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ON SIMULTANEOUS APPROXIMATION AND COMBINATIONS OF LUPAS TYPE OPERATORS

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Abstract. The purpose of the present paper is to study a sequence of linear and positive operators which was introduced by A. Lupas. First, we obtain estimate of moments of the operators and then prove a basic convergence theorem for simultaneous approximation. Further, we find error in approximation in terms of modulus of continuity of function. Finally, we establish a Voronovskaja asymptotic formula for linear combination of the above operators.

1. Introduction

At the International Dortmund Meeting held in Written (Germany, March, 1995), A. Lupas [11] introduced the following Linear positive operators for $f:[0,\infty)\to\mathbb{R}$ as

$$L_n(f,x) =(1-a)^{nx}\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu f\left(\frac{\nu}{n}\right),\quad x\geq 0, \tag{1.1}$$

$$(1-a)^{-\alpha} =\sum_{\nu=0}^{\infty}\frac{(\alpha)_\nu}{\nu!}a^\nu, \tag{1.2}$$

where

$$|a|<1,\quad (\alpha)_0=1,\quad (\alpha)_\nu=\alpha(\alpha+1)\cdots(\alpha+\nu-1),\quad \nu\geq 1.$$

If we impose that $L_n(t,x)=x$, we find that $a=1/2$. Therefore, operator (1.1) becomes

$$L_n(f,x)=2^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{2^\nu \nu!}f\left(\frac{\nu}{n}\right),\quad x\geq 0.$$

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It was seen that these opeartors are positive and linear and preseve linear functions. Bernstein polynomials [10] exhibit the property of simultaneous approximation. Simultaneous approximation for Baskakov operators, modified by Durrmeyer, was studied by Heilmann and Müller [8]. Another modification of Baskakov operators for simultaneous approximation was investigated by Sinha et al. [18]. Yet another modification of Baskakov operators viz., integral modification of Baskakov operators shows simultaneous approximation property in Thamer et al. [20]. This was studied for Durrmeyer modification of Bernstein polynomials by Gonska and Zhou [4]. In the summation-integral type operators Gupta et al. [7] explored the simultanoeus approximation. So far research work was done for linear positive operators ([3],[6],[9],[12]-[15], [19]). Singh and Agrawal [17] proved simultaneous approximation by a linear combination of Bernstein-Durrmeyer type polynomials. Gupta [5] studied the differences of operators of Lupas type. So, the Lupas operators play very important role to approximate functions for $f \in C[0, \infty)$.

It turns out that the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. Therefore, in order to improve the order of approximation by the operators (1.1), we apply the technique of linear combination introduced by Butzer [2] and Rathore [16].

The approximation process for linear combination is defined as follows.

Let $d_0, d_1, \ldots, d_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. Then, the linear combination $L_n(f, k, x)$ of $L_{d_j n}(f, x), j = 0, 1, 2, \ldots, k$, is given by

$$L_n(f,k,x) = \frac{1}{\Delta} \begin{vmatrix} L_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-k} \\ L_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \cdots & d_k^{-k} \end{vmatrix},$$

where Δ is the Vandermonde determinant defined as

$$\Delta = \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_k^{-1} & d_k^{-2} & \cdots & d_k^{-k} \end{vmatrix}.$$

On simplification, we have

$$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k) L_{d_j n}(f,x), \tag{1.3}$$

where

$$C(j,k) = \prod_{i=0, i\neq j}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \text{ and } C(0,0) = 1.$$

2. Moment Estimates

Lemma 2.1 ([5])**.** *The following relations hold:*

$$L_n(1,x)=1,\quad L_n(t-x,x)=\frac{2a-1}{1-a}x, L_n((t-x)^2,x)=\frac{n^2x^2(2a-1)^2+nax}{n^2(1-a)^2}.$$

Now, we define mth order moment

$$\mu_m(x)=L_n((t-x)^m,x)=(1-a)^{nx}\left\{\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu\left(\frac{\nu}{n}-x\right)^m\right\}.$$

Lemma 2.2. *$\mu_m(x)$ is a polynomial in x of degree $[m/2]$. Moreover*

$$\mu_m(x)=O\left(\frac{1}{n^{[\frac{m+1}{2}]}}\right),\quad n\to\infty.$$

Proof. By definition of moments of mth order, we have

$$\begin{aligned}\mu_m(x)=&(1-a)^{nx}\sum_{\nu=0}^{\infty}(-1)^m a^\nu\frac{(nx)_\nu}{\nu!}\left\{\sum_{r=0}^{m}\binom{m}{r}(-1)^r x^{m-r}\left(\frac{\nu}{n}\right)^r\right\}\\ =&(-1)^m(1-a)^{nx}\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu\left\{\sum_{r=0}^{m}\binom{m}{r}\left(-\frac{1}{n}\right)^r x^{m-r}\right.\\ &\left.\times\left(\nu^{(r)}+p_2\nu^{(r-1)}+p_4\nu^{(r-2)}+\cdots\right)\right\},\end{aligned}\tag{2.1}$$

where $\nu^{(r)}=\nu(\nu-1)(\nu-2)\cdots(\nu-r+1)$, p_2 is a polynomial in r of second degree, p_4 is a polynomial in r of fourth degree and so on.

It follows from (1.2) upon s times differentiation in a that

$$\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}\nu(\nu-1)\cdots(\nu-s+1)a^{\nu-s}=(nx)_s(1-a)^{-nx-s}.\tag{2.2}$$

Making an use of (2.2) in (2.1)

$$\begin{aligned}\mu_m(x)=&(-1)^m\sum_{r=0}^{m}\binom{m}{r}\left(-\frac{1}{n}\right)^r x^{m-r}\left\{\frac{a^r}{(1-a)^r}(nx)_r+p_2\frac{a^{r-1}}{(1-a)^{r-1}}(nx)_{r-1}\right.\\ &\left.+p_4\frac{a^{r-2}}{(1-a)^{r-2}}(nx)_{r-2}+\cdots\right\}.\end{aligned}$$

Again,

$$\frac{(nx)_r}{n^r x^r}=1+\frac{q_2}{nx}+\frac{q_4}{(nx)^2}+\frac{q_6}{(nx)^3}+\cdots,$$

where q_j as before is a polynomial in r of degree j.

Therefore, taking $a = 1/2$ and using fact that $\sum_{r=0}^{m} \binom{m}{r}(-1)^r r^s = 0$, $s < m$, we find that

$$\mu_m(x) = (-1)^m x^m \left\{ \frac{C}{(nx)^{[\frac{m+1}{2}]}} + \cdots \text{higher order terms} \right\}.$$

Therefore, $\mu_m(x)$ is a polynomial in x of degree $[m/2]$. This completes the proof of lemma. □

3. Simultaneous Approximation

Theorem 3.1. *Let $f' \in C_B[0,\infty)$. Then, sequence $\left\{\frac{d}{dx}(L_n(f,x))\right\}_{n=1}^{\infty}$ converges to $f'(x)$ pointwise on $[0,\infty)$. Moreover, if S is a compact subset of $[0,\infty)$ then sequence $\left\{\frac{d}{dx}(L_n(f,x))\right\}_{n=1}^{\infty}$ converges to $f'(x)$ uniformly on S.*

Proof. We expand

$$f(w) = f(x) + (w-x)f'(x) + \int_x^w (f'(t) - f'(x))dt.$$

Operating $L_n(\cdot, y)$ on both sides of above equation and in view of Lemma 2.1, we obtain

$$L_n(f,y) = f(x) + \left(\frac{ay}{1-a} - x\right) f'(x) + (1-a)^{ny} \left\{ \sum_{\nu=0}^{\infty} \frac{(ny)_\nu}{\nu!} a^\nu R_\nu \right\},$$

where $R_\nu = \int_x^{\nu/n} (f'(t) - f'(x))dt$. Thus,

$$\begin{aligned} \frac{d}{dx} L_n(f,x) =& \frac{a}{1-a} f'(x) + n(1-a)^{nx} \\ & \times \left\{ \ln(1-a) \sum_{\nu=0}^{\infty} \frac{(nx)_\nu}{\nu!} a^\nu R_\nu + \sum_{\nu=1}^{\infty} \frac{d(nx)_\nu}{d(nx)} \frac{a^\nu}{\nu!} R_\nu \right\}. \end{aligned} \tag{3.1}$$

We put $nx = \alpha$ and differentiate (1.2) w.r.t. α. Further, we equate coefficient of a^ν on both sides, we get

$$\frac{1}{\nu!} \cdot \frac{d(\alpha)_\nu}{d\alpha} = \frac{\alpha_{\nu-1}}{(\nu-1)!} + \frac{1}{2} \cdot \frac{\alpha_{\nu-2}}{(\nu-2)!} + \frac{1}{3} \cdot \frac{\alpha_{\nu-3}}{(\nu-3)!} + \cdots + \frac{1}{\nu} \cdot \frac{\alpha_0}{0!}. \tag{3.2}$$

Using (3.2) in (3.1), we get

$$\begin{aligned}
&\frac{d}{dx}L_n(f,x) - \frac{a}{1-a}f'(x)\\
=&n(1-a)^\alpha \Bigg[a(R_1-R_0) + a^2\left\{\frac{(\alpha)_1}{1!}(R_2-R_1) + \frac{(\alpha)_0}{2}(R_2-R_0)\right\}\\
&+ a^3\left\{\frac{(\alpha)_2}{2!}(R_3-R_2) + \frac{1}{2}\cdot\frac{(\alpha)_1}{1!}(R_3-R_1) + \frac{1}{3}(\alpha)_0(R_3-R_0)\right\} + \cdots\\
&+ a^\nu\Bigg\{\frac{(\alpha)_{\nu-1}}{(\nu-1)!}(R_\nu - R_{\nu-1}) + \frac{(\alpha)_{\nu-2}}{(\nu-2)!}\cdot\frac{1}{2}(R_\nu - R_{\nu-2})\\
&+ \frac{(\alpha)_{\nu-3}}{(\nu-3)!}\cdot\frac{1}{3}(R_\nu - R_{\nu-3}) + \cdots + \frac{(\alpha)_1}{1!}\cdot\frac{1}{\nu-1}(R_\nu - R_1)\\
&+ \frac{1}{\nu}\cdot\frac{(\alpha)_0}{1}(R_\nu - R_0)\Bigg\} + \cdots\Bigg]\\
=&n(1-a)^\alpha\Bigg[a\Bigg\{(\alpha)_0(R_1-R_0) + \frac{a(\alpha)_1}{1!}(R_2-R_1) + \frac{a^2(\alpha)_2}{2!}(R_3-R_2)\\
&+ \frac{a^3(\alpha)_3}{3!}(R_4-R_3) + \cdots\Bigg\} + a^2\Bigg\{\frac{(\alpha)_0}{2}(R_2-R_0) + \frac{a(\alpha)_1}{1!}\cdot\frac{1}{2}(R_3-R_1)\\
&+ \frac{a^2(\alpha)_2}{2!}\cdot\frac{1}{2}(R_4-R_2) + \frac{a^3(\alpha)_3}{3!}\cdot\frac{1}{2}(R_5-R_3) + \cdots\Bigg\}\\
&+ a^3\Bigg\{(\alpha)_0\frac{1}{3}(R_3-R_0) + \frac{a(\alpha)_1}{1!}\cdot\frac{1}{3}(R_4-R_1) + \frac{a^2(\alpha)_2}{2!}\cdot\frac{1}{3}(R_5-R_2) + \cdots\Bigg\}\\
&+\cdots\Bigg]
\end{aligned}$$

$$=n(1-a)^\alpha\left[\Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots\right], \quad \text{say.} \tag{3.3}$$

The continuity of $f'(\cdot)$ at point x implies that for a given $\epsilon > 0$ there exists a $\delta = \delta(x)$, (depending on x) such that $|f'(t) - f'(x)| < \epsilon$ if $|t-x| < \delta$. We break $R_p - R_q$ in two parts depending upon $|t-x| < \delta$ and $|t-x| \geq \delta$. In the second part, there may be two terms, where $|f'(t)-f'(x)| \leq 2\|f'\|_{C_B[0,\infty)}\cdot\frac{1}{\delta^2}(t-x)^2$.

Using Lemma 2.1, we get

$$\begin{aligned}
|\Sigma_1| &\leq a\frac{\epsilon}{n}\left(\sum_{k=0}^{\infty}\frac{a^k}{k!}(\alpha)_k\right) + \frac{2\cdot 2\|f'\|_{C_B[0,\infty)}}{\delta^2}\cdot\frac{a}{n}\left\{\sum_{k=0}^{\infty}\frac{a^k}{k!}(\alpha)_k\left(\frac{k}{n}-x\right)^2\right\}\\
&= a\frac{\epsilon}{n}(1-a)^{-\alpha} + \frac{4\|f'\|_{C_B[0,\infty)}}{\delta^2}\cdot\frac{a}{n}\cdot\left\{\frac{nx^2(2a-1)^2+ax}{n(1-a)^2}\right\}(1-a)^{-\alpha}.
\end{aligned} \tag{3.4}$$

Now,

$$|\Sigma_2| \leq a^2 \frac{\epsilon}{n} \left\{ \sum_{k=0}^{\infty} \frac{a^k}{k!} (\alpha)_k \right\} + \frac{4\|f'\|_{C_B[0,\infty)}}{\delta^2} \cdot \frac{a^2}{n} \left\{ \sum_{k=0}^{\infty} \frac{a^k}{k!} (\alpha)_k \left(\frac{k}{n} - x \right)^2 \right\}$$

$$= a^2 \frac{\epsilon}{n} (1-a)^{-\alpha} + \frac{4\|f'\|_{C_B[0,\infty)}}{\delta^2} \cdot \frac{a^2}{n} \left\{ \frac{nx^2(2a-1)^2 + ax}{n(1-a)^2} \right\} (1-a)^{-\alpha}. \tag{3.5}$$

The similar estimates for $\Sigma_3, \Sigma_4, \ldots$ are combined in (3.3) and we take $a = \frac{1}{2}$ due to Agratini [1]. Finally,

$$\left| \frac{d}{dx} L_n(f,x) - f'(x) \right| \leq Cn \left(\frac{\epsilon}{n} + \frac{1}{n^2} \right).$$

This completes the proof of the first part.

Proof of second part of Theorem 3.1. Let S be a compact subset of $[0,\infty)$. The pointwise continuity of function $f'(\cdot)$ at points of S, imply, by virtue of compactness of S, that $f'(\cdot)$ is now uniformly continuous on S. Thus, δ is now independent of x. Moreover S, being compact, is a bounded subset of $[0,\infty)$. Thus $x \in S$ implies $|x| < C_1$, a constant. This implies by (3.4) and (3.5) that convergence is uniform. □

Theorem 3.2. *Let $f' \in C_B[0,\infty)$. Then for $\delta > 0$ and $[a,b] \subset (a_1,b_1)$ we have*

$$\sup_{x\in[a,b]} |L_n'(f,x) - f'(x)| \leq \omega(f',\delta,[a_1,b_1]) + \frac{C}{n}\|f'\|_{C_B[0,\infty)}.$$

Proof. We proceed in similar way as in the proof of Theorem 3.1. In the steps following (3.3) if $|t-x| < \delta$, then $|f'(t) - f'(x)| \leq \omega(f',\delta,[a_1,b_1])$. When $|t-x| \geq \delta$, using boundedness of f' the total contribution is of order $\|f'\|_{C_B[0,\infty)} O\left(\frac{1}{n}\right)$ as $n \to \infty$, by Lemma 2.1. Hence, the proof follows. □

4. Linear Combinations

Theorem 4.1. *Let $f^{(2k+2)} \in C_B[0,\infty)$. Then there holds for each $x \in [0,\infty)$, pointwise:*

$$\frac{d}{dx} L_n(f,k,x) - f'(x) = \frac{1}{n^{k+1}} \left\{ \sum_{j=k+2}^{2k+2} q_j(x) f^{(j)}(x) \right\} + o\left(\frac{1}{n^{k+1}} \right), \quad n \to \infty. \tag{4.1}$$

Moreover, if S is a compact subset of $[0,\infty)$, then convergence (4.1) is uniform on S.

Proof. Using Taylor's series expansion, we write

$$\begin{aligned}f(w) =& f(x) + (w-x)f'(x) + \frac{(w-x)^2}{2!}f^{(2)}(x) + \cdots \\ &+ \frac{(w-x)^{2k+2}}{(2k+2)!}f^{(2k+2)}(x) \\ &+ \int_x^w \int_x^{t_1} \int_x^{t_2} \cdots \int_x^{t_{2k+1}} (f^{(2k+2)}(u) - f^{(2k+2)}(x))dt_{2k+1}\, dt_{2k} \cdots dt_1 du.\end{aligned}$$

Operating $L_n(\cdot, y)$ on both sides of above equation and in view of Lemma 2.1, we obtain

$$\begin{aligned}L_n(f,y) =& f(x) + \left(\frac{ay}{1-a} - x\right) f'(x) + \frac{f^{(2)}(x)}{2!}p_2(1/n, y) \\ &+ \frac{f^{(3)}(x)}{3!}p_3(1/n,y) + \cdots + \frac{f^{(2k+2)}(x)}{(2k+2)}p_{2k+2}(1/n,y) \\ &+ (1-a)^{ny}\left(\sum_{\nu=0}^{\infty}\frac{(ny)_\nu}{\nu!}a^\nu R_\nu\right),\end{aligned}$$

where

$$R_\nu = \int_x^{\nu/n}\int_x^{t_1}\int_x^{t_2}\cdots\int_x^{t_{2k+1}}(f^{(2k+2)}(u) - f^{(2k+2)}(x))dt_{2k+1}dt_{2k}\cdots dt_1 du$$

and $p_j\left(\frac{1}{n}, y\right)$ is a polynomial in y of degree j and in $\frac{1}{n}$ of degree $(j-1)$.

This implies that

$$\begin{aligned}\frac{d}{dx}L_n(f,x) =& \frac{a}{1-a}f'(x) + \frac{f^{(2)}(x)}{2!}p_2'(1/n,x) \\ &+ \frac{f^{(3)}(x)}{3!}p_3'(1/n,x) + \cdots + \frac{f^{(2k+2)}(x)}{(2k+2)}p_{2k+2}'(1/n,x) \\ &+ n(1-a)^{nx}\left\{\log(1-a)\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu R_\nu + \sum_{\nu=1}^{\infty}\frac{d(nx)_\nu}{d(nx)}\cdot\frac{a^\nu}{\nu!}R_\nu\right\}.\end{aligned} \tag{4.2}$$

Let $\phi(n,x) = n(1-a)^{nx}\left\{\log(1-a)\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu R_\nu + \sum_{\nu=1}^{\infty}\frac{d(nx)_\nu}{d(nx)}\frac{a^\nu}{\nu!}R_\nu\right\}$. Now, taking linear combinations on (4.2) and using their properties (1.3), we have

$$\frac{d}{dx}L_n(f,k,x) - \left(\frac{a}{1-a}\right)f'(x) = \left\{\sum_{j=k+2}^{2k+2}q_j(x)f^{(j)}(x)\right\}\frac{1}{n^{k+1}} + \sum_{j=0}^{k}C(j,k)\phi(d_j n, x).$$

We analyze last term as in (3.1) and obtain the required result.

The proof of the second part of theorem follows from the proof of the second part of Theorem 3.1. □

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