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FAULT-TOLERANT METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF CAYLEY GRAPHS

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Abstract. Metric dimension and fault-tolerant metric dimension of any graph G is subject to size of resolving set. It has become more important in modern GPS and sensors based world as resolving set ensures that in case of semi outage system is still scalable using redundant interfaces. Metric dimension of several interesting classes of graphs have been investigated like Cayley digraphs, Cartesian product of graphs, wheel graphs, convex polytopes and certain networks for categorical product of graphs. In this paper we used the phenomena of barycentric subdivision of graph and proved that fault-tolerant metric dimension of barycentric subdivision of Cayley graph is constant.

1. Introduction

Concept of metric dimension in graph theory was first introduced by Slater [18], Harary and Melter [10] in mid 70's. In a connected graph G, the *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. *Metric dimension* of any graph G can be defined as $S \subseteq V(G)$ with minimum cardinality where all other vertices of G are uniquely determined by their distances to the vertices in S. A vertex x resolves two vertices u and v if $d(x,u) \neq d(x,v)$, hence minimum cardinality of a resolving set of G is called the metric dimension and is denoted by $\beta(G)$. Similarly a resolving set R is said to be fault-tolerant, if $R \backslash \{x\}$ is also a resolving set for every $x \in R$ that is why *fault-tolerant metric dimension* is the minimum cardinality of a fault-tolerant resolving set of G. The fault-tolerant metric

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dimension of graph G is denoted by $\beta'(G)$. A fault-tolerant resolving set of order $\beta'(G)$ is also called a fault-tolerant metric basis of G.

Lot of work has been done in the area of metric dimension and has used in different domains of scientific research. Work of Slater on fault-tolerant metric dimension of graphs carried out in different dimensions like resolvability of crystal structures, network analysis, chemical structures of Methylene, mathematical formalization of woven structures and most significant in Fast-Cluster for removing redundant sequences. Concept of metric dimension using radio navigation by considering the vertices as sonar or loran station ruled in last three decades but now its obsolete and is replaced by GPS and sensor and ad-hoc networks. Through fault-tolerant resolving, a system can continue operating somehow even in case of any failure in one or more of its components. In case of semi outage that leads to graceful degradation of service, system tries to act as scalable system by discovering redundant network interfaces. Fault-tolerant metric dimension can support physical connectivity and link discovery in distributed network based systems.

Metric dimension of several interesting classes of graphs have been investigated: Johnson and Kneser graph [2], Grassmann graphs [3], Cayley digraphs [7] and Cartesian product of graphs [5]. Siddiqui et al. [17] investigated the metric dimension of some infinite families of wheel-related graphs. Kratica et al. [16] studied the metric dimension problem of convex polytopes. Imran et al. [13] studied further the metric dimension of convex polytopes generated by wheel-related graphs. Ahmad et al. [1] studied the metric dimension of Cayley graph of certain finite groups. Vetrik et al. [19] studied the metric dimension problem for certain networks which can be obtained as the categorical product of graphs. In [4], it has been shown that metric dimension of a graphs is not necessarily a finite natural number. They proved that some infinite graphs have infinite metric dimension. The computational complexity of these problems is studied in [8]. Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor memory pairs, each of which is called a processing vertex. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multiprocessor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips.

By inserting a new vertex at any edge to split it into two equi-halves this phenomena is known as edge subdivision. If edge subdivision is applied on multiple edges then it is called graph subdivision, whereas if all edges are subdivided then it is called barycentric subdivision of graph. Gross and Yellen [9] explained nice properties that barycentric subdivided graph will be bipartite, loopless and any loopless graph will be simple as well. Gary and Johnson [8] put an argument that problem of determining $\beta(G) < k$ is NP-Complete problem. In this paper we determined that fault-tolerant metric dimension of barycentric subdivision of Cayley graph is constant and four vertices are sufficient to resolve all the vertices of graph.

2. Results

Let P_n be a path of n vertices, Chartrand et al. [6] determined the metric dimension in the following theorem.

Theorem 2.1 ([6]). *A connected graph G has metric dimension 1 if and only if $G \cong P_n$.*

By considering the two endpoints of the path, the fault-tolerant metric basis obtained. It is easy to observe that $\beta(P_n) = 1$ and $\beta'(P_n) = 2$, for path P_n, $n \geq 2$. From this result and the definition of the fault-tolerant metric dimension the following inequality holds

$$\beta'(G) \geq \beta(G) + 1.$$

Javaid et al. [14] proved in the following theorem that the difference between metric dimension and fault-tolerant of a graph can be arbitrary large.

Theorem 2.2 ([14]). *For every positive integer n, there exists a graph such that $\beta'(G) - \beta(G) \geq n$.*

Let SG be a semigroup, and let H be a nonempty subset of SG. The Cayley graph $Cay(SG, H)$ of SG relative to H is defined as the graph with vertex set SG and edge set $E(SG)$ consisting of those ordered pairs (x, y) such that $hx = y$ for some $h \in H$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. The Cayley graph $Cay(SG, H)$ of a group SG is symmetric or undirected if and only if $H = H^{-1}$.

The Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$, $n \geq 3$, $m \geq 2$, is a graph which can be obtained as the Cartesian product $P_m \Box C_n$ of a path on m vertices with a cycle on n vertices. The vertex set and the edge set of $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$ are defined as: $V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)) = \{(a_s, b_t) : 1 \leq s \leq n, 1 \leq t \leq m\}$ and $E(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)) = \{(a_s, b_t)(a_{s+1}, b_t) : 1 \leq s \leq n, 1 \leq t \leq m\} \cup \{(a_s, b_t)(a_s, b_{t+1}) : 1 \leq s \leq n, 1 \leq t \leq m-1\}$. We have $|V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))| = mn$, $|E(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))| = (2m-1)n$, where $|V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))|$, $|E(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))|$ denote the number of vertices, edges of the Cayley graphs $Cay(Z_n \oplus Z_m)$, respectively.

The metric dimension of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ has been determined in [5] while the metric dimension of Cayley graphs $Cay(\mathbb{Z}_n : H)$ for all $n \geq 7$ and $H = \{\pm 1, \pm 3\}$ has been determined in [15].

The barycentric subdivision graph $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ can be obtained by adding a new vertex (c_s, d_t) between (a_s, b_t) and (a_{s+1}, b_t) and adding a new vertex (u_s, v_t) between (a_s, b_t) and (a_s, b_{t+1}). Clearly, $S(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ has $3nm - n$ vertices and $4nm - 2n$ edges.

The metric dimension of $P_m \Box C_n$ has been determined in [5] and Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ is actually the Cartesian product of $P_2 \Box C_n$. In the next theorem, we prove that the fault-tolerant metric dimension of barycentric subdivision $BS(Cay(Z_n \oplus Z_m))$ is constant and only four vertices appropriately chosen suffice to resolve all the vertices of the $BS(Cay(Z_n \oplus Z_m))$ for $n \geq 6$ and $m \geq 2$.

Theorem 2.3. *Let $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ be the barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$. Then the fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is 4 for $n \geq 6$ and $m \geq 2$.*

Proof. Theorem will be proved for equality using double inequality.

Case 1: $n \equiv 0 \pmod 2$. Let

$$R = \{(a_1, b_1), (a_2, b_1), (a_{\frac{n}{2}+1}, b_1), (a_n, b_1)\} \subseteq V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)))$$

that shows R is a fault-tolerant resolving set for this case. With respect to R a representation for the vertices of BS(Cay($\mathbb{Z}_n \oplus \mathbb{Z}_m$)) is as follows.

For $1 \leq t \leq m$,

$$\gamma((a_s, b_t)|R) = \begin{cases} (2t-2, 2t, 2t+n-2, 2t), & \text{for } s = 1, \\ (2t, 2t-2, 2t+n-4, 2t+2), & \text{for } s = 2, \\ (2t+2s-2, 2t+2s-6, 2t-2s+n, 2t+2s-2), & \text{for } 3 \leq s \leq \frac{n}{2}, \\ (2t+n-2, 2t+n-4, 2t-2, 2t+n-4), & \text{for } s = \frac{n}{2}+1, \\ (2t-2s+2n, 2t-2s+2n+2, 2t+2s-n-4, & \\ 2t-2s+2n-2), & \text{for } \frac{n}{2}+2 \leq s \leq n. \end{cases}$$

For $1 \leq t \leq m$,

$$\gamma((c_s, d_t)|R) = \begin{cases} (2t-1, 2t-1, 2t+n-3, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n-1, 2t+2s-1), & \text{for } 2 \leq s \leq \frac{n}{2}-1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n-1, 2t+2s-3), & \text{for } s = \frac{n}{2}, \\ (2t+n-3, 2t+n-3, 2t-1, 2t+n-5), & \text{for } s = \frac{n}{2}+1, \\ 2t-2s+2n-1, 2t-2s+2n+1, 2t+2s-n-3, & \\ 2t-2s+2n-3), & \text{for } \frac{n}{2}+2 \leq s \leq n. \end{cases}$$

For $1 \leq t \leq m-1$,

$$\gamma((u_s, v_t)|R) = \begin{cases} (2t-1, 2t+1, 2t+n-1, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n+1, 2t+2s-1), & \text{for } 2 \leq s \leq \frac{n}{2}, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n+1, 2t+2s-5), & \text{for } s = \frac{n}{2}+1, \\ (2t-2s+2n+1, 2t-2s+2n+3, 2t+2s-n-3, & \\ 2t-2s+2n-1), & \text{for } \frac{n}{2}+2 \leq s \leq n. \end{cases}$$

These vertex representation are distinct, so R is the fault-tolerant resolving set of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$. Therefore fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is less than equal to 4 that means $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) \leq 4$.

Imran [12] showed that metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$ is 3, for $m = 2$ and Ahmad et al. [1] proved that metric dimension of

$BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)) = 3$ for $m \geq 3$, therefore the fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ must be greater than 3 that means $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) \geq 4$. Hence proved that fault-tolerant metric dimension is $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) = 4$ for $n \geq 6$ and $m \geq 2$.

Case 2: $n \equiv 1 \pmod 2$. Let

$$R = \{(a_1, b_1), (a_2, b_1), (a_{\lceil \frac{n}{2} \rceil}, b_1), (a_n, b_1)\} \subseteq V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)))$$

that shows R is a fault-tolerant resolving set for this case. With respect to R a representation for the vertices of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is as follows.

For $1 \leq t \leq m$,

$$\gamma((a_s, b_t)|R) = \begin{cases} (2t-2, 2t, 2t+n-3, 2t), & \text{for } s = 1, \\ (2t, 2t-2, 2t+n-5, 2t+2), & \text{for } s = 2, \\ (2t+2s-4, 2t+2s-6, 2t-2s+n-1, \\ 2t+2s-2), & \text{for } 3 \leq s \leq \lceil \frac{n}{2} \rceil - 1, \\ (2t+n-3, 2t+n-5, 2t-2, 2t+n-3), & \text{for } s = \lceil \frac{n}{2} \rceil, \\ (2t+n-3, 2t+n-3, 2t, 2t+n-5), & \text{for } s = \lceil \frac{n}{2} \rceil + 1, \\ (2t-2s+2n, 2t-2s+2n+2, \\ 2t+2s-n-3, 2t-2s+2n-2), & \text{for } \lceil \frac{n}{2} \rceil + 2 \leq s \leq n. \end{cases}$$

For $1 \leq t \leq m$,

$$\gamma((c_s, d_t)|R) = \begin{cases} (2t-1, 2t-1, 2t+n-4, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n-2, \\ 2t+2s-1), & \text{for } 2 \leq s \leq \lceil \frac{n}{2} \rceil - 1, \\ (2t+n-2, 2t+n-4, 2t-1, 2t+n-4), & \text{for } s = \lceil \frac{n}{2} \rceil, \\ (2t-2s+2n-1, 2t-2s+2n+1, \\ 2t+2s-n-2, 2t-2s+2n-3), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq s \leq n-1, \\ (2t-1, 2t+1, 2t+n-2, 2t-1), & \text{for } s = n. \end{cases}$$

For $1 \leq t \leq m-1$,

$$\gamma((u_s, v_t)|R) = \begin{cases} (2t-1, 2t+1, 2t+n-2, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n, 2t+2s-1), & \text{for } 2 \leq s \leq \lceil \frac{n}{2} \rceil - 1, \\ (2t+n-2, 2t+n-4, 2t-1, 2t+n-2), & \text{for } s = \lceil \frac{n}{2} \rceil, \\ (2t+n-2, 2t+n-2, 2t+1, 2t+n-4), & \text{for } s = \lceil \frac{n}{2} \rceil + 1, \\ (2t-2s+2n+1, 2t-2s+2n+3, \\ 2t+2s-n-2, 2t-2s+2n-1), & \text{for } \lceil \frac{n}{2} \rceil + 2 \leq s \leq n. \end{cases}$$

These vertex representations are distinct, so R is the fault-tolerant resolving set of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$. Therefore, fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is less than or equal to 4 that means $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) \leq 4$.

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