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GENERALIZATIONS OF SOME BERNSTEIN-TYPE INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. In this paper, we establish some new Bernstein-type bounds for the polar derivative of constrained polynomials on the unit circle in the plane. The obtained results sharpen some known estimates for the ordinary derivative of polynomials as special cases.

1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{v=0}^{n} c_v z^v$ of degree n. The extremal problems of functions of complex variables and the results where some approches to obtaining the classical inequalities are developed on using various methods of the geometric function theory are known for various norms and for many classes of functions such as polynomials with various constraints, and on various regions of the complex plane. A classical result due to Bernstein [2], that relates an estimate of the size of the derivative and the polynomial for the sup-norm on the unit circle states that: if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The above inequality (1.1) was proved by Bernstein in 1912. Later in 1985, Frappier, Rahman and Ruscheweyh [3] strengthened (1.1), by proving that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq n \max_{1\leq l\leq 2n}\left|P(e^{\frac{il\pi}{n}})\right|. \tag{1.2}$$

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Clearly (1.2) represents a refinement of (1.1), since the maximum of $|P(z)|$ on $|z|=1$ may be larger than the maximum of $|P(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z)=z^n+ia,\ a>0$. Following the approach of Frappier, Rahman and Ruscheweyh [3], Aziz [1] showed that the bound in (1.2) can be considerably improved. In fact, Aziz proved that if $P\in\mathbb{P}_n$, then

$$\max_{|z|=1}\Big|P'(z)\Big|\le\frac{n}{2}(M_\alpha+M_{\alpha+\pi}), \tag{1.3}$$

where

$$M_\alpha=\max_{1\le l\le n}\Big|P(e^{i(\alpha+2l\pi)/n})\Big|, \tag{1.4}$$

for all real α.

In the same paper, Aziz obtained a lower bound for the maximum of $|P'(z)|$ on $|z|=1$, by proving that if $P\in\mathbb{P}_n$, then

$$\max_{|z|=1}\Big|P'(z)\Big|\ge\frac{n}{2}\Big\{2\max_{|z|=1}\Big|P(z)\Big|-\Big(M_0+M_\pi\Big)\Big\}. \tag{1.5}$$

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then (1.1) can be replaced by

$$\max_{|z|=1}\Big|P'(z)\Big|\le\frac{n}{2}\max_{|z|=1}\Big|P(z)\Big|, \tag{1.6}$$

whereas, if $P(z)$ has no zeros in $|z|>1$, then

$$\max_{|z|=1}\Big|P'(z)\Big|\ge\frac{n}{2}\max_{|z|=1}\Big|P(z)\Big|. \tag{1.7}$$

Inequality (1.6) was conjectured by Erdős and later proved by Lax [6], whereas inequality (1.7) is due to Turán [18]. Ideally, it is natural to look for improving results in (1.3) when $P(z)$ does not vanish in the unit disk, and accordingly Aziz [1] proved that if $P\in\mathbb{P}_n$, and $P(z)\ne0$ in $|z|<1$, then for every real number α,

$$\max_{|z|=1}\Big|P'(z)\Big|\le\frac{n}{2}\Big\{M_\alpha^2+M_{\alpha+\pi}^2\Big\}^{\frac{1}{2}}, \tag{1.8}$$

where M_α is defined by (1.4).

It is important to mention that different versions of the Bernstein and Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. These inequalities have their own significance and importance in Approximation theory. One of such generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative. Before proceeding to our main results, let us remind that the polar derivative $D_\beta P(z)$ of $P(z)$ where $P\in\mathbb{P}_n$, with respect to the point β is defined as

$$D_\beta P(z):=nP(z)+(\beta-z)P'(z).$$

Note that $D_\beta P(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $P(z)$ with respect to β (see [7]). It generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta P(z)}{\beta}\right\} := P'(z),$$

uniformly with respect to z for $|z| \le R,\ R > 0$.

More information on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [9] and Rahman and Schmeisser [17].

Over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z = 0$, the modulus of largest root of $P(z)$, restrictions on coefficients, using higher order derivatives, etc. Many of these generalizations involve the comparison of polar derivative $D_\beta P(z)$ with various choices of $P(z)$, β and other parameters. The latest research and development on this topic can be found in the papers ([5, 8, 10, 11, 13–16, 20]).

The main purpose of this paper is to obtain some upper bound estimates for the maximal modulus of polar derivative of a polynomial on a disk under the assumption that the polynomial has no zeros either in the disk $|z| < k$ or in $|z| > k,\ k > 0$. The obtained results sharpen as well generalize some already known estimates for the ordinary derivative of polynomials as special cases.

2. Main Results

Theorem 2.1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \ge 1$, then for every complex number β with $|\beta| \ge 1$*

$$\begin{aligned}\max_{|z|=1}\big|D_\beta P(z)\big| \le & \frac{n}{2}\Bigg[2\max_{|z|=1}\big|P(z)\big| + \big(|\beta|-1\big)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2 \\ & - \frac{2}{(1+k)}\left((k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right)\big|P(z)\big|^2\Bigg\}^{\frac{1}{2}}\Bigg], \end{aligned} \tag{2.1}$$

where M_α is defined by (1.4).

The result is best possible for $k = 1$ and equality in (2.1) *holds for $P(z) = z^n + 1$, with real $\beta \ge 1$.*

By taking $\alpha = 0$ in (2.1), we get the following result.

Corollary 2.1. *Let $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$. If $t_1, t_2, \ldots, t_n$ are the zeros of $z^n + 1$ and $s_1, s_2, \ldots, s_n$ are the zeros of $z^n - 1$, then for $|\beta| \geq 1$*

$$\max_{|z|=1}\left|D_\beta P(z)\right| \leq \frac{n}{2}\left[2\max_{|z|=1}\left|P(z)\right| + \Big(|\beta| - 1\Big)\left\{\Big(\max_{1\leq j\leq n}|P(t_j)|\Big)^2 + \Big(\max_{1\leq j\leq n}|P(s_j)|\Big)^2\right.\right.$$
$$\left.\left. - \frac{2}{(1+k)}\left((k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right)\left|P(z)\right|^2\right\}^{\frac{1}{2}}\right]. \tag{2.2}$$

The result is best possible for $k = 1$ and equality in (2.2) holds for $P(z) = z^n + 1$, with real $\beta \geq 1$.

Dividing both sides of inequality (2.1) by $|\beta|$ and letting $|\beta| \to \infty$, we get the following result.

Corollary 2.2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then we have for every real α*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right]\left|P(z)\right|^2\right\}^{\frac{1}{2}},$$

where M_α is defined by (1.4).

It is easy to verify that Corollary 2.2 generalizes as well as sharpens inequality (1.8). Taking $k = 1$ in Corollary 2.2, we get the following result.

Corollary 2.3. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then we have for every real α,*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{n}\left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\left|P(z)\right|^2\right\}^{\frac{1}{2}},$$

where M_α is defined by (1.4).

The bound obtained in Corollary 2.3 is always sharper than the bound obtained from inequality (1.8), for this it needs to show that

$$\frac{|c_0| - |c_n|}{|c_0| + |c_n|} \geq 0,$$

which is equivalent to

$$|c_0| \geq |c_n|,$$

which is true as $P(z) \neq 0$ in $|z| < 1$.

If we divide both sides of inequality (2.2) by $|\beta|$ and let $|\beta| \to \infty$, we get the following result.

Corollary 2.4. *Let $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$. If $t_1, t_2, \ldots, t_n$ are the zeros of $z^n + 1$, and $s_1, s_2, \ldots, s_n$ are the zeros of $z^n - 1$, then*

$$\max_{|z|=1}\left|P'(z)\right| \leq \frac{n}{2}\Bigg\{\left(\max_{1\leq j\leq n}|P(t_j)|\right)^2 + \left(\max_{1\leq j\leq n}|P(s_j)|\right)^2$$

$$- \frac{2}{(1+k)}\left((k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right)\left|P(z)\right|^2\Bigg\}^{\frac{1}{2}}. \tag{2.3}$$

The result is best possible for $k = 1$ and equality in (2.3) holds for $P(z) = z^n + 1$.

Remark 2.1. It is easy to see that Corollary 2.4 generalizes the following result due to Wali and Shah [19, Corollary 1].

Theorem 2.2. *Let $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$. If $t_1, t_2, \ldots, t_n$ are the zeros of $z^n + 1$, and $s_1, s_2, \ldots, s_n$ are the zeros of $z^n - 1$, then for $|z| = 1$, we have*

$$\left|P'(z)\right| \leq \frac{n}{2}\left\{\left(\max_{1\leq j\leq n}\left|P(t_j)\right|\right)^2 + \left(\max_{1\leq j\leq n}\left|P(s_j)\right|\right)^2 - \frac{2}{n}\left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\left|P(z)\right|^2\right\}^{\frac{1}{2}}. \tag{2.4}$$

Equality in (2.4) holds for $P(z) = z^n + 1$.

If $P(z)$ has all its zeros on $|z| = k$, $k > 1$, then from Theorem 2.1, we get the following result.

Corollary 2.5. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros on $|z| = k$, $k > 1$, then for every complex number β, with $|\beta| \geq 1$*

$$\max_{|z|=1}\left|D_\beta P(z)\right| \leq \frac{n}{2}\left[2\max_{|z|=1}\left|P(z)\right| + \left(|\beta| - 1\right)\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\left(\frac{k-1}{k+1}\right)|P(z)|^2\right\}^{\frac{1}{2}}\right],$$

where M_α is defined by (1.4).

Next as an application of Theorem 2.1, we prove the following result.

Theorem 2.3. *Let $P(z) = \sum_{v=0}^{n} c_v z^v \in \mathbb{P}_n$, $c_0 \neq 0$, with $P(z) \neq 0$ in $|z| > k$, $k \leq 1$, then for every complex number γ with $|\gamma| \leq 1$, we have for $|z| = 1$*

$$\left|D_\gamma P(z)\right| \leq \frac{n}{2}\Bigg[2|\gamma|\max_{|z|=1}\left|P(z)\right| + \left(1 - |\gamma|\right)$$

$$\times\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right]\left|P(z)\right|^2\right\}^{\frac{1}{2}}\Bigg], \tag{2.5}$$

where M_α is defined by (1.4).

The result is best possible for $k = 1$ and equality in (2.5) holds for $P(z) = z^n + 1$, with real $\gamma \leq 1$.

Remark 2.2. If we take $\gamma = 0$ in (2.5), we get for $|z| = 1$

$$\begin{aligned}&\Big|nP(z) - zP'(z)\Big|\\ (2.6)\qquad &\leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right]\Big|P(z)\Big|^2\right\}^{\frac{1}{2}}.\end{aligned}$$

If $\max\limits_{|z|=1}\Big|P(z)\Big| = \Big|P(e^{i\phi})\Big|$, we get from (2.6) that

$$\begin{aligned}\Big|P'(e^{i\phi})\Big| \geq &\frac{n}{2}\Bigg[2\max_{|z|=1}\Big|P(z)\Big|\\ (2.7)\qquad &- \left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left((1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right)\Big|P(z)\Big|^2\right\}^{\frac{1}{2}}\Bigg].\end{aligned}$$

Since $\max\limits_{|z|=1}\Big|P'(z)\Big| \geq \Big|P'(e^{i\phi})\Big|$, we get from (2.7) that

$$\begin{aligned}\max_{|z|=1}\Big|P'(z)\Big| \geq &\frac{n}{2}\Bigg[2\max_{|z|=1}\Big|P(z)\Big|\\ (2.8)\qquad &- \left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left((1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right)\Big|P(z)\Big|^2\right\}^{\frac{1}{2}}\Bigg].\end{aligned}$$

Taking $k = 1$ in (2.8), we immediately get a refinement of (1.5) when all the zeros of $P(z)$ lie in $|z| \leq 1$.

Remark 2.3. It may be remarked here that for $k = 1$, Theorems 2.1 and 2.3 were recently established by Mir [11].

3. Lemmas

We need the following lemmas to prove our theorems.

Lemma 3.1. *If* $x_v,\ v = 1, 2, \ldots, n$ *is a sequence of real numbers such that* $x_v \geq 1$ *for all* $v \in \mathbb{N}$, *then*

$$\sum_{v=1}^{n}\frac{x_v - 1}{x_v + 1} \geq \frac{\prod_{v=1}^{n} x_v - 1}{\prod_{v=1}^{n} x_v + 1}, \quad \textit{for all } n \in \mathbb{N}.$$

Proof of Lemma 3.1. We prove this result with the help of mathematical induction and we use induction on n. The result is trivially true for $n = 1$.

For $n = 2$

$$\frac{x_1 - 1}{x_1 + 1} + \frac{x_2 - 1}{x_2 + 1} \geq \frac{x_1x_2 - 1}{x_1x_2 + 1},$$

if

$$\frac{2(x_1x_2-1)}{1+x_1+x_2+x_1x_2} \geq \frac{x_1x_2-1}{x_1x_2+1},$$

i.e., if $(x_1-1)(x_2-1) \geq 0$, which is true, since $x_1, x_2 \geq 1$. This shows that the result holds for $n=2$. Assume the result is true for $n = r \in \mathbb{N}$. Now since $\prod_{v=1}^{r} x_v \geq 1$, we have

$$\begin{aligned}\sum_{v=1}^{r+1} \frac{x_v-1}{x_v+1} =& \sum_{v=1}^{r} \frac{x_v-1}{x_v+1} + \frac{x_{r+1}-1}{x_{r+1}+1} \\ \geq& \frac{\prod_{v=1}^{r} x_v - 1}{\prod_{v=1}^{r} x_v + 1} + \frac{x_{r+1}-1}{x_{r+1}+1} \quad \text{(by induction hypothesis)} \\ \geq& \frac{\prod_{v=1}^{r+1} x_v - 1}{\prod_{v=1}^{r+1} x_v + 1} \quad \text{(by the case } n=2\text{)}.\end{aligned}$$

This shows that the result holds for $n = r+1$ as well. Therefore by the principle of mathematical induction, it follows that the result holds for all $n \in \mathbb{N}$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for each point z on $|z| = 1$ for which $P(z) \neq 0$, we have*

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{1}{1+k}\left\{n - \left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right\}. \tag{3.1}$$

Proof of Lemma 3.2. Recall that $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \geq k$, $k \geq 1$. If $z_1, z_2, \dots, z_n$ are the zeros of $P(z) = \sum_{v=0}^{n} c_v z^v$ of degree n, then $|z_v| \geq k$, $k \geq 1$, and we can write $P(z) = c_n \prod_{v=1}^{n}(z - z_v)$. This gives

$$\frac{zP'(z)}{P(z)} = \sum_{v=1}^{n} \frac{z}{z - z_v}.$$

Now for the points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with $P(e^{i\theta}) \neq 0$, we have

$$\begin{aligned}\operatorname{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) =& \sum_{v=1}^{n} \operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta} - z_v}\right) \\ \leq& \sum_{v=1}^{n} \frac{1}{1+|z_v|} \\ =& \frac{n}{1+k} - \frac{1}{1+k}\sum_{v=1}^{n} \frac{|z_v| - k}{|z_v| + 1} \\ \leq& \frac{n}{1+k} - \frac{1}{1+k}\sum_{v=1}^{n} \frac{|z_v| - k}{|z_v| + k} \quad \text{(as } k \geq 1\text{)} \\ =& \frac{n}{1+k} - \frac{1}{1+k}\sum_{v=1}^{n} \frac{|z_v|/k - 1}{|z_v|/k + 1}.\end{aligned}$$

Since $|z_v|/k \geq 1,\ v = 1, 2, \ldots, n$, we get on using Lemma 3.1 for the points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with $P(e^{i\theta}) \neq 0$,

$$\begin{aligned}\operatorname{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) \leq& \frac{n}{1+k} - \frac{1}{1+k}\left(\frac{\prod_{v=1}^{n}|z_v|/k - 1}{\prod_{v=1}^{n}|z_v|/k + 1}\right)\\ =& \frac{n}{1+k} - \frac{1}{1+k}\left(\frac{|c_0|/k^n|c_n| - 1}{|c_0|/k^n|c_n| + 1}\right),\end{aligned}$$

which is equivalent to (3.1). This completes the proof of Lemma 3.2. □

Lemma 3.3. *If $P \in \mathbb{P}_n$, then for $|z| = 1$, and for any real number α,*

$$\left|P'(z)\right|^2 + \left|Q'(z)\right|^2 \leq \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right),$$

where M_α is defined by (1.4).

The above lemma is due to Aziz [1].

Lemma 3.4. *If $P \in \mathbb{P}_n$, then for $|z| = 1$,*

$$\left|P'(z)\right| + \left|Q'(z)\right| \leq n \max_{|z|=1}|P(z)|.$$

The above lemma is a special case of a result due to Govil and Rahman [4].

4. Proof of the Theorems

Proof of Theorem 2.1. Recall that $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \geq k$, $k \geq 1$. First, we suppose that $P(z)$ has no zeros on $|z| = k,\ k > 1$ and therefore, all the zeros of $P(z)$ lie in $|z| > k$, we have by Lemma 3.2 for $|z| = 1$

$$2\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{2}{1+k}\left\{n - \left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right\}. \tag{4.1}$$

Also it easily follows that

$$\left|Q'(z)\right| = \left|nP(z) - zP'(z)\right|, \quad \text{for } |z| = 1, \tag{4.2}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$. This implies

$$\left|\frac{zQ'(z)}{P(z)}\right|^2 = \left|n - \frac{zP'(z)}{P(z)}\right|^2 = n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2n\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right),$$

which by using (4.1) yields for $|z| = 1$

$$2\left|P'(z)\right|^2 \leq \left|P'(z)\right|^2 + \left|Q'(z)\right|^2 + \left[\frac{2n^2}{1+k} - \frac{2n}{1+k}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right) - n^2\right]|P(z)|^2. \tag{4.3}$$

On combining Lemma 3.3 and inequality (4.3), we get for $|z|=1$

$$\left|P'(z)\right| \leq \frac{n}{2}\left\{M_\alpha^2+M_{\alpha+\pi}^2-\frac{2}{(1+k)}\left[(k-1)+\frac{2}{n}\left(\frac{|c_0|-k^n|c_n|}{|c_0|+k^n|c_n|}\right)\right]\left|P(z)\right|^2\right\}^{\frac{1}{2}}. \tag{4.4}$$

The above inequality (4.4) trivially holds for $k=1$ as well as for points z on $|z|=1$ for which $P(z)=0$ by (1.8). Using the definition of polar derivative of a polynomial $P\in\mathbb{P}_n$ with respect to the complex number β, we have

$$\begin{aligned}\left|D_\beta P(z)\right| =&|nP(z)+(\beta-z)P'(z)|\\ \leq&\left|nP(z)-zP'(z)\right|+|\beta|\left|P'(z)\right|\\ =&\left|Q'(z)\right|+|\beta|\left|P'(z)\right| \quad \text{(using (4.2))}\\ \leq& n\max_{|z|=1}|P(z)|+(|\beta|-1)\left|P'(z)\right| \quad \text{(by Lemma 3.4)}.\end{aligned} \tag{4.5}$$

Inequality (4.5) in conjunction with inequality (4.4) gives,

$$\begin{aligned}\max_{|z|=1}\left|D_\beta P(z)\right| \leq&\frac{n}{2}\Bigg[2\max_{|z|=1}\left|P(z)\right|+\Big(|\beta|-1\Big)\Bigg\{M_\alpha^2+M_{\alpha+\pi}^2\\ &-\frac{2}{(1+k)}\left((k-1)+\frac{2}{n}\left(\frac{|c_0|-k^n|c_n|}{|c_0|+k^n|c_n|}\right)\right)|P(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg].\end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. By hypothesis, the polynomial $P(z)=\sum_{v=0}^{n}c_vz^v$, $c_0\neq 0$ has all its zeros in $|z|\leq k$, $k\leq 1$, therefore, the polynomial $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$ has no zeros in $|z|<1/k$, $1/k\geq 1$. Applying Theorem 2.1 to the polynomial $Q(z)$, we get for $|\beta|\geq 1$ and $|z|=1$

$$\begin{aligned}\left|D_\beta Q(z)\right| \leq&\frac{n}{2}\Bigg[2\max_{|z|=1}|Q(z)|+\Big(|\beta|-1\Big)\Bigg\{Y_\alpha^2+Y_{\alpha+\pi}^2\\ &-\frac{2}{(1+1/k)}\left[(1/k-1)+\frac{2}{n}\left(\frac{|c_n|-1/k^n|c_0|}{|c_n|+1/k^n|c_0|}\right)\right]|Q(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg].\end{aligned} \tag{4.6}$$

Since $|P(z)|=|Q(z)|$ for $|z|=1$, it follows that

$$Y_\alpha=\max_{1\leq l\leq n}\left|Q(e^{i(\alpha+2l\pi)/n})\right|=\max_{1\leq l\leq n}\left|P(e^{i(\alpha+2l\pi)/n})\right|=M_\alpha.$$

Using this in (4.6), we get for $|\beta| \geq 1$ and $|z| = 1$

$$\begin{aligned}\Big|D_\beta Q(z)\Big| \leq &\frac{n}{2}\Bigg[2\max_{|z|=1}|P(z)| + \Big(|\beta|-1\Big)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2\\ &- \frac{2}{(1+k)}\bigg[(1-k) + \frac{2k}{n}\bigg(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\bigg)\bigg]|P(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg].\end{aligned} \tag{4.7}$$

For $|z| = 1$, we have $z\overline{z} = 1$, then it is easy to verify (for example, see [11]), that for $|\alpha| \neq 0$

$$\Big|D_\beta Q(z)\Big| = \Big|\bar{\beta}\Big|\Big|D_{1/\bar{\beta}}P(z)\Big|.$$

Replacing $1/\bar{\beta}$ by γ, so that $|\gamma| \leq 1$, we obtain from (4.7), that

$$\begin{aligned}\Big|D_\gamma P(z)\Big| \leq &\frac{n}{2}\Bigg[2|\gamma|\max_{|z|=1}|P(z)| + \Big(1-|\gamma|\Big)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2\\ &- \frac{2}{(1+k)}\bigg[(1-k) + \frac{2k}{n}\bigg(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\bigg)\bigg]|P(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg],\end{aligned}$$

for $|z| = 1$ and $|\gamma| \leq 1$.

This completes the proof of Theorem 2.3. □

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References

[1] A. Aziz, *A refinement of an inequality of S. Bernstein,* J. Math. Anal. Appl. **144** (1989), 226–235. https://doi.org/10.1016/0022-247X(89)90370-3

[2] S. Bernstein, *Sur l'ordre de la meilleure approximation des functions continues par des polynomes de degré donné*, Mem. Acad. R. Belg. **4** (1912), 1–103.

[3] C. Frappier, Q. I. Rahman and St. Ruscheweyh, *New inequalities for polynomials,* Trans. Amer. Math. Soc. **288** (1985), 69–99. https://doi.org/10.1090/S0002-9947-1985-0773048-1

[4] N. K. Govil and Q. I. Rahman, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc. **137** (1969), 501–517. https://doi.org/10.1090/S0002-9947-1969-0236385-6

[5] A. Hussain, A. Mir and A. Ahmad, *On Bernstein-type inequalities for polynomials involving the polar derivative,* J. Class. Anal. **16** (2020), 9–15. https://doi.org/10.7153/jca-2020-16-02

[6] P. D. Lax, *Proof of a conjecture of P. Erdős on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513. https://doi.org/10.1090/S0002-9904-1944-08177-9

[7] M. Marden, *The TeXbook,* Mathematical Surveys, American Mathematical Society, Providence, USA, 1966.

[8] G. V. Milovanović and A. Mir, *Generalizations of Zygmund-type integral inequalities for the polar derivative of a complex polynomial*, J. Inequal. Appl. **2020** (2020), 1–12. https://doi.org/10.1186/s13660-020-02404-x

[9] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *The TeXbook,* World Scientific Publishing Co. Singapore, 1994.
[10] A. Mir, *On an operator preserving inequalities between polynomials*, Ukrainian Math. J. **69** (2018), 1234–1247. https://doi.org/10.1007/s11253-017-1427-2
[11] A. Mir, *Some sharp upper bound estimates for the maximal modulus of polar derivative of a polynomial*, Annali Dell 'Universitá Di Ferrara **65** (2019), 327–336. https://doi.org/10.1007/s11565-019-00317-2
[12] A. Mir, *New integral norm estimates of Bernstein-type inequalities for a certain class of polynomials*, Indian J. Pure Appl. Math. **51** (2020), 1357–1371. https://doi.org/10.1007/s13226-020-0470-0
[13] A. Mir and B. Dar, *On the polar derivative of a polynomial*, J. Ramanujan Math. Soc. **29** (2014), 403–412.
[14] A. Mir and I. Hussain, *On the Erdős-Lax inequality concerning polynomials*, C. R. Math. Acad. Sci. Paris **355** (2017), 1055–1062. https://doi.org/10.1016/j.crma.2017.09.017
[15] A. Mir and A. H. Malik, *Inequalities concerning the rate of growth of polynomials involving the polar derivative,* Ann. Univ. Mariae Curie-Sk Lodowska Sect. A **74** (2020), 67–75. http://dx.doi.org/10.17951/a.2020.74.1.67-75
[16] A. Mir and A. Wani, *A note on two recent results about polynomials with restricted zeros*, J. Math. Inequal. **14** (2020), 45–50. https://doi.org/10.7153/jmi-2020-14-04
[17] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press, New York, 2002.
[18] P. Turán, *Über die Ableitung von polynomen,* Compos. Math. **7** (1939), 89–95. http://eudml.org/doc/88754
[19] S. L. Wali and W. M. Shah, *Bernstien type inequalities for polynomials with restricted zeros,* J. Anal. **29** (2021), 1083–1091. https://doi.org/10.1007/s41478-020-00296-0
[20] A. Zireh and M. Bidkham, *Inequalities for the polar derivative of a polynomial with restricted zeros*, Kragjevac J. Math. **40** (2016), 113–124. https://doi.org/10.5937/KgJMath1601113Z

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