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ESSENTIAL NORM OF GENERALIZED INTEGRATION OPERATOR BETWEEN ZYGMUND TYPE SPACES

MOSTAFA HASSANLOU[1*] AND FARIBA ALIGHADR[2]

Abstract. Considering the generalized integration operator

$$\left(C^n_{\varphi,g} f\right)(z) = \int_0^z f^{(n)}(\varphi(\xi)) g(\xi)\, d\xi,$$

between two Zygmund type spaces, the essential norm of the operator will be estimated. Here φ is an analytic self-map on $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$. As a result, a criteria for the compactness of the above operator is given in the paper.

1. Introduction

First we bring some notation and information will be used in the paper. A main problem concerning composition operator or other classical operators is to relate the function-theoretic properties of the inducing function of the operator and operator-theoretic properties of it's own operator while acts on various Banach spaces of analytic functions. For example, it is well known that the composition operator C_φ is bounded on the classical Hardy, Bergman and Bloch spaces. See the excellent books [2, 13] for complete study on the composition operators and classical spaces of analytic functions.

Throughout the paper $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Every analytic self-map φ of the unit disk $\mathbb{D}$ induces through composition a linear composition operator C_φ from $H(\mathbb{D})$ to itself which is defined as

$$C_\varphi(f)(z) = f(\varphi(z)).$$

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The above operator is generalized by Li and Stević in [7] as follows:

$$\left(C_{\varphi}^{g} f\right)(z)=\int_0^z f'(\varphi(\xi))g(\xi)\, d\xi, \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $g \in H(\mathbb{D})$. The following operator is a generalization of C_{φ}^{g} on the unit disk

$$\left(C_{\varphi,g}^{n} f\right)(z)=\int_0^z f^{(n)}(\varphi(\xi))g(\xi)\, d\xi, \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $n \in \mathbb{N}$. If $n=1$, then we write $C_{\varphi,g}^{n}=C_{\varphi}^{g}$. This integral type operator has been investigated by many authors, see, e.g., [1, 5–8, 19, 20].

Some characterizations of the boundedness, compactness and estimating the essential norm of generalized integration operators or other classical operators on or into the Zygmund type spaces can be found in [1, 4–7, 9–12, 19–22].

The class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{\left|f\left(e^{i(\theta+h)}\right)+f\left(e^{i(\theta-h)}\right)-2f\left(e^{i\theta}\right)\right|}{h} < +\infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h>0$, is denoted by $\mathcal{Z}$ and called Zygmund space. The Closed Graph Theorem together [3, Theorem 5.3] implies that $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f''(z)| < +\infty. \tag{1.1}$$

The space $\mathcal{Z}$ is a Banach space with the following norm

$$\|f\|_{\mathcal{Z}} = |f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f''(z)|. \tag{1.2}$$

Motivated by the above definition, the Zygmund type space $\mathcal{Z}_\alpha$, $\alpha>0$, is defined as the space of all analytic functions f on $\mathbb{D}$ for which $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)|<+\infty$ and the norm on this spaces is

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha|f''(z)|.$$

The little Zygmund type space of $\mathbb{D}$, denoted by $\mathcal{Z}_{\alpha,0}$, is the closed subspace of $\mathcal{Z}_\alpha$ consisting of functions f with

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha|f''(z)| = 0.$$

Zygmund type spaces can be generalized on the set the functions defined on or the weight function instead of $(1-|z|^2)^\alpha$. We refer, for some researches considered Zygmund type spaces, to [5–8, 11, 12, 15, 19–22].

Also, for $\alpha>0$ the Bloch type space $\mathcal{B}^\alpha$ is the space of the functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)|+\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha|f'(z)| < +\infty.$$

The essential of the operator $T : X \to Y$ between Banach spaces is the distance of T from the space of compact operators from X to Y. In other words

$$\|T\|_{e,X\to Y} = \inf \|T - S\|,$$

where the infimum is taken all over the compact operators $S : X \to Y$ and $\|.\|$ is the operator norm. The operator is compact if and only if $\|T\|_{e,X\to Y} = 0$.

The aim of the paper is to approximate the essential norm of the operator $C^n_{\varphi,g}$ between Zygmund type space $\mathcal{Z}_\alpha$. Before that, we characterize bounded generalized integration operators on these spaces. It should be noted that the operator norm can be computed from boundedness conditions.

In this paper, for real scalars A and B, the notation $\preceq$ means $A \le cB$ for some positive constant c. Also, the notation $A \approx B$ means $A \preceq B$ and $B \preceq A$.

2. Bounded Operator

In this section, we first state some facts about functions in Zygmund type spaces and then characterize the bounded operators. If $f \in \mathcal{Z}_\alpha$, then

$$|f''(z)| \le \frac{\|f\|_{\mathcal{Z}_\alpha}}{(1-|z|^2)^\alpha}. \tag{2.1}$$

It is known that for $f \in \mathcal{B}^\alpha$

$$|f^{(k)}(z)| \le \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+k-1}},$$

where $k \ge 2$, see for example [17]. So, for $f \in \mathcal{Z}_\alpha$ we get

$$|f^{(k+1)}(z)| \le \frac{\|f\|_{\mathcal{Z}_\alpha}}{(1-|z|^2)^{\alpha+k-1}}. \tag{2.2}$$

For estimating $|f(z)|$ and $|f'(z)|$ the following lemma is applied, see [4].

Lemma 2.1. *For every $f \in \mathcal{Z}_\alpha$ where $\alpha > 0$ we have*

(i) $|f'(z)| \le \frac{2}{1-\alpha}||f||_{\mathcal{Z}_\alpha}$ *and* $|f(z)| \le \frac{2}{1-\alpha}||f||_{\mathcal{Z}_\alpha}$ *for every* $0 < \alpha < 1$;

(ii) $|f'(z)| \le 2||f||_{\mathcal{Z}} \log \frac{1}{1-|z|}$ *and* $|f(z)| \le ||f||_{\mathcal{Z}}$ *for* $\alpha = 1$;

(iii) $|f'(z)| \le \frac{2}{\alpha-1} \cdot \frac{||f||_{\mathcal{Z}_\alpha}}{(1-|z|)^{\alpha-1}}$, *for every* $\alpha > 1$;

(iv) $|f(z)| \le \frac{2}{(\alpha-1)(2-\alpha)}||f||_{\mathcal{Z}_\alpha}$, *for every* $1 < \alpha < 2$;

(v) $|f(z)| \le 2||f||_{\mathcal{Z}_2} \log \frac{2}{1-|z|}$, *for* $\alpha = 2$;

(vi) $|f(z)| \le \frac{2}{(\alpha-1)(\alpha-2)} \cdot \frac{||f||_{\mathcal{Z}_\alpha}}{(1-|z|)^{\alpha-2}}$, *for every* $\alpha > 2$.

Because of the above estimations, we divide the results for $n \ge 2$ and $n = 1$.

Theorem 2.1. *Let $n \ge 2$ and $0 < \alpha, \beta < +\infty$. Then, $C^n_{\varphi,g} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded if and only if*

$$M_1 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta\,|\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}} < +\infty, \quad M_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta\,|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}} < +\infty.$$

Proof. Suppose that $C^n_{\varphi,g} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded. So there exists a positive constant C such that for every $f \in \mathcal{Z}_\alpha$

$$\|C^n_{\varphi,g} f\|_{\mathcal{Z}_\beta} \leq C\|f\|_{\mathcal{Z}_\alpha}.$$

We prove that $M_1 < +\infty$. The proof of the other one is similar. Since any polynomial belongs to $\mathcal{Z}_\alpha$, by taking the function $f_1(z) = z^n$ in the above inequality, we obtain

$$R_1 = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta |g'(z)| < +\infty. \tag{2.3}$$

Similarly by using $f_2(z) = z^{n+1}$, we get

$$\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta |\varphi'(z)g(z) + \varphi(z)g'(z)| < +\infty. \tag{2.4}$$

According to (2.3), (2.4) and boundedness of φ, we obtain that

$$R_2 = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta |\varphi'(z)| \cdot |g(z)| < +\infty. \tag{2.5}$$

Fix $a \in \mathbb{D}$ and define the functions F_a as follows

$$F_a(z) = C_F \frac{(1 - |a|^2)^3}{(1 - \overline{a}z)^{\alpha+1}} - D_F \frac{(1 - |a|^2)^2}{(1 - \overline{a}z)^{\alpha}}, \tag{2.6}$$

where $C_F = \frac{\alpha}{\overline{a}^{n+1}}$ and $D_F = \frac{\alpha(\alpha+n)}{\overline{a}^{n+1}}$. It can be checked that $F_a \in \mathcal{Z}_\alpha$, $F_a^{(n)}(a) = 0$ and

$$F_a^{(n+1)}(a) = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{(1 - |a|^2)^{\alpha+n-1}}.$$

Also, $\sup_{\frac{1}{2}<|a|<1} \|F_a\|_{\mathcal{Z}_\alpha} < +\infty$. Then, using the boundedness of the operator, we have

$$\begin{aligned}\|C^n_{\varphi,g} F_{\varphi(a)}\|_{\mathcal{Z}_\beta} \geq& \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta \left|\left(C^n_{\varphi,g} F_{\varphi(a)}\right)''(z)\right| \\ \geq& \left(1 - |a|^2\right)^\beta \left|\varphi'(a)g(a)F^{(n+1)}_{\varphi(a)}(\varphi(a))\right| - \left(1 - |a|^2\right)^\beta \left|g'(a)F^{(n)}_{\varphi(a)}(\varphi(a))\right| \\ =& \frac{\alpha(\alpha+1)\cdots(\alpha+n)\,(1 - |a|^2)^\beta\,|\varphi'(a)| \cdot |g(a)|}{(1 - |\varphi(a)|^2)^{\alpha+n-1}}.\end{aligned}$$

Hence,

$$\sup_{1/2<|\varphi(a)|<1} \frac{(1 - |a|^2)^\beta\,|\varphi'(a)| \cdot |g(a)|}{(1 - |\varphi(a)|^2)^{\alpha+n-1}} \preceq \sup_{1/2<|\varphi(a)|<1} \|C^n_{\varphi,g} F_{\varphi(a)}\|_{\mathcal{Z}_\beta} < +\infty. \tag{2.7}$$

On the other hand

$$\sup_{|\varphi(a)|\leq 1/2} \frac{(1 - |a|^2)^\beta\,|\varphi'(a)| \cdot |g(a)|}{(1 - |\varphi(a)|^2)^{\alpha+n-1}} \preceq \sup_{|\varphi(a)|\leq 1/2} \left(1 - |a|^2\right)^\beta |\varphi'(a)| \cdot |g(a)| \leq R_2 < +\infty. \tag{2.8}$$

From (2.7) and (2.8), $M_1 < +\infty$. In order to prove $M_2 < +\infty$ we use the functions

$$G_a(z) = C_G \frac{\left(1-|a|^2\right)^3}{\left(1-\overline{a}z\right)^{\alpha+1}} - D_G \frac{\left(1-|a|^2\right)^2}{\left(1-\overline{a}z\right)^{\alpha}}, \tag{2.9}$$

where $C_G = \frac{\alpha}{\overline{a}^n}$ and $D_G = \frac{\alpha+n+1}{\overline{a}^n}$. Then, $G_a^{(n+1)}(a) = 0$ and

$$G_a^{(n)}(a) = -\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{(1-|a|^2)^{\alpha+n-2}}.$$

In the rest of the proof is used the same method and do not bring here.

Conversely, suppose that M_1 and M_2 are finite. Let $f \in \mathcal{Z}_\alpha$. The equation (2.2) implies that

$$\begin{aligned}
\|C_{\varphi,g}^n f\|_{\mathcal{Z}_\beta} =& |f^{(n)}(\varphi(0))| \cdot |g(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \Big|\varphi'(z)g(z)f^{(n+1)}(\varphi(z)) \\
&+ g'(z)f^{(n)}(\varphi(z))\Big| \\
\leq& \frac{|g(0)|}{(1-|\varphi(0)|^2)^{\alpha+n-2}}\|f\|_{\mathcal{Z}_\alpha} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}\|f\|_{\mathcal{Z}_\alpha} \\
&+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}\|f\|_{\mathcal{Z}_\alpha}.
\end{aligned}$$

So, $C_{\varphi,g}^n : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded. □

Theorem 2.2. *Let $n = 1$. Then, $C_\varphi^g = C_{\varphi,g}^1 : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded if and only if*

$$M_3 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^\alpha} < +\infty$$

and

(i) *for $0 < \alpha < 1$, $g \in \mathcal{B}^\beta$;*
(ii) *for $\alpha = 1$, $\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(z)| \log(1-|\varphi(z)|^2)^{-1} < +\infty$;*
(iii) *for $\alpha > 1$, $\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} < +\infty$.*

Proof. To prove that $M_3 < +\infty$, we use the same method as in the proof of Theorem 2.1. Also the proof of part (iii) is similar, take $n = 1$ in the proof of Theorem 2.1. Moreover the part (ii) comes from [7] with a simple modification for $\beta > 0$.

Now suppose that $M_3 < +\infty$ and $g \in \mathcal{B}^\beta$. Then,

$$\begin{aligned}
\|C_\varphi^g f\|_{\mathcal{Z}_\beta} \leq& |g(0)f'(\varphi(0))| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |\varphi'(z)g(z)f''(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |g'(z)f'(\varphi(z))| \\
\leq& \frac{2}{1-\alpha}|g(0)|\cdot\|f\|_{\mathcal{Z}_\alpha} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\|f\|_{\mathcal{Z}_\alpha}
\end{aligned}$$

$$+\frac{2}{1-\alpha}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}|g'(z)|\cdot\|f\|_{\mathcal{Z}_\alpha}.$$

So, $C_\varphi^g:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ is bounded. Now we assume that the operator is bounded. We just prove that $g\in\mathcal{B}^\beta$ and this comes from applying C_φ^g to the function $f_0(z)=z$. □

3. Essential Norm

In the following theorem we bring an approximation for the essential norm, as a tool for investigate the compactness of $C_{\varphi,g}^n:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$.

Theorem 3.1. *Let $n\geq 2$ and $\alpha,\beta>0$. If $C_{\varphi,g}^n:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ is bounded, then*

$$\|C_{\varphi,g}^n\|_e\approx\max\left\{\limsup_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)^{\beta}|\varphi'(z)||g(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha+n-1}},\limsup_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)^{\beta}|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha+n-2}}\right\}.\tag{3.1}$$

As a result of this theorem, we can find a characterization for compactness of the operator: $C_{\varphi,g}^n:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ is compact if and only if

$$\limsup_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)^{\beta}|\varphi'(z)|\cdot|g(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha+n-1}}=0,\quad\limsup_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)^{\beta}|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha+n-2}}=0.$$

Proof of Theorem 3.1. Define the operator $K_r:\mathcal{Z}_\alpha\to\mathcal{Z}_\alpha$ by $K_rf=f_r$, where $0<r<1$ and $f_r(z)=f(rz)$. Then, K_r is a compact operator. Let $\{r_j\}$ be a sequence in $(0,1)$ and $r_j\to1$ as $j\to+\infty$. So $C_{\varphi,g}^nK_{r_j}$ is compact $\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$. Let $f\in\mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha}\leq1$. Then,

$$\begin{aligned}
&\|(C_{\varphi,g}^n-C_{\varphi,g}^nK_{r_j})f\|_{\mathcal{Z}_\beta}\\
=&|g(0)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(0))\right|\\
&+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}\left|\varphi'(z)g(z)\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))+g'(z)\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\
\leq&|g(0)|\cdot\left|(f-f_{r_j})^{(n)}(\varphi(0))\right|\\
&+\sup_{|\varphi(z)|\leq r_N}\left(1-|z|^2\right)^{\beta}|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\
&+\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^{\beta}|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\
&+\sup_{|\varphi(z)|\leq r_N}\left(1-|z|^2\right)^{\beta}|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\
&+\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^{\beta}|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|,
\end{aligned}$$

where $N\in\mathbb{N}$ such that $r_j\geq2/3$ for $j\geq N$. Since $f_{r_j}\to f$ uniformly on compact subsets of $\mathbb{D}$, then

$$\limsup_{j\to+\infty}|g(0)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(0))\right|=0$$

and also

$$\begin{aligned}&\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left(1-|z|^2\right)^\beta|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\ \le& R_2\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|=0\end{aligned}$$

and

$$\begin{aligned}&\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left(1-|z|^2\right)^\beta|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\ \le& R_1\limsup_{j\to+\infty}\sup_{|\varphi(z)|\le r_N}\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|=0,\end{aligned}$$

where R_1 and R_2 come from equations (2.3) and (2.5). So, we have

$$\begin{aligned}&\limsup_{j\to+\infty}\|(C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j})f\|_{\mathcal{Z}_\beta}\\ \le&\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^\beta|\varphi'(z)|\cdot|g(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n+1)}(\varphi(z))\right|\\ &+\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\left(1-|z|^2\right)^\beta|g'(z)|\cdot\left|\left(f-f_{r_j}\right)^{(n)}(\varphi(z))\right|\\ \preceq& 2\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^\beta|\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}\\ &+2\limsup_{j\to+\infty}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^\beta|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}.\end{aligned}$$

Definition of essential norm implies that

$$\begin{aligned}\|C^n_{\varphi,g}\|_e\le&\limsup_{j\to+\infty}\|C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j}\|\\ =&\limsup_{j\to+\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\|(C^n_{\varphi,g}-C^n_{\varphi,g}K_{r_j})f\|_{\mathcal{Z}_\beta}\\ \preceq&\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|\varphi'(z)|\cdot|g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}+\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}},\end{aligned}$$

and the upper bound for the essential norm finds. In order to prove the lower bound, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to1$, $k\to+\infty$. We use the sequence of functions $f_k(z)=F_{\varphi(z_k)}(z)$ as in (2.6). Then, $\{f_k\}$ is bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, $\sup_k\|f_k\|_{\mathcal{Z}_\alpha}<+\infty$. So for any compact operator $K:\mathcal{Z}_\alpha\to\mathcal{Z}_\beta$ we have

$$\begin{aligned}\|C^n_{\varphi,g}-K\|\succeq&\limsup_{k\to+\infty}\|(C^n_{\varphi,g}-K)f_k\|_{\mathcal{Z}_\beta}\\ \ge&\limsup_{k\to+\infty}\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\beta}-\limsup_{k\to+\infty}\|Kf_k\|_{\mathcal{Z}_\beta}\\ =&\limsup_{k\to+\infty}\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\beta}\end{aligned}$$

$$\geq \limsup_{k\to+\infty} \left(1-|z_k|^2\right)^{\beta} |\varphi'(z_k)| \cdot |g(z_k)| \cdot \left|f_k^{(n+1)}(\varphi(z_k))\right|$$
$$\approx \limsup_{k\to+\infty} \frac{(1-|z_k|^2)^{\beta} |\varphi'(z_k)| \cdot |g(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}}.$$

Hence,

$$\|C^n_{\varphi,g}\|_e = \inf_K \|C^n_{\varphi,g} - K\| \succeq \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}.$$

If we use the sequence $g_k(z) = G_{\varphi(z_k)}(z)$ as in (2.9), then it can be obtained

$$\|C^n_{\varphi,g}\|_e \succeq \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-2}}.$$

Now the proof of theorem is completed. □

In the case $n = 1$, the following theorem can be applied.

Theorem 3.2. *Let $\alpha, \beta > 0$. If $C^g_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded, then the following hold.*

(i) *If $0 < \alpha < 1$, then*

$$\|C^g_\varphi\|_e \approx \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}.$$

(ii) *If $\alpha = 1$, then*

$$\|C^g_\varphi\|_e \approx \max \left\{ \begin{array}{l} \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^{\beta} |g'(z)| \log (1-|\varphi(z)|^2)^{-1}, \\ \limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)^{\beta} |\varphi'(z)|\cdot|g(z)|}{1-|\varphi(z)|^2} \end{array} \right\}.$$

(iii) *If $\alpha > 1$, then*

$$\|C^g_\varphi\|_e \approx \max \left\{ \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}, \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \right\}.$$

Proof. (i) The upper bound can be obtained from the proof of Theorem 3.1 using Lemma 2.1 and the lower bound can be obtained from the proof of Theorem 3.1.

(ii) The upper bound can be obtained from the proof of Theorem 3.1 using Lemma 2.1 and the lower bound can be obtained from the proof of Theorem 3.1 using the functions $f_k(z) = F_{\varphi(z_k)}(z)$ and

$$g_k(z) = \frac{\overline{\varphi(z_k)}z - 1}{\overline{\varphi(z_k)}} \left[\left(1 + \log \frac{1}{1-\overline{\varphi(z_k)}z}\right)^2 + 1 \right] \left(\log \frac{1}{1-|\varphi(z_k)|^2} \right)^{-1}.$$

(iii) The proof is similar to the proof of Theorem 3.1. □

As a result, we can find a characterization for compactness. The operator $C^g_\varphi : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is compact if and only if

(i) $0 < \alpha < 1$:

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0;$$

(ii) $\alpha = 1$:

$$\limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right)^{\beta} |g'(z)| \log\left(1-|\varphi(z)|^2\right)^{-1} = 0,$$

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |\varphi'(z)| \cdot |g(z)|}{1-|\varphi(z)|^2} = 0,$$

which is Theorem 6 of [7], $\beta = 1$;

(iii) $\alpha > 1$:

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |\varphi'(z)| \cdot |g(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0, \quad \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta} |g'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0.$$

Remark 3.1. Let $g(z) = \varphi'(z)$. Then, $\left(C_{\varphi}^{g} f\right)(z) = \int_0^z f'(\varphi(\xi))\varphi'(\xi)\, d\xi = C_{\varphi} f(z)$, the composition operator. So, the essential norm and also characterization for compactness of composition operator $C_{\varphi} : \mathcal{Z}_{\alpha} \to \mathcal{Z}_{\beta}$ can be obtained which is known in the literature.

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[1]Engineering Faculty of Khoy,
Urmia University of Technology,
Urmia, Iran
Email address: m.hassanlou@uut.ac.ir
ORCID iD: https://orcid.org/0000-0002-9213-2574

[2]Faculty of Mathematics, Statistics and Computer Science,
University of Tabriz,
Tabriz, Iran
Email address: faribaalighadr@gmail.com
ORCID iD: https://orcid.org/0000-0001-5682-9334

*Corresponding author