N}\right\},$$

where $S(v_i, r_i)$ is the open ball centered at v_i and radius r_i for each $i = 1,2,\dots,n$. One may refer to [8, 11, 17, 27, 32, 34] for more details on compact operators and Hausdorff measure of non-compactness. We need following lemmas for our investigation.

Lemma 5.1. *$\ell_1^\beta = \ell_\infty$, $\ell_p^\beta = \ell_q$ and $\ell_\infty^\beta = \ell_1$, where $1 < p < \infty$. Further, if $V \in \{\ell_1, \ell_p, \ell_\infty\}$, then $\|a\|_V^* = \|a\|_{V^\beta}$ holds for all $a \in V^\beta$, where $\|\cdot\|_{V^\beta}$ is the natural norm on V^β.*

Lemma 5.2. ([39, Theorem 4.2.8]). *Let V and W be BK-spaces. Then we have $(V,W) \subset B(V,W)$, that is, every $A \in (V,W)$ defines a linear operator $L_A \in B(V,W)$, where $L_A(v) = A(v)$ for all $v \in V$.*

Lemma 5.3. ([28, Theorem 2.25, Corollary 2.26]). *Let V and W be Banach spaces and $L \in B(V,W)$. Then we have*

$$\|L\|_\chi = \chi(L(S(V))) = \chi(L(B(V))) \tag{5.2}$$

and

$$L \in C(V,W) \quad \text{if and only if} \quad \|L\|_\chi = 0. \tag{5.3}$$

Lemma 5.4. ([28, Theorem 1.23]). *Let $V \supset \varphi$ be a BK space. If $A \in (V,W)$ then $\|L_A\| = \|A\|_{(V,W)} = \sup_n \|A_n\|_V^* < \infty$.*

Lemma 5.5. ([28, Theorem 2.15]). *Let Q be a bounded subset of the normed space V, where V is ℓ_p, $1 \leq p < \infty$, or c_0. If $P_r : V \to V$ is the operator defined by $P_r(v_0, v_1, v_2 \ldots) = (v_0, v_1, v_2 \ldots, v_r, 0, 0, \ldots)$ for all $v = (v_k) \in V$, then*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{v\in Q}\|(I-P_r)(v)\|\right), \qquad \text{where } I \text{ is the identity operator on } V.$$

Lemma 5.6. ([33, Theorem 3.7]). *Let $V \supset \varphi$ be a BK-space. Then the following statements hold.*

(a) *If $A \in (V, c_0)$, then $\|L_A\|_\chi = \limsup_{n\to\infty}\|A_n\|_V^*$ and L_A is compact if and only if $\lim_{n\to\infty}\|A_n\|_V^* = 0$.*

(b) *If V has AK and $A \in (V, c)$, then*

$$\frac{1}{2}\limsup_{n\to\infty}\|A_n-\alpha\|_V^* \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\|A_n-\alpha\|_V^*$$

and L_A is compact if and only if $\lim_{n\to\infty}\|A_n-\alpha\|_V^* = 0$, where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_{n\to\infty} a_{nk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (V, \ell_\infty)$, then $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\|A_n\|_V^*$ and L_A is compact if and only if $\lim_{n\to\infty}\|A_n\|_V^* = 0$.*

Lemma 5.7. ([33, Theorem 3.11]). *Let $V \supset \varphi$ be a BK-space. If $A \in (V, \ell_1)$, then*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N}A_n\right\|_V^*\right) \leq \|L_A\|_\chi \leq 4\cdot\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N}A_n\right\|_V^*\right)$$

and L_A is compact if and only if $\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N}A_n\right\|_V^*\right) = 0$, where $\mathcal{N}_r$ is the subcollection of $\mathcal{N}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

Lemma 5.8. ([33, Theorem 4.4, Corollary 4.5]). *Let $V \supset \varphi$ be a BK-space and let $\|A_n\|_{bs}^{[n]} = \left\|\sum_{m=0}^{n}A_m\right\|_V^*$. Then, the following statements hold.*

(a) *If $A \in (V, cs_0)$, then $\|L_A\|_\chi = \limsup_{n\to\infty}\|A_n\|_{(V,bs)}^{[n]}$ and L_A is compact if and only if $\lim_{n\to\infty}\|A_n\|_{(V,bs)}^{[n]} = 0$.*

(b) *If V has AK and $A \in (V, cs)$, then*

$$\frac{1}{2}\limsup_{n\to\infty}\left\|\sum_{m=0}^{n}A_m - a\right\|_V^* \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left\|\sum_{m=0}^{n}A_m - a\right\|_V^*$$

and L_A is compact if and only if $\lim_{n\to\infty}\|\sum_{m=0}^{n}A_m - a\|_V^* = 0$, where $a = (a_k)$, with $a_k = \lim_{n\to\infty}\sum_{m=0}^{n}a_{mk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (V, bs)$, then $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\|A\|_{(V,bs)}^{[n]}$ and L_A is compact if and only if $\lim_{n\to\infty}\|A\|_{(V,bs)}^{[n]} = 0$.*

Define an associated matrix $F = (f_{nk})$ of the infinite matrix $A = (a_{nk})$ by

$$f_{nk} = \left(\frac{a_{nk}}{q_k} + \sum_{j=k+1}^{\infty} (-1)^{j-k} a_{nj} \sum_{i=k}^{k+1} \frac{\Gamma(-\alpha+1)}{(j-i)!\Gamma(-\alpha-j+i+1)q_i} \right) Q_k, \tag{5.4}$$

for all $n, k \in \mathbb{N}$.

Lemma 5.9. *Let V be a sequence space and $A = (a_{nk})$ be an infinite matrix. If $A \in (r_p^q(\Delta^{B\alpha}), V)$, then $F \in (\ell_p, V)$ and $Av = Fu$ for all $v \in r_p^q(\Delta^{B\alpha})$, where A and F are related by* (5.4) *and* $1 \leq p \leq \infty$.

Theorem 5.1. *Let $1 < p < \infty$ and $s = \frac{p}{p-1}$. Then we have the following.*

(a) *If $A \in (r_p^q(\Delta^{B\alpha}), c_0)$, then* $\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}}$.

(b) *If $A \in (r_p^q(\Delta^{B\alpha}), c)$, then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_k |f_{nk} - f_k|^s \right)^{\frac{1}{s}} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k |f_{nk} - f_k|^s \right)^{\frac{1}{s}},$$

where $f = (f_k)$ and $f_k = \lim_{n\to\infty} f_{nk}$ for each $k \in \mathbb{N}$.

(c) *If $A \in (r_p^q(\Delta^{B\alpha}), \ell_\infty)$, then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}}$.

(d) *If $A \in (r_p^q(\Delta^{B\alpha}), \ell_1)$, then*

$$\lim_{r\to\infty} \|A\|^{[r]}_{(r_p^q(\Delta^{B\alpha}),\ell_1)} \leq \|L_A\|_\chi \leq 4 \lim_{r\to\infty} \|A\|^{[r]}_{(r_p^q(\Delta^{B\alpha}),\ell_1)},$$

where $\|A\|^{[r]}_{(r_p^q(\Delta^{B\alpha}),\ell_1)} = \sup_{N\in\mathcal{N}_r} \left(\sum_k \left|\sum_{n\in N} f_{nk}\right|^s\right)^{\frac{1}{s}}$, $r \in \mathbb{N}$.

(e) *If $A \in (r_p^q(\Delta^{B\alpha}), cs_0)$, then* $\|L_A\|_\chi = \limsup\limits_{n\to\infty} \left(\sum_k \left|\sum_{m=0}^n f_{mk}\right|^s\right)^{\frac{1}{s}}$.

(f) *If $A \in (r_p^q(\Delta^{B\alpha}), cs)$, then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^n f_{mk} - \tilde{f}_k \right|^s \right)^{\frac{1}{s}} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^n f_{mk} - \tilde{f}_k \right|^s \right)^{\frac{1}{s}},$$

where $\tilde{f} = (\tilde{f}_k)$ with $\tilde{f}_k = \lim_{n\to\infty} \left(\sum_{m=0}^n f_{mk}\right)$ for each $k \in \mathbb{N}$.

(g) *If $A \in (r_p^q(\Delta^{B\alpha}), bs)$, then* $0 \leq \|L_A\|_\chi \leq \limsup\limits_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^n f_{mk} \right|^s \right)^{\frac{1}{s}}$.

Proof. (a) Using Lemma 5.1, one can notice that

$$\|A_n\|^*_{r_p^q(\Delta^{B\alpha})} = \|F_n\|^*_{\ell_p} = \|F_n\|_{\ell_s} = \left(\sum_k |f_{nk}|^s \right)^{\frac{1}{s}}, \quad \text{for } n \in \mathbb{N}.$$

Hence, using Lemma 5.6 (a), we get the desired result.

(b) We have

$$|F_n - f|^*_{\ell_p} = |F_n - f|_{\ell_s} = \left(\sum_k |f_{nk} - f_k|^s \right)^{\frac{1}{s}}, \quad \text{for each } n \in \mathbb{N}.$$

Now, let $A \in (r_p^q(\Delta^{B\alpha}), c)$, then from Lemma 5.1, we have $F \in (\ell_p, c)$. Then we write, using Lemma 5.6 (b),

$$\frac{1}{2}\limsup_{n\to\infty} \|F_n - f\|^*_{\ell_p} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \|F_n - f\|^*_{\ell_p}.$$

This implies

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_k |f_{nk} - f_k|^s\right)^{\frac{1}{s}} \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_k |f_{nk} - f_k|^s\right)^{\frac{1}{s}},$$

which is the desired result.

(c) The proof is similar to that of (a) and (b) except that we employ Lemma 5.6 (c) instead of Lemma 5.6 (a) or 5.6 (b).

(d) Clearly,

$$\left\|\sum_{n\in\mathbb{N}} F_n\right\|^*_{\ell_p} = \left\|\sum_{n\in\mathbb{N}} F_n\right\|_{\ell_s} = \left(\sum_k \left|\sum_{n\in\mathbb{N}} f_{nk}\right|^s\right)^{\frac{1}{s}}.$$

Let $A \in (r_p^q(\Delta^{B\alpha}), \ell_1)$. Then $F \in (\ell_p, \ell_1)$ by Lemma 5.9. Hence, using Lemma 5.7, we get

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N} F_n\right\|^*_{\ell_p}\right) \leq \|L_A\|_\chi \leq 4\cdot\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left\|\sum_{n\in N} F_n\right\|^*_{\ell_p}\right).$$

This implies

$$0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_k \|f_{nk}\|^s\right)^{\frac{1}{s}},$$

as desired.

(e) It is clear that

$$\left\|\sum_{m=0}^{n} A_m\right\|^*_{r_p^q(\Delta)} = \left\|\sum_{m=0}^{n} F_m\right\|^*_{\ell_p} = \left\|\sum_{m=0}^{n} F_m\right\|_{\ell_s} = \left(\sum_k \left|\sum_{m=0}^{n} f_{mk}\right|^s\right)^{\frac{1}{s}}.$$

Hence, by using Lemma 5.8 (a), we get the desired result.

(f) This is similar to the proof of part (e) with part (b) of Lemma 5.8 instead of part (a) of Lemma 5.8.

(g) This is similar to the proof of Part (e) with part (c) of Lemma 5.8 instead of Part (a) of Lemma 5.8. □

Now, we have the following corollaries.

Corollary 5.1. *Let* $1 < p < \infty$.

(a) *Let* $A \in (r_p^q(\Delta^{B\alpha}), c_0)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty}\left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}} = 0$.

(b) *Let* $A \in (r_p^q(\Delta^{B\alpha}), c)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sum_k |f_{nk} - f_k|^s\right)^{\frac{1}{s}} = 0.$$

(c) *Let* $A \in (r_p^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if* $\lim_{n\to\infty}\left(\sum_k |f_{nk}|^s\right)^{\frac{1}{s}} = 0$.

(d) *Let* $A \in (r_p^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{N}_r}\left(\sum_k\left|\sum_{n\in N} f_{nk}\right|^s\right)^{\frac{1}{s}}\right) = 0.$$

(e) *Let* $A \in (r_p^q(\Delta^{B\alpha}), cs_0)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk}\right|^s\right)^{\frac{1}{s}} = 0.$$

(f) *Let* $A \in (r_p^q(\Delta^{B\alpha}), cs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk} - \tilde{f}\right|^s\right)^{\frac{1}{s}} = 0.$$

(g) *Let* $A \in (r_p^q(\Delta^{B\alpha}), bs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk}\right|^s\right)^{\frac{1}{s}} = 0.$$

Theorem 5.2. *The following statements hold.*

(a) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), c_0)$, *then* $\|L_A\|_\chi = \limsup\limits_{n\to\infty}\sum\limits_k |f_{nk}|$.

(b) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), c)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_k |f_{nk} - f_k|\right) \le \|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_k |f_{nk} - f_k|\right),$$

where $f = (f_k)$ *and* $f_k = \lim\limits_{n\to\infty} f_{nk}$ *for each* $k \in \mathbb{N}$.

(c) *If* $A \in (r_\infty^q(\Delta^{(\alpha)}), \ell_\infty)$, *then* $0 \le \|L_A\|_\chi \le \limsup_{n\to\infty}\sum_k |f_{nk}|$.

(d) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), \ell_1)$, *then*

$$\lim_{r\to\infty}\|A\|^{[r]}_{(r_\infty^q(\Delta^{B\alpha}),\ell_1)} \le \|L_A\|_\chi \le 4\lim_{r\to\infty}\|A\|^{[r]}_{(r_\infty^q(\Delta^{B\alpha}),\ell_1)},$$

where $\|A\|^{[r]}_{(r_\infty^q(\Delta^{B\alpha}),\ell_1)} = \sup_{N\in\mathcal{N}_r}\left(\sum_k\left|\sum_{n\in N} f_{nk}\right|\right)$, $r \in \mathbb{N}$.

(e) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), cs_0)$, *then* $\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^n f_{mk}\right|\right)$.

(f) *If* $A \in (r_\infty^q(\Delta^{B\alpha}), cs)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk} - \tilde{f}_k\right|\right) \le \|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_k\left|\sum_{m=0}^{n} f_{mk} - \tilde{f}_k\right|\right),$$

where $\tilde{f} = (\tilde{f}_k)$ *with* $\tilde{f}_k = \lim_{n\to\infty} \left(\sum_{m=0}^{n} f_{mk}\right)$ *for each* $k \in \mathbb{N}$.

(g) *If* $A \in (r^q_\infty(\Delta^{B\alpha}), bs)$*, then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_k \left|\sum_{m=0}^{n} f_{mk}\right|\right)$.

Proof. The proof is analogous to the proof of Theorem 5.1. □

Similarly, we have the following result.

Corollary 5.2. *The following statements hold.*

(a) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), c_0)$*, then* L_A *is compact if and only if* $\lim_{n\to\infty} \sum_k |f_{nk}| = 0$.

(b) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), c)$*, then* L_A *is compact if and only if* $\lim_{n\to\infty} \left(\sum_k |f_{nk} - f_k|\right) = 0$.

(c) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), \ell_\infty)$*, then* L_A *is compact if and only if* $\lim_{n\to\infty} \sum_k |f_{nk}| = 0$.

(d) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), \ell_1)$*, then* L_A *is compact if and only if*

$$\lim_{r\to\infty} \left(\sup_{N\in\mathcal{N}_r} \left(\sum_k \left| \sum_{n\in N} f_{nk} \right| \right) \right) = 0.$$

(e) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), cs_0)$*, then* L_A *is compact if and only if*

$$\limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^{n} f_{mk} \right| \right) = 0.$$

(f) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), cs)$*, then* L_A *is compact if and only if*

$$\limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^{n} f_{mk} - \tilde{f} \right| \right) = 0.$$

(g) *Let* $A \in (r^q_\infty(\Delta^{B\alpha}), bs)$*, then* L_A *is compact if and only if*

$$\limsup_{n\to\infty} \left(\sum_k \left| \sum_{m=0}^{n} f_{mk} \right| \right) = 0.$$

Theorem 5.3. *The following statements hold.*

(a) *If* $A \in (r^q_1(\Delta^{B\alpha)}), c_0)$*, then* $\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sup_k |f_{nk}|\right)$.

(b) *If* $A \in (r^q_1(\Delta^{B\alpha}), c)$*, then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sup_k |f_{nk} - f_k| \right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_k |f_{nk} - f_k| \right),$$

where $f = (f_k)$ *and* $f_k = \lim_{n\to\infty} f_{nk}$ *for each* $k \in \mathbb{N}$.

(c) *If* $A \in (r^q_1(\Delta^{B\alpha}), \ell_\infty)$*, then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_k |f_{nk}|\right)$.

(d) *If* $A \in (r^q_1(\Delta^{B\alpha}), \ell_1)$*, then* $\|L_A\|_\chi = \lim_{r\to\infty} \left(\sup_k \sum_{n=r}^{\infty} |f_{nk}|\right)$.

(e) *If* $A \in (r^q_1(\Delta^{B\alpha}), cs_0)$*, then* $\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right)$.

(f) *If* $A \in (r^q_1(\Delta^{B\alpha}), cs)$*, then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sup_k \left| \sum_{m=0}^{n} f_{mk} - \tilde{f}_k \right| \right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_k \left| \sum_{m=0}^{n} f_{mk} - \tilde{f}_k \right| \right),$$

where $\tilde{f} = (\tilde{f}_k)$ *with* $\tilde{f}_k = \lim_{n\to\infty} \left(\sum_{m=0}^{n} f_{mk}\right)$ *for each* $k \in \mathbb{N}$.

(g) *If* $A \in (r_1^q(\Delta^{B\alpha}), bs)$, *then* $0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_k \left|\sum_{m=0}^n f_{mk}\right|\right)$.

Proof. The proof is analogous to the proof of Theorem 5.1. □

Similarly, we have the following result.

Corollary 5.3. *The following statements hold.*

(a) *Let* $A \in (r_1^q(\Delta^{B\alpha}), c_0)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sup_k |f_{nk}|\right) = 0.$$

(b) *Let* $A \in (r_1^q(\Delta^{B\alpha}), c)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sup_k |f_{nk} - f_k|\right) = 0.$$

(c) *Let* $A \in (r_1^q(\Delta^{B\alpha}), \ell_\infty)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sup_k |f_{nk}|\right) = 0.$$

(d) *Let* $A \in (r_1^q(\Delta^{B\alpha}), \ell_1)$, *then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |f_{nk}|\right) = 0.$$

(e) *Let* $A \in (r_1^q(\Delta^{B\alpha}), cs_0)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right) = 0.$$

(f) *Let* $A \in (r_1^q(\Delta^{B\alpha}), cs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sup_k \left|\sum_{m=0}^{n} f_{mk} - \tilde{f}\right|\right) = 0.$$

(g) *Let* $A \in (r_1^q(\Delta^{B\alpha}), bs)$, *then* L_A *is compact if and only if*

$$\limsup_{n\to\infty}\left(\sup_k \left|\sum_{m=0}^{n} f_{mk}\right|\right) = 0.$$

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