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GENERALIZATION OF LUPAŞ-KANTOROVICH OPERATORS CONNECTED WITH PÓLYA DISTRIBUTION

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Abstract. The motive of this paper is to introduce the generalization of Lupaş-Kantorovich operators connected with Pólya distribution and establish the rate of convergence in terms of modulus of continuity. Furthermore, a Voronovskaja type asymptotic formula for these operators is studied. In the end, few numerical examples with graphical representation are added to depict the effect of convergence of the operators.

1. Introduction

About a decade ago, Gurdek et al. [20] defined the Baskakov operators for functions of two variables and analysed the approximation degree and differential properties of these operators. Agrawal et al. [8, 9] considered the bivariate form of the Lupaş Durrmeyer operators with Pólya distribution which was considered by Gupta and Rassias in [19]. In 2010, Gadjiev and Gorbanalizadeh [15] constructed the two dimensional extension of Bernstein–Stancu type polynomials and investigated the degree of convergence of these polynomials. The Kantrovich variants of various operators have been intensively studied in [1–4, 11, 16] and [23]. Very recently, Agrawal et al. [7] discussed the approximation features of the Kantorovich modification of the operators proposed by Stancu [26] and introduced their bivariate extension. For more related work, we suggest the readers (see [5, 14, 17, 18, 22, 24, 25, 27]). Inspired by the above work, we now introduce the bivariate form of the operators defined in [6] and given as:

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$$\left(\widetilde{Q}_n^{(1/n)} f\right)(x) = (1+n)\sum_{j=0}^{n} \widetilde{q}_{n,j}^{(1/n)}(x) \int_{I_{j,n}} f(\kappa)d\kappa, \quad x \in \left[\frac{1}{4}, \frac{3}{4}\right], \tag{1.1}$$

where

$$\widetilde{q}_{n,j}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{j}\left(\frac{2x(n+1)-1}{2}\right)_j \left(\frac{2n(1-x)-2x+1}{2}\right)_{n-j},$$

$I_{j,n} = \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$ and $(nx)_j = \prod_{i=0}^{j-1}(nx+i)$.

These operators preserve the linear functions along with the constants. In [6], the authors have provided moments and established some direct results for the operators defined by (1.1).

2. Preliminary Results

Let J be the interval $[\frac{1}{4}, \frac{3}{4}]$. Then on $J^2 = J \times J$, The space of continuous functions with real values is denoted by $C(J^2)$. The norm for this space is $||g||_{C(J^2)} = \sup\limits_{(x,y)\in J^2} |g(x,y)|$.

For $f \in C(J^2)$ and $(x,y) \in J^2$, we define

$$\begin{aligned}\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\right)(x,y) =&(1+n_1)(1+n_2)\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} \widetilde{q}_{n_1,n_2,j_1,j_2}^{(1/n_1,1/n_2)}(x,y)\\ &\times \int_{I_{j_1,n_1}}\int_{I_{j_2,n_2}} f(u,v)dudv,\end{aligned}$$

where

$$\begin{aligned}\widetilde{q}_{n_1,n_2,j_1,j_2}^{(1/n_1,1/n_2)}(x,y) =&\frac{2(n_1!)}{(2n_1)!}\frac{2(n_2!)}{(2n_2)!}\binom{n_1}{j_1}\binom{n_2}{j_2}\left(\frac{2x(n_1+1)-1}{2}\right)_{j_1}\\ &\times \left(\frac{2n_1(1-x)-2x+1}{2}\right)_{n_1-j_1}\left(\frac{2y(n_2+1)-1}{2}\right)_{j_2}\\ &\times \left(\frac{2n_2(1-y)-2y+1}{2}\right)_{n_2-j_2}.\end{aligned}$$

The following lemmas are helpful in determining the key outcomes.

Lemma 2.1 ([6])**.** *For $x \in [\frac{1}{4}, \frac{3}{4}]$ and $n = 1, 2, 3, \ldots,$ we have*

$$\begin{aligned}\left(\widetilde{Q}_n^{(1/n)} e_0\right)(x) =&1, \quad \left(\widetilde{Q}_n^{(1/n)} e_1\right)(x) = x,\\ \left(\widetilde{Q}_n^{(1/n)} e_2\right)(x) =&\frac{1}{12(1+n)^3}\Big\{12n^3x^2 + 12n^2x(x+2) + n\left(-12x^2+48x-11\right)\\ &-12x^2+24x-5\Big\}.\end{aligned}$$

Lemma 2.2 ([6]). *For $x \in [\frac{1}{4}, \frac{3}{4}]$ and $n = 1, 2, 3, \dots,$ we have*

$$\begin{aligned}\Big(\widetilde{Q}_n^{(1/n)}(e_1 - xe_0)\Big)(x) =& 0,\\ \Big(\widetilde{Q}_n^{(1/n)}(e_1 - xe_0)^2\Big)(x) =& \frac{1}{12(1+n)^3}\Big\{ - 24n^2(x-1)x\\ &+ n\left(-48x^2 + 48x - 11\right) - 24x^2 + 24x - 5\Big\}.\end{aligned}$$

Lemma 2.3. *If we denote $e_{ij} = x^i y^j$, where $i, j = 0, 1, 2$ and $i + j \le 2$, then*

$$\begin{aligned}\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{00}\Big)(x,y) =& 1, \quad \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{10}\Big)(x,y) = x,\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{01}\Big)(x,y) =& y,\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{20}\Big)(x,y) =& \frac{1}{12(1+n_1)^3}\Big\{12n_1^3x^2 + 12n_1^2x(x+2)\\ &+ n_1\left(-12x^2 + 48x - 11\right) - 12x^2 + 24x - 5\Big\},\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{02}\Big)(x,y) =& \frac{1}{12(1+n_2)^3}\Big\{12n_2^3y^2 + 12n_2^2y(y+2)\\ &+ n_2\left(-12y^2 + 48y - 11\right) - 12y^2 + 24y - 5\Big\}.\end{aligned}$$

Lemma 2.4. *The following result holds:*

$$\begin{aligned}\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-x)\Big)(x) =& 0, \quad (\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-y))(x) = 0,\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-x)^2\Big)(x,y) =& \frac{1}{12(1+n_1)^3}\Big\{ - 24n_1^2(x-1)x\\ &+ n_1\left(-48x^2 + 48x - 11\right) - 24x^2 + 24x - 5\Big\},\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-y)^2\Big)(x,y) =& \frac{1}{12(1+n_2)^3}\Big\{ - 24n_2^2(y-1)y\\ &+ n_2\left(-48y^2 + 48y - 11\right) - 24y^2 + 24y - 5\Big\}\\ &= O\left(\frac{1}{n}\right), \quad \text{when } n \to +\infty.\end{aligned}$$

Also,

$$\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-x)^4\Big)(x,y) = O\left(\frac{1}{n^2}\right), \quad \text{when } n \to +\infty$$

and

$$\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-y)^4\Big)(x,y) = O\left(\frac{1}{n^2}\right), \quad \text{when } n \to +\infty.$$

3. Rate of Convergence

For $f \in C(J^2)$, the full modulus of continuity with respect to x and y is given as

$$\overline{\omega}(f,h) = \max\Big\{|f(x_1,y_1) - f(x_2,y_2)| : (x_1,y_1) \text{ and } (x_2,y_2) \in J^2\Big\}, \quad h > 0,$$

with the condition that

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \leq h.$$

And the partial moduli of continuity is given as

$$\omega_1(f,h) = \max\Big\{|f(x,y_1) - f(x,y_2)| : (x,y_1) \text{ and } (x,y_2) \in J^2 \text{ with } |y_1-y_2| \leq h\Big\}$$

and

$$\omega_2(f,h) = \max\Big\{|f(x_1,y) - f(x_2,y)| : (x_1,y) \text{ and } (x_2,y) \in J^2 \text{ with } |x_1-x_2| \leq h\Big\},$$

respectively.

They meet the well-known features of the usual modulus of continuity, as defined in [10]. Various results related to the partial moduli of continuity have been studied by researchers (for instance, one may refer [21]).

Theorem 3.1. *If $f \in C(J^2)$, the operators $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f$ converge uniformly to f on J^2.*

Proof. Clearly,

$$\lim_{n_1\to+\infty, n_2\to+\infty} \widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{ij} = e_{ij},$$

for (i,j) taking the values $(0,0)$, $(1,0)$ and $(1,1)$, and

$$\lim_{n_1\to+\infty, n_2\to+\infty} \widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} (e_{02}+e_{20}) = e_{02}+e_{20}.$$

Thus, on applying [12, Theorem 2.1], we obtain the desired result. □

Theorem 3.2. *For $f \in C(J^2)$ and $\zeta,\eta \in J^2$, we have*

$$\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\Big)(\zeta,\eta) - f(\zeta,\eta)\Big| \leq 2\Big\{\omega_1\Big(f, \frac{1}{\sqrt{n_1+1}}\Big) + \omega_2\Big(f, \frac{1}{\sqrt{n_2+1}}\Big)\Big\}.$$

Proof. From the property of partial moduli of continuity, we get

$$\begin{aligned}
\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\Big)(\zeta,\eta) - f(\zeta,\eta)\Big| &\leq \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,v) - f(\zeta,\eta)|\Big)(\zeta,\eta)\\
&\leq \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,v) - f(u,\eta)|\Big)(\zeta,\eta)\\
&\quad + \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,\eta) - f(\zeta,\eta)|\Big)(\zeta,\eta)\\
&\leq \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} \omega_2(f,|v-\eta|)\Big)(\zeta,\eta)\\
&\quad + \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} \omega_1(f,|u-\zeta|)\Big)(\zeta,\eta)\\
&\leq \Big(1 + h_{n_2}^{-1}(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|v-\eta|)(\eta)\Big)\,\omega_2(f,h_{n_2})\\
&\quad + \Big(1 + h_{n_1}^{-1}(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|u-\zeta|)(\zeta)\Big)\,\omega_1(f,h_{n_1}),
\end{aligned}$$

where $h_{n_1}, h_{n_2} > 0$.

Making use of Cauchy-Schwarz inequality, we may write

$$\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\Big)(\zeta,\eta)-f(\zeta,\eta)\Big| \le \left(1+h_{n_2}^{-1}\sqrt{\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-\eta)^2\Big)(\eta)}\right)\omega_2(f,h_{n_2}) + \left(1+h_{n_1}^{-1}\sqrt{\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-\zeta)^2\Big)(\zeta)}\right)\omega_1(f,h_{n_1}).$$

Thus, by choosing $h_{n_1} = \frac{1}{\sqrt{n_1+1}}$ and $h_{n_2} = \frac{1}{\sqrt{n_2+1}}$, we reach the required result. □

Theorem 3.3. *For $f \in C(J^2)$and $\zeta, \eta \in J^2$, we have*

$$\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\Big)(\zeta,\eta)-f(\zeta,\eta)\Big| \le ||f_\zeta||\sqrt{\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-\zeta)^2\Big)(\zeta,\eta)} + ||f_\eta||\sqrt{\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-\eta)^2\Big)(\zeta,\eta)}.$$

Proof. If $(\zeta, \eta) \in J^2$, then

$$f(u,v)-f(\zeta,\eta)=\int_\zeta^u f_s(s,v)ds+\int_\eta^v f_t(\zeta,t)dt.$$

Applying the operators $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}$ on both sides of the sides of above inequality, we get

$$\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\Big)(u,v)-f(\zeta,\eta)\Big| \le \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}\int_\zeta^u f_s(s,v)ds\Big)(\zeta,\eta) + \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}\int_\eta^v f_t(\zeta,t)dt\Big)(\zeta,\eta),$$

as

$$\Big|\int_\zeta^u f_s(s,v)ds\Big| \le ||f_\zeta||\cdot|u-\zeta|$$

and

$$\Big|\int_\eta^v f_t(\zeta,t)dt\Big| \le ||f_\eta||\cdot|v-\eta|,$$

therefore,

$$\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\Big)(u,v)-f(\zeta,\eta)\Big| \le ||f_\zeta||\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|u-\zeta|\Big)(\zeta,\eta) + ||f_\eta||\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|v-\eta|\Big)(\zeta,\eta).$$

We got the desired conclusion by using the Cauchy-Schwarz inequality. □

For $(u, v), (\zeta, \eta) \in J^2$, we define the Lipschitz class (as defined in [13]), $\mathrm{Lip}_K\,\alpha$, as follows:

$$\mathrm{Lip}_K\,\alpha=\Big\{f\in C(J^2):|f(u,v)-f(\zeta,\eta)|\le K\Big\{(u-\zeta)^2+(v-\eta)^2\Big\}^{\frac{\alpha}{2}};\ \alpha\in(0,1]\Big\}.$$

Theorem 3.4. *If $f \in \mathrm{Lip}_K\,\alpha$, then the following conclusion is correct:*

$$\left|\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\right)(\zeta,\eta)-f(\zeta,\eta)\right| \le K\left\{\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-\zeta)^2\right)(\zeta,\eta) + \left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-\eta)^2\right)(\zeta,\eta)\right\}^{\frac{\alpha}{2}}.$$

Proof. If $f \in \mathrm{Lip}_K\,\alpha$, then we may write

$$\begin{aligned}\left|\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\right)(\zeta,\eta)-f(\zeta,\eta)\right| &\le\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|f(u,v)-f(\zeta,\eta)|\right)(\zeta,\eta)\\ &\le K\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}\left\{|u-\zeta|^2+|v-\eta|^2\right\}^{\frac{\alpha}{2}}\right)(\zeta,\eta).\end{aligned}$$

Using the Hölder's inequality and $v_1=\frac{2}{\alpha}$ and $w_1=\frac{2}{2-\alpha}$, we obtain

$$\left|\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\right)(\zeta,\eta)-f(\zeta,\eta)\right| \le K\left\{\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-\zeta)^2\right)(\zeta,\eta) + \left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-\eta)^2\right)(\zeta,\eta)\right\}^{\frac{\alpha}{2}}.$$

Hence, the required result follows. □

4. Voronovskaja–type Theorem

Let $C^2(J^2)$ be the space containing the functions f that have the property $f \in C(J^2)$ and $f^{(i,j)} \in C(J^2)$, $0 \le i+j \le 2$.

Here,

$$f^{(i,j)}=\left\{\frac{\partial^i f}{\partial\zeta^i},\frac{\partial^i f}{\partial\eta^i}\,:\; i=1,2\right\},\quad \zeta,\eta\in J^2.$$

The space $C^2(J^2)$ is equipped with the norm

$$||f||_{C^2(J^2)}=||f||_{C(J^2)}+\left\|\frac{\partial f}{\partial\zeta}\right\|_{C(J^2)}+\left\|\frac{\partial f}{\partial\eta}\right\|_{C(J^2)}+\left\|\frac{\partial^2 f}{\partial\zeta^2}\right\|_{C(J^2)}+\left\|\frac{\partial^2 f}{\partial\eta^2}\right\|_{C(J^2)}.$$

Theorem 4.1. *Let $f\in C^2(J^2)$, then*

$$\lim_{n\to+\infty} n\left\{\left(\widetilde{Q}_{n,n}^{(1/n,1/n)}f\right)(\zeta,\eta)-f(\zeta,\eta)\right\}=\zeta(\zeta-1)f_{\zeta\zeta}(\zeta,\eta)+\eta(\eta-1)f_{\eta\eta}(\zeta,\eta).$$

Proof. Let $(\zeta,\eta),(u,v)\in J^2$. Applying Taylor's expansion, we get

$$\begin{aligned}f(u,v)=&f(\zeta,\eta)+f_\eta(\zeta,\eta)(v-\eta)+f_\zeta(\zeta,\eta)(u-\zeta)+\frac{1}{2}\Big\{f_{\eta\eta}(\zeta,\eta)(v-\eta)^2\\ &+f_{\zeta\zeta}(\zeta,\eta)(u-\zeta)^2+2f_{\zeta\eta}(\zeta,\eta)(u-\zeta)(v-\eta)\Big\}\\ &+\xi(u,v)\Big\{(u-\zeta)^2+(v-\eta)^2\Big\},\end{aligned}$$

where $\xi(u,v)$ vanishes as $(u,v)\to(\zeta,\eta)$.

By the linearity of $\widetilde{Q}_{n,n}^{(1/n,1/n)}$, we have

$$\begin{aligned}(\widetilde{Q}_{n,n}^{(1/n,1/n)}f)(u,v) =& f(\zeta,\eta) + f_{\eta}(\zeta,\eta)(\widetilde{Q}_{n}^{(1/n)}(v-\eta))(\eta) + f_{\zeta}(\zeta,\eta)(\widetilde{Q}_{n}^{(1/n)}(u-\zeta))(\zeta)\\ &+ \frac{1}{2}\Big\{ f_{\eta\eta}(\widetilde{Q}_{n}^{(1/n)}(v-\eta)^2)(\eta) + f_{\zeta\zeta}(\widetilde{Q}_{n}^{(1/n)}(u-\zeta)^2)(\zeta)\\ &+ 2f_{\zeta\eta}(\zeta,\eta)(\widetilde{Q}_{n}^{(1/n)}(u-\zeta))(\zeta)(\widetilde{Q}_{n}^{(1/n)}(v-\eta))(\eta)\Big\}\\ &+ \widetilde{Q}_{n,n}^{(1/n,1/n)}\Big\{\xi(u,v)\Big((u-\zeta)^2+(v-\eta)^2\Big)\Big\}.\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}&\left|\widetilde{Q}_{n,n}^{(1/n,1/n)}\Big\{\xi(u,v)\Big((u-\zeta)^2+(v-\eta)^2\Big)\Big\}\right|\\ \le&\Big\{\widetilde{Q}_{n,n}^{(1/n,1/n)}\xi^2(u,v)(\zeta,\eta)\Big\}^{\frac{1}{2}}\Big\{\Big(\widetilde{Q}_{n,n}^{(1/n,1/n)}\Big((u-\zeta)^2+(v-\eta)^2\Big)^2\Big)(\zeta,\eta)\Big\}^{\frac{1}{2}}\\ \le&\sqrt{2}\Big\{\widetilde{Q}_{n,n}^{(1/n,1/n)}\xi^2(u,v)(\zeta,\eta)\Big\}^{\frac{1}{2}}\Big\{(\widetilde{Q}_{n,n}^{(1/n,1/n)}(u-\zeta)^4)(\zeta)+(\widetilde{Q}_{n,n}^{(1/n,1/n)}(v-\eta)^4)(\eta)\Big\}^{\frac{1}{2}}.\end{aligned}$$

In view of Theorem 3.1, we have

$$\lim_{n\to+\infty}\widetilde{Q}_{n,n}^{(1/n,1/n)}\xi^2(u,v)(\zeta,\eta)=0.$$

Using Lemma 2.4, we may write

$$\lim_{n\to+\infty} n\widetilde{Q}_{n,n}^{(1/n,1/n)}\Big\{\xi(u,v)\Big((u-\zeta)^2+(v-\eta)^2\Big)\Big\}(\zeta,\eta)=0.$$

Finally, on using the values from Lemma 2.4, the proof of the theorem follows. □

5. Graphical Analysis

For validating the convergence results obtained in the above sections, we provide few numerical examples involving illustrative graphics.

Example 5.1. For $f(x,y)=x^2-x+y^2-y$, we show the convergence of $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}$ to $f(x,y)=x^2-x+y^2-y$ for $n_1=n_2=50$ and $n_1=n_2=200$ in Figure 1 and Figure 2, respectively.

Example 5.2. For $f(x,y)=-\sqrt{7}\,(x^2+2xy-2x+y^2-2y+1)+x^2-10xy$, we show the convergence of $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}$ to $f(x,y)$ for $n_1=n_2=5$ and $n_1=n_2=50$ in Figure 3 and Figure 4, respectively.

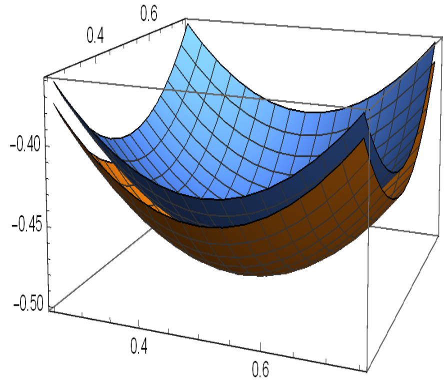

FIGURE 1. Graphs of $\widetilde{Q}_{50,50}^{(1/50,1/50)}$ (blue) and $f(x,y) = x^2 - x + y^2 - y$ (yellow).

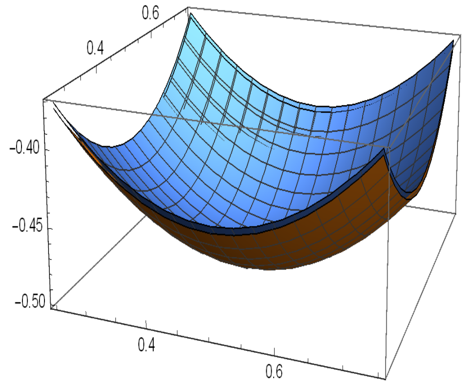

FIGURE 2. Graphs of $\widetilde{Q}_{200,200}^{(1/200,1/200)}$ (blue) and $f(x,y) = x^2 - x + y^2 - y$ (yellow).

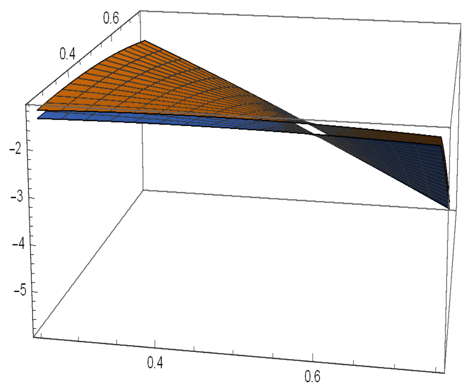

FIGURE 3. Graphs of $\widetilde{Q}_{5,5}^{(1/5,1/5)}$ (blue) and $f(x,y) = -\sqrt{7}\,(x^2 + 2xy - 2x + y^2 - 2y + 1) + x^2 - 10xy$ (yellow).

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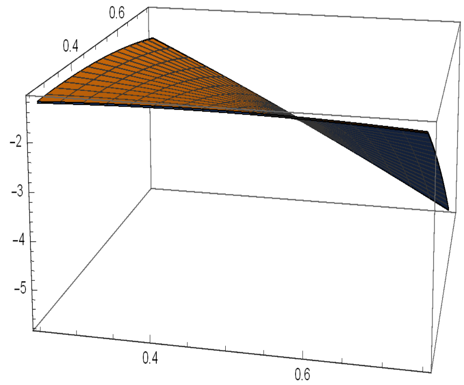

FIGURE 4. Graphs of $\widetilde{Q}_{50,50}^{(1/50,1/50)}$ (blue) and $f(x,y) = -\sqrt{7}\,(x^2+2xy-2x+y^2-2y+1)+x^2-10xy$ (yellow).

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