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ON UNIQUE MINIMAL SOBOLEV NORM ELEMENT OF BANACH SPACES OF FUNCTIONS WHICH TAKES A GIVEN VALUE IN A FIXED POINT

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ABSTRACT. First, it will be shown that some Banach spaces V of functions, which are subspaces subspaces of Sobolev spaces satisfy the c-minimal norm property, i.e., in any set

$$V_{z,c} := \{f \in V \mid f(z) = c\},$$

if non-empty, there is exactly one element with t minimal Sobolev norm. Later, it will be proved that this element depends continuously on the deformation of the norm and on an increasing sequence of domains in a precisely defined sense. We conclude with applications to the theory of linear partial differential equations.

1. INTRODUCTION

Classical theory states that the solution of a partial differential equation is unique in many cases if the boundary or initial conditions are given. Here we introduce a different idea - if we want a solution of a homogeneous linear partial differential equation to be unique, we can consider the solution with the minimal Sobolev norm among the solutions which take given value at a fixed point.

First, we prove a theorem showing sufficient conditions for the Banach space of functions to have the minimal norm property, i.e., the property of the existence of a unique element with minimal norm in the subset of elements taking a certain value at a fixed point. Then we show that some Banach spaces of functions with Sobolev norm satisfy the assumptions of this theorem. Next, we prove our main results - theorems

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stating that the unique minimal norm element of a Banach space of functions with Sobolev norm continuously depends on an integration weight, i.e., on the deformation of the norm and on an increasing sequence of domains.

2. Preliminaries

It is known that a Hilbert space V of functions defined on a domain D is a reproducing kernel Hilbert space if and only if it satisfies the minimal norm property, e.g., in any set

$$V_z := \{f \in V \mid f(z) = 1\},$$

if it is not empty, there is exactly one element with the minimal norm. In fact, the property of the minimal norm is equivalent to a formally stronger property: c-minimal norm property, i.e., in every set

$$V_{z,c} := \{f \in V \mid f(z) = c\},$$

if non-empty, there is exactly one element with minimal norm. Indeed, if V satisfies the property of c-minimal norm, then it also satisfies the property of minimal norm. On the other hand, if φ_z is unique element of V_z with the minimal norm, then $c\varphi_z$ is the unique element of $V_{z,c}$ with the minimal norm.

For more details and proofs see e.g. [6]. For examples of Hilbert spaces that do not satisfy the minimal norm property, i.e., are not reproducing kernel Hilbert spaces, see e.g. [6] or [3].

The aim of this paper is to show that some subspaces of functions of Sobolev spaces, which are not necessarily Hilbert spaces, also satisfy the minimal norm property. Moreover, we will show that the unique minimal norm element in the set V_z depends continuously on a weight function and on the integration domain. These results can be somehow treated as a continuation of the research introduced in [6] and [4].

Throughout the paper, the unique minimal norm element of V_z will be denoted by φ_z without further recalling.

3. Proof that some Subspaces of Functions of Sobolev Spaces Satisfy the Minimal Norm Property

Let V be a Banach space of functions which is a subspace of a Sobolev space $W^{k,p}(U)$, $U \subset \mathbb{R}^n$, U an open bounded domain with boundary of class C^1, $1 < p < +\infty$, equipped with the same norm, i.e.,

$$||f|| := ||f||_{W^{k,p}(U)} = \left(\sum_{|i| \le k} ||D^i f||^p_{L^p(U)} \right)^{\frac{1}{p}},$$

where $i = (i_1, i_2, \ldots, i_m)$, $|i| = i_1 + i_2 + \cdots + i_m$ and

$$D^i f = \frac{\partial^{|i|} f}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}.$$

It is well-known that each space L^p, $1 < p < +\infty$, is strictly convex, therefore also V is strictly convex as a closed subspace of L^p. Moreover, the following theorem holds.

Theorem 3.1 ([1], Theorem 6 in Section 5.6.3.). *If $f \in W^{k,p}(U)$, with additional assumptions above and $k > \frac{n}{p}$, then f is also an element of Hölder's space $C^{k-\left[\frac{n}{p}\right]-1,\gamma}(U)$, where γ is equal to $\left[\frac{n}{p}\right] + 1 - \frac{n}{p}$, if $\frac{n}{p} \notin \mathbb{Z}$ and is any positive number lower than 1, otherwise.*

Moreover, the following holds.

$$||f||_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})} \leq C||f||_{W^{k,p}(U)}. \tag{3.1}$$

A direct consequence of this theorem is the fact that the functionals of the point evaluation in V are continuous if only its assumptions are fulfilled, i.e. for every $z \in U$ there exists constant C_z, such that

$$|f(z)| \leq C_z||f||_{W^{k,p}(U)},$$

for any $f \in V$.

Now we are ready to prove the main theorems of this section.

Theorem 3.2. *Let V be a Banach space of functions defined on U satisfying the following conditions:*

(i) *functionals of point evaluation are continuous;*

(ii) *V is strictly convex.*

Then, for any $z \in U$, if there exist $f_z \in V$ such that $f_z(z) \neq 0$, we have

$$\inf_{\{f \in V | f(z) = 1\}} ||f|| > 0. \tag{3.2}$$

Moreover, if the infimum is reached, then it is only reached for one function, i.e, then V satisfies the minimal norm property.

Proof. Let $z \in U$ be a point for which there exist functions in V that take a nonzero value at z, and let $f \in V$ be such a function. Then, by continuity of functionals of the point evaluation we have

$$0 < \frac{1}{C_z} \leq \left|\left|\frac{f}{f(z)}\right|\right|.$$

Therefore, (3.2) holds.

Now, let us assume that the infimum in (3.2) is reached for some functions $f, g \in V_z$. We will show that $f = g$. First, let us define

$$h = \frac{1}{2}(f + g) \in V_z,$$

i.e., $h(z) = 1$. Next, by the triangle inequality and the fact that $||f|| = ||g||$, we obtain

$$||h|| = \left|\left|\frac{1}{2}(f + g)\right|\right| \leq \frac{1}{2}(||f|| + ||g||) = ||f||.$$

On the other hand, by our assumptions,

$$||h|| \geq ||f||,$$

therefore

$$||h|| = ||f||. \tag{3.3}$$

Since V is strictly convex and the triangle inequality in the above case is in fact equality, we conclude that there exists $\alpha \in \mathbb{C}$, such that $f = \alpha g$. Then,

$$||h|| = \left|\left|\frac{1}{2}(f+g)\right|\right| = \frac{1}{2}(\alpha+1)||f||.$$

By (3.3) we get that $\alpha = 1$ and in conclusion $f = g$. □

Corollary 3.1. *Let V be a Banach space of functions equipped with the Sobolev norm, which satisfies assumptions of Theorem* 3.1. *Then, V satisfies the minimal norm property.*

Proof. As it was stated before, V is strictly convex and functionals of point evaluation for that space are continuous. Therefore,

$$\inf_{f \in V_z} ||f||_{W^{k,p}(U)} > 0,$$

and if it is reached, then it is reached only for one function. It is sufficient to show that the infimum above is in fact minimum. Let $\{f_n\} \subset V_z$ be a sequence which approximates this infimum, i.e., such that

$$\lim_{n \to +\infty} ||f_n||_{W^{k,p}(U)} = \inf_{f \in V_z} ||f||_{W^{k,p}(U)}.$$

It is obvious that $\{f_n\}$ is bounded in the $|| \cdot ||_{W^{k,p}(U)}$ norm. By Theorem 3.1, it is also bounded in a Hölder space. Therefore, the sequence $\{f_n\}$ satisfies the assumptions of the Arzelà-Ascoli theorem and there is a convergent subsequence of it. Let us denote its limit by f_0. By Fatou's lemma $||f_0||_{W^{k,p}(U)} \leq \inf_{f \in V_z} ||f||_{W^{k,p}(U)}$, so f_0 is the unique element of $W^{k,p}(U)$, such that

$$||f_0||_{W^{k,p}(U)} = \min_{f \in V_z} ||f||_{W^{k,p}(U)}.$$

□

Lemma 3.1. *Let V be a Banach space of functions defined on D which satisfies the minimal norm property. Then, the functionals of the point evaluation for that space are continuous. Moreover, for any $z \in D$ and any functional $E_z : V \ni f \mapsto f(z) \in \mathbb{C}$,*

$$||E_z||^* = \frac{1}{||\varphi_z||}.$$

Proof. By the minimal norm property for any $z \in D$ there exists $C_z > 0$ such that for any $f \in V$ for which $f(z) \neq 0$ we have

$$C_z \leq \left|\left|\frac{f}{f(z)}\right|\right|.$$

Therefore,

$$|f(z)| \leq C_z||f||.$$

If $f(z) = 0$, then the above inequality is trivially satisfied. Moreover, for any g such that $g(z) = 1$, we have

$$|g(z)| \leq \frac{1}{||\varphi_z||}||g||.$$

For any function $f \in V$, such that $f(z) = c$, $c \neq 0, 1$ there exists function $g \in V$, such that $g(z) = 1$ and $f = cg$. So, we have

$$|f(z)| = |cg(z)| \leq \frac{1}{||\varphi_z||}||cg|| = \frac{1}{||\varphi_z||}||f||.$$

Therefore, $||E_z||^* \leq \frac{1}{||\varphi_z||}$. Since the above inequality for $f = \varphi_z$ is in fact equality, we conclude that $||E_z||^* = \frac{1}{||\varphi_z||}$. □

4. Dependence of Minimal Norm Element on a Weight of Integration

Let $\mu_1, \mu_2, \ldots, \mu_r$ be measurable positive functions defined on a domain U such that there exist positive constants c_1, c_2 for which

$$c_1 < \mu_i(w) < c_2, \tag{4.1}$$

for any $i = 1, 2, \ldots, r$. $\mu := (\mu_1, \mu_2, \ldots, \mu_r)$ will be called a weight. We will say that a sequence of weights $\{\mu^N\}$ converges almost everywhere to the weight μ if the coefficients of μ^N converge almost everywhere to the corresponding coefficients of μ.

Let

$$||f||_{L^p_{\mu_i}(U)} = \left(\int_U |f(w)|^p \mu_i(w) \mathrm{d}w\right)^{\frac{1}{p}}.$$

Now let us define weighted Sobolev norm $W^{k,p}_\mu$

$$||f|| := ||f||_{W^{k,p}_\mu(U)} = \left(\sum_{|i|\leq k} ||D^i f||^p_{L^p_{\mu_i}(U)}\right)^{\frac{1}{p}}.$$

It is clear that in $W^{k,p}_\mu$ functionals of point evaluation are continuous. Indeed, using Theorem 3.1, we get that

$$||u||_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})} \leq C||u||_{W^{k,p}(U)} \leq C\frac{1}{c_1^{\frac{1}{p}}}||u||_{W^{k,p}_\mu(U)}.$$

A direct consequence of this inequality is the fact that any sequence that is bounded in the weighted Sobolev norm has a convergent subsequence, by Arzelá-Ascoli Theorem. Therefore, by Theorem 3.2, Banach space of functions with weighted Sobolev norm with the additional assumptions of Theorem 3.1 satisfies the minimal norm property.

Theorem 4.1. *Let μ^N be a sequence of weights that converges to the weight μ almost everywhere. Let V_0 be a Banach space of functions with weighted Sobolev norm $||\cdot||_{W^{k,p}_{\mu}(U)}$ satisfying the assumptions of Theorem 3.1. For any $N \in \mathbb{N}$, let V^N be the same as a vector space as V_0, but equipped with the weighted Sobolev norm $||\cdot||_{W^{k,p}_{\mu^N}(U)}$ and analogously let V be the same as a vector space as V_0 but equipped with the weighted Sobolev norm $||\cdot||_{W^{k,p}_{\mu}(U)}$. Let $\varphi^{\mu^N}_{z,c}$ denote the minimal norm element in the set*

$$\{f \in V^N \mid f(z) = c\},$$

and similarly $\varphi^{\mu}_{z,c}$ denote the minimal norm element in the set $\{f \in V \mid f(z) = c\}$. Then,

$$\lim_{N\to+\infty} \varphi^{\mu^N}_{z,c} = \varphi^{\mu}_{z,c}$$

and the limit above is uniform on U.

By (4.1) all spaces V^N, V and V^0 are the same as topological vector spaces. In particular they are all complete, i.e., they are Banach spaces.

Note that by the minimal norm property it is sufficient to show this theorem for $c = 1$.

Lemma 4.1. *If $\mu^1 := (\mu^1_1, \mu^1_2, \dots, \mu^1_r)$ and $\mu^2 := (\mu^2_1, \mu^2_2, \dots, \mu^2_r)$ are weights such that $\mu^1_i \le \mu^2_i$ for any $1 \le i \le r$, then*

$$||\varphi^{\mu^1}_{z,c}||_{W^{k,p}_{\mu^1}(U)} \le ||\varphi^{\mu^2}_{z,c}||_{W^{k,p}_{\mu^2}(U)}.$$

Proof. By its very own definition

$$||\varphi^{\mu^1}_{z,c}||_{W^{k,p}_{\mu^1}(U)} \le ||\varphi^{\mu^2}_{z,c}||_{W^{k,p}_{\mu^1}(U)} \le ||\varphi^{\mu^2}_{z,c}||_{W^{k,p}_{\mu^2}(U)}.$$

□

Now we are going to give the proof of the main theorem.

Proof. First, let us show that the limit of $\varphi^{\mu^N}_z$ exists. Let us take

$$\mu^0 := (2c_2, 2c_2, \dots, 2c_2)$$

(see (4.1)). By (4.1) and Lemma 4.1,

$$||\varphi^{\mu^N}_z||_{W^{k,p}(U)} \le \frac{1}{c_1^{\frac{1}{p}}} ||\varphi^{\mu^N}_z||_{W^{k,p}_{\mu^N}(U)} \le ||\varphi^{\mu^0}_z||_{W^{k,p}_{\mu^0}(U)}.$$

So the sequence $||\varphi^{\mu^N}_z||_{W^{k,p}(U)}$ is bounded. Therefore, by (3.1) sequence $\varphi^{\mu^N}_{z,c}$ satisfies the assumptions of the Arzelá-Ascoli theorem and we conclude that there exists a uniformly convergent subsequence of it. Without loss of generality, we can identify it with the entire sequence. Let us denote its limit by g. By the property of the minimal norm, it is sufficient to show that

$$||g||_{W^{k,p}_{\mu}(U)} \le ||\varphi^{\mu}_z||_{W^{k,p}_{\mu}(U)}. \tag{4.2}$$

By Fatou's lemma

$$||g||_{W^{k,p}_{\mu}(U)} \leq \liminf_{N\to+\infty} ||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}. \tag{4.3}$$

Note that since μ^N for any N and μ are bounded from below and above by non-zero constants, all spaces $W^{k,p}_{\mu^N}(U)$ are equal as topological vector spaces (and in particular as sets). Therefore, for any f that is an element of any of these spaces, by Lemma 3.1, we have

$$|f(z)| \leq \frac{1}{||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}} ||f||_{W^{k,p}_{\mu^N}(U)}.$$

Taking the limit on the right hand side we get that

$$|f(z)| \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}} ||f||_{W^{k,p}_{\mu}(U)}.$$

We can use Lebesgue dominated convergence theorem to show that $||f||_{W^{k,p}_{\mu^N}(U)} \to ||f||_{W^{k,p}_{\mu}(U)}$. Moreover, by Lemma 3.1, again

$$\frac{1}{||\varphi_z^{\mu}||_{W^{k,p}_{\mu}(U)}} \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}}. \tag{4.4}$$

Combining (4.3) with (4.4), we obtain (4.2). □

5. Dependence of Minimal Norm Element on Increasing Sequence of Domains

Theorem 5.1. *Let $\{U_N\}$ be an increasing sequence of domains with boundaries of class C^1 and $U = \bigcup_{N=1}^{+\infty} U_N$. Let V_1 be the Banach space of functions defined on U_1 equipped with the Sobolev norm $||\cdot||_{W^{k,p}(U_1)}$ satisfying the assumptions of Theorem 3.1. Let V^N for $N > 1$ be the Banach space of functions equipped with the Sobolev norm $||\cdot||_{W^{k,p}(U_N)}$, such that for any $f \in V^N$ we have $f_{|U_{N-k}} \in V_{N_k}$, for $k = 1, 2, \dots, N-1$. Similarly, let V be the Banach space of functions equipped with the Sobolev norm $||\cdot||_{W^{k,p}(U)}$, such that for any $f \in V$ we have $f_{|U_N} \in V_N$ for any N. Let $\varphi_{z,c}^{U_N}$ denote the unique minimal norm element of the set*

$$\{f \in V^N \mid f(z) = c\}$$

and similarly $\varphi_{z,c}^{U}$ denote the unique minimal norm element of the set

$$\{f \in V \mid f(z) = c\}.$$

Then,

$$\lim_{N\to+\infty} \varphi_{z,c}^{U^N} = \varphi_{z,c}^{U}$$

and the limit above is locally uniform on U.

As in the previous section, by the minimal norm property, it is sufficient to show the above theorem only for $c = 1$. We can also define $\varphi_{z,c}^{U^N}(w) \equiv 0$ for $w \in U \setminus U^N$.

Lemma 5.1. *If U_1 and U_2 are domains such that $U_1 \subseteq U_2$, then $||\varphi_{z,c}^{U_1}||_{W^{k,p}(U_1)} \leq ||\varphi_{z,c}^{U_2}||_{W^{k,p}(U_2)}$.*

Proof. By its very own definition $||\varphi_{z,c}^{U_1}||_{W^{k,p}(U_1)} \leq ||\varphi_{z,c}^{U_2}||_{W^{k,p}(U_1)} \leq ||\varphi_{z,c}^{U_2}||_{W^{k,p}(U_2)}$, it is clear that $\varphi_{z,c}^{U_2} \in W^{k,p}(U_1)$. □

We will now prove the main theorem.

Proof of Theorem 5.1. First, let us show that the locally uniform limit of $\varphi_z^{U^N}$ exists on U. For any compact set $X \subset U$ there exists $j \in \mathbb{N}$ such that $X \subset U_N$ for $N > j$. By properties of the integral and Lemma 4.1 for $N > j$ we have

$$||\varphi_z^{U_N}||_{W^{k,p}(X)} \leq ||\varphi_z^{U_N}||_{W^{k,p}(U_N)} \leq ||\varphi_z^{U}||_{W^{k,p}(U)}.$$

So, the sequence $||\varphi_z^{U_N}||_{W^{k,p}(X)}$ is bounded. Therefore, by (3.1) sequence $\varphi_z^{\mu^N}$ satisfies the assumptions of the Arzelá-Ascoli theorem on X and we conclude that there exists a locally uniformly convergent subsequence of it. Without loss of generality, we can identify it with the entire sequence. Let us denote its limit by g. By the property of the minimal norm, it is sufficient to show that

$$||g||_{W^{k,p}(U)} \leq ||\varphi_z^{U}||_{W^{k,p}(U)}. \tag{5.1}$$

By Fatou's lemma

$$||g||_{W^{k,p}(U)} \leq \liminf_{N\to+\infty} ||\varphi_z^{U^N}||_{W^{k,p}(U_N)}. \tag{5.2}$$

Note that we have $W^{k,p}(U) \subseteq W^{k,p}(U_{j+1}) \subseteq W^{k,p}(U_j)$. Therefore, for any $f \in W^{k,p}(U)$, by Lemma 3.1, we have

$$|f(z)| \leq \frac{1}{||\varphi_z^{U_N}||_{W^{k,p}(U_N)}} ||f||_{W^{k,p}(U_N)}.$$

Taking the limit on the right hand side we get that

$$|f(z)| \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{U_N}||_{W^{k,p}(U_N)}} ||f||_{W^{k,p}(U)}.$$

(We can use Lebesgue dominated convergence theorem to show that

$$||f||_{W^{k,p}(U_N)} \to ||f||_{W^{k,p}(U)}.$$

We just need to consider integrals $\int_U |f(w)|^p \mathbb{I}_{U_n} \mathrm{d}w$, where $\mathbb{I}_X$ is indicator function of set X, and notice that the integrated functions can be dominated by $|f|^p$.) Moreover, by Lemma 3.1, again

$$\frac{1}{||\varphi_z^{U}||_{W^{k,p}(U)}} \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{U_N}||_{W^{k,p}(U_N)}}. \tag{5.3}$$

Combining (5.2) with (5.3), we obtain (5.1). □

6. Examples of Applications

Usually, we want the solutions of differential equations to be elements of some Sobolev spaces. Classical theory states that in many cases, if boundary or initial conditions are given, then the solution is unique. If the only set of solutions of the differential equation under consideration is a vector space that is closed in the corresponding Sobolev norm, and we want to have a unique solution, we can look for the solution with minimal possible Sobolev norm in the set of functions that take given value at a fixed point.

In the rest of the section, we assume without further recalling that our Sobolev norm satisfies the assumptions of Theorem 3.1. By solution we mean a classical solution, i.e., a sufficiently smooth function.

Note also that the term 'minimal solution' of a differential equation is usually used in a different sense (see e.g. [2, 5]).

Example 6.1. Let us consider the set of solutions of Laplace's equation that have a finite Sobolev norm. It is clear that this set forms a vector space. We will show that this vector space is closed in the Sobolev norm topology.

Let (f_n) be a sequence of harmonic functions converging to f in the Sobolev norm topology on a bounded domain U. Then, f_n in particular converges to f in the $L^p(U)$ topology. It is known that the L^p-limit of harmonic functions implies a locally uniform limit of the same sequence to the same limit. Indeed, by Mean Value Theorem for subharmonic functions,

$$|f(x)|^p \leq C_x \int_{B(x,r)} |f(x)|^p \mathrm{d}x,$$

where $B(x,r)$ is the ball with centre in x and sufficiently small radius $r > 0$ to lie with its boundary in U. Moreover, $C_x = \frac{1}{L(r)}$, where $L(r)$ is Lebesgue measure of a ball of radius r, i.e. C_x depends only on distance of x to the boundary. It means that for any compact set $X \subset U$ there is constant C_X, such that for any $x \in X$

$$|f(x)|^p \leq C_X \int_{B(x,r)} |f(x)|^p \mathrm{d}x.$$

Therefore, if f_n converges to f in the topology $L^p(U)$, it also converges to the same limit in the topology of locally uniform convergence. It is known that the locally uniform limit of harmonic functions is also a harmonic function, therefore the vector space of solutions of the Laplace equation with finite Sobolev norm is closed in the topology $W^{k,p}(U)$. This means that there is exactly one harmonic function which takes a given value in a fixed point with minimal Sobolev norm. Moreover, that extremal function depends continuously on a weight of integration and an increasing sequence of domains in precisely defined sense. Note that $V_{z,c}$ is never empty in this example, since the constant function is always its element.

Example 6.2. Now let $p = 2$, i.e., let our Sobolev norm be a norm generated by the inner product. Moreover, let us consider any homogeneous elliptic partial differential equation with C^∞ coefficients. It is obvious that the set of solutions of such equation, which have a finite $W^{k,2}$ norm, is a vector space. Let (f_n) be a sequence of elements of that space that converges to f in the topology $W^{k,2}(U)$, where U is a bounded domain. Then, in particular, f_n converges to f in $L^2(U)$. It can be shown that the L^2-limit of the classical solutions of an elliptic equation with smooth coefficients is also a classical solution of the same equation (see [6] for the proof). Therefore, the vector space of solutions with finite $W^{k,2}$ norm of the homogeneous elliptic equation with C^∞coefficients is closed in $W^{k,2}(U)$ topology. This means that if we want our solution of the elliptic homogeneous equation with C^∞ coefficients to be unique, we may look for the solution with the minimal $W^{k,2}(U)$ norm among the solutions which take given value at a fixed point. Moreover, if the equation also has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y = 0,$$

then $V_{z,c}$ is never empty, because constant function is always its element.

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