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CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH QUASI-SUBORDINATION INVOLVING (p,q)-DERIVATIVE OPERATOR

Ş. ALTINKAYA[1] AND S. YALÇIN[2]

Abstract. In this present paper, as applications of the post-quantum calculus known as the (p,q)-calculus, we construct a new class $\boldsymbol{D}^k_{p,q}(\gamma,\zeta,\Psi)$ of bi-univalent functions of complex order defined in the open unit disk. Coefficients inequalities and several special consequences of the results are obtained.

1. Introduction and Preliminaries

The q-calculus as well as the fractional q-calculus provide important tools that have been used in the fields of special functions and many other areas. Historically speaking, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [30]). In fact, the theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, [3, 19, 21, 26]). In particular, Purohit and Raina [20] investigated applications of fractional q-calculus operators to define several classes of functions which are analytic in the open unit disk. On the other hand, Mohammed and Darus [14] studied approximation and geometric properties of these q-operators in regard to some subclasses of analytic functions in a compact disk.

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Further the possibility of extension of the q-calculus to post-quantum calculus denoted by the (p,q)-calculus. The (p,q)-calculus which have many applications in areas of science and engineering was introduced in order to generalize the q-series by Gasper and Rahman [8]. The (p,q)-series is derived as corresponding extensions of q-identities (for example [2, 6]).

We begin by providing some basic definitions and concept details of the (p,q)-calculus which are used in this paper.

The (p,q)-number is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad p \neq q,$$

which is a natural generalization of the q-number (see [11]), that is

$$\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

It is clear that the notation $[n]_{p,q}$ is symmetric, that is,

$$[n]_{p,q} = [n]_{q,p} \,.$$

Let p and q be elements of complex numbers and $D = D_{p,q} \subset \mathbb{C}$ such that $x \in D$ implies $px \in D$ and $qx \in D$. Here, in this investigation, we give the following two definitions which involve a post-quantum generalization of Sofonea's work [27].

Definition 1.1. Let $0 < |q| < |p| \leq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called (p,q)-differentiable under the restriction that, if $0 \in D_{p,q}$, then $f'(0)$ exists.

Definition 1.2. Let $0 < |q| < |p| \leq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called (p,q)-differentiable of order n, if and only if $0 \in D_{p,q}$, then $f^{(n)}(0)$ exists.

Definition 1.3 ([6])**.** The (p,q)-derivative of a function f is defined as

$$(D_{p,q}f)(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided $f'(0)$ exists.

As with ordinary derivative, the action of the (p,q)-derivative of a function is a linear operator. More precisely, for any constants a and b,

$$D_{p,q}(af(z) + bg(z)) = aD_{p,q}f(z) + bD_{p,q}g(z).$$

The (p,q)-derivative fulfils the following product rules

$$\begin{aligned} D_{p,q}(f(z)g(z)) =& f(pz)D_{p,q}g(z) + g(qz)D_{p,q}f(z), \\ D_{p,q}(f(z)g(z)) =& g(pz)D_{p,q}f(z) + f(qz)D_{p,q}g(z). \end{aligned}$$

Further, the (p,q)-derivative fulfils the following product rules

$$D_{p,q}\left(\frac{f(z)}{g(z)}\right)=\frac{g(qz)D_{p,q}f(z)-f(qz)D_{p,q}g(z)}{g(pz)g(qz)},$$
$$D_{p,q}\left(\frac{f(z)}{g(z)}\right)=\frac{g(pz)D_{p,q}f(z)-f(pz)D_{p,q}g(z)}{g(pz)g(qz)}.$$

Let A indicate an analytic function family, which is normalized under the condition of $f(0)=f'(0)-1=0$ in $\Delta=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$ and given by the following Taylor-Maclaurin series:

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n.$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ.

If f is of the form (1.1), then

$$(D_{p,q}f)(z)=1+\sum_{n=2}^{\infty}[n]_{p,q}\,a_nz^{n-1}.$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0)=0, |w(z)|<1, \quad z\in\Delta,$$

such that

$$f(z)=g(w(z)), \quad z\in\Delta.$$

We denote this subordination by

$$f\prec g \ \text{ or } \ f(z)\prec g(z), \quad z\in\Delta.$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0)=g(0), \quad f(\Delta)\subset g(\Delta).$$

In the year 1970, Robertson [23] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is said to be quasi-subordinate to g in Δ and written as

$$f(z)\prec_\rho g(z), \quad z\in\Delta,$$

if there exists an analytic function $|h(z)|\leq 1$ such that $\frac{f(z)}{h(z)}$ analytic in Δ and

$$\frac{f(z)}{h(z)}\prec g(z), \quad z\in\Delta,$$

that is, there exists a Schwarz function $w(z)$ such that $f(z)=h(z)g(w(z))$. Observe that if $h(z)=1$, then $f(z)=g(w(z))$ so that $f(z)\prec g(z)$ in Δ. Also notice that if $w(z)=z$, then $f(z)=h(z)g(z)$ and it is said that is majorized by g and written $f(z)\ll g(z)$ in Δ. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization (see, e.g., [13, 22, 23] for works related to quasi-subordination).

The Koebe-One Quarter Theorem [7] ensures that the image of Δ under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ $\left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.1). For a brief history and interesting examples in the class Σ, see [29] (see also [4, 5, 12, 16]). Furthermore, judging by the remarkable flood of papers on the subject (see, for example, [10, 17, 28]). Not much is known about the bounds on the general coefficient $|a_n|$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([1, 9, 15, 31]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$, $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

Recently for $f \in A$, Selvaraj et al. [25] defined and discussed (p, q)-analogue of Salagean differential operator as given below:

$$\begin{aligned}
\boldsymbol{D}^0_{p,q} f(z) =& f(z) \\
\boldsymbol{D}^1_{p,q} f(z) =& z\left(\boldsymbol{D}_{p,q} f(z)\right) \\
&\vdots \\
\boldsymbol{D}^k_{p,q} f(z) =& z\boldsymbol{D}_{p,q}(\boldsymbol{D}^{k-1}_{p,q} f(z)) \\
\boldsymbol{D}^k_{p,q} f(z) =& z + \sum_{n=2}^{\infty} [n]^k_{p,q} a_n z^n, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \Delta.
\end{aligned}$$

If we let $p = 1$ and $q \to 1^-$, then $\boldsymbol{D}^k_{p,q} f(z)$ reduces to the well-known Salagean differential operator (see [24]).

Making use of the differential operator $\boldsymbol{D}^k_{p,q}$, we introduce a new class of analytic bi-univalent functions as follows.

Definition 1.4. A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$\boldsymbol{D}^k_{p,q}(\gamma, \zeta, \Psi), \quad \gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq \zeta < 1, k \in \mathbb{N}_0,\ 0 < q < p \leq 1,\ z, w \in \Delta,$$

if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}^k_{p,q} f(z)\right)'}{(1-\zeta)\boldsymbol{D}^k_{p,q} f(z) + \zeta z\left(\boldsymbol{D}^k_{p,q} f(z)\right)'} - 1\right) \prec_\rho (\Psi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}_{p,q}^{k}g(w)\right)'}{(1-\zeta)\boldsymbol{D}_{p,q}^{k}g(w)+\zeta w\left(\boldsymbol{D}_{p,q}^{k}g(w)\right)'}-1\right)\prec_{\rho}\left(\Psi(w)-1\right),$$

where the function g is given by (1.2).

Remark 1.1. For $p = 1$ and $q \to 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\boldsymbol{D}^{k}\left(\gamma, \zeta, \Psi\right)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}^{k}f(z)\right)'}{(1-\zeta)\boldsymbol{D}^{k}f(z)+\zeta z\left(\boldsymbol{D}^{k}f(z)\right)'}-1\right)\prec_{\rho}\left(\Psi(z)-1\right), \quad z\in\Delta$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}^{k}g(w)\right)'}{(1-\zeta)\boldsymbol{D}^{k}g(w)+\zeta w\left(\boldsymbol{D}^{k}g(w)\right)'}-1\right)\prec_{\rho}\left(\Psi(w)-1\right), \quad z\in\Delta,$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq \zeta < 1,\ k \in \mathbb{N}_0$ and the function g is given by (1.2).

Remark 1.2. For $\zeta = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\boldsymbol{D}_{p,q}^{k}\left(\gamma, \Psi\right)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}_{p,q}^{k}f(z)\right)'}{\boldsymbol{D}_{p,q}^{k}f(z)}-1\right)\prec_{\rho}\left(\Psi(z)-1\right), \quad z\in\Delta$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}_{p,q}^{k}g(w)\right)'}{\boldsymbol{D}_{p,q}^{k}g(w)}-1\right)\prec_{\rho}\left(\Psi(w)-1\right), \quad z\in\Delta,$$

where $k \in \mathbb{N}_0,\ 0 < q < p \leq 1$ and the function g is given by (1.2).

Remark 1.3. For $\zeta = k = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $S_{\Sigma}\left(\gamma, \Psi\right)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec_{\rho}\left(\Psi(z)-1\right), \quad z\in\Delta$$

and

$$\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}-1\right)\prec_{\rho}\left(\Psi(w)-1\right), \quad z\in\Delta,$$

where the function g is given by (1.2).

2. Main Result and its Consequences

Firstly, we will state the Lemma 2.1 to obtain our result.

Lemma 2.1 ([18])**.** *If $s \in P$, then $|s_i| \leq 2$ for each i, where P is the family of all functions s, analytic in Δ, for which*

$$\operatorname{Re}(s(z)) > 0,$$

where

$$s(z) = 1 + s_1 z + s_2 z^2 + \cdots .$$

Through out this paper it is assumed that Ψ is analytic in Δ with $\Psi(0) = 1$ and let

$$\Psi(z) = 1 + C_1 z + C_2 z^2 + \cdots, \quad C_1 > 0. \tag{2.1}$$

Also let

$$h(z) = D_0 + D_1 z + D_2 z^2 + \cdots, \quad |h(z)| \leq 1, \ z \in \Delta. \tag{2.2}$$

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\boldsymbol{D}_{p,q}^{k}(\gamma, \zeta, \Psi)$ proposed by Definition 1.4.

Theorem 2.1. *Let f of the form* (1.1) *be in the class $\boldsymbol{D}_{p,q}^{k}(\gamma, \zeta, \Psi)$. Then*

$$|a_2| \leq \frac{|\gamma|\,|D_0|\,C_1\sqrt{C_1}}{\sqrt{(1-\zeta)\left|2\,[3]_{p,q}^{k}\,\gamma C_1^2 D_0 - [2]_{p,q}^{2k}\left[(1-\zeta)(C_2 - C_1) + (1+\zeta)\gamma C_1^2 D_0\right]\right|}}$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2\, C_1^2}{(1-\zeta)^2\,[2]_{p,q}^{2k}} + \frac{|\gamma D_1|\,C_1}{2(1-\zeta)\,[3]_{p,q}^{k}} + \frac{|\gamma D_0|\,C_1}{2(1-\zeta)\,[3]_{p,q}^{k}}.$$

Proof. If $f \in \boldsymbol{D}_{p,q}^{k}(\gamma, \zeta, \Psi)$ then, there are two analytic functions $u, v : \Delta \to \Delta$ with $u(0) = v(0) = 0$, $|u(z)| < 1, |v(w)| < 1$ and a function h given by (2.2), such that

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}_{p,q}^{k} f(z)\right)'}{(1-\zeta)\boldsymbol{D}_{p,q}^{k} f(z) + \zeta z\left(\boldsymbol{D}_{p,q}^{k} f(z)\right)'} - 1\right) = h(z)\left(\Psi(u(z)) - 1\right) \tag{2.3}$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}_{p,q}^{k} g(w)\right)'}{(1-\zeta)\boldsymbol{D}_{p,q}^{k} g(w) + \zeta w\left(\boldsymbol{D}_{p,q}^{k} g(w)\right)'} - 1\right) = h(w)\left(\Psi(v(w)) - 1\right). \tag{2.4}$$

Determine the functions s_1 and s_2 in P given by

$$s_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + t_1 z + t_2 z^2 + \cdots$$

and

$$s_2(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_1 w + q_2 w^2 + \cdots .$$

Thus,

$$u(z) = \frac{s_1(z)-1}{s_1(z)+1} = \frac{1}{2}\left(t_1 z + \left(t_2 - \frac{t_1^2}{2}\right) z^2 + \cdots\right) \tag{2.5}$$

and

$$v(w) = \frac{s_2(w)-1}{s_2(w)+1} = \frac{1}{2}\left(q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \cdots\right). \tag{2.6}$$

The fact that s_1 and s_2 are analytic in Δ with $s_1(0) = s_2(0) = 1$. Since $u, v : \Delta \to \Delta$, the functions s_1, s_2 have a positive real part in Δ, and the relations $|t_i| \leq 2$ and $|q_i| \leq 2$ are true. Using (2.5) and (2.6) together with (2.1) and (2.2) in the right hands of the relations (2.3) and (2.4), we obtain

$$\begin{aligned} h(z)\left(\Psi\left(u(z)\right)-1\right) =& \frac{1}{2} D_0 C_1 t_1 z \\ &+ \left(\frac{1}{2} D_1 C_1 t_1 + \frac{1}{2} D_0 C_1 \left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4} D_0 C_2 t_1^2\right) z^2 + \cdots \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} h(w)\left(\Psi\left(v(w)\right)-1\right) =& \frac{1}{2} D_0 C_1 q_1 w \\ &+ \left(\frac{1}{2} D_1 C_1 q_1 + \frac{1}{2} D_0 C_1 \left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4} D_0 C_2 q_1^2\right) w^2 + \cdots. \end{aligned} \tag{2.8}$$

In the light of (2.3) and (2.4), we get

$$\frac{(1-\zeta)\,[2]_{p,q}^{k}}{\gamma} a_2 = \frac{D_0 C_1 t_1}{2}, \tag{2.9}$$

$$\frac{2(1-\zeta)\,[3]_{p,q}^{k}\, a_3 - (1-\zeta^2)\,[2]_{p,q}^{2k}\, a_2^2}{\gamma} = \frac{D_1 C_1 t_1}{2} + \frac{D_0 C_1}{2}\left(t_2 - \frac{t_1^2}{2}\right) + \frac{D_0 C_2 t_1^2}{4} \tag{2.10}$$

and

$$-\frac{(1-\zeta)\,[2]_{p,q}^{k}}{\gamma} a_2 = \frac{D_0 C_1 q_1}{2}, \tag{2.11}$$

$$\frac{2(1-\zeta)\,[3]_{p,q}^{k}\,(2a_2^2 - a_3) - (1-\zeta^2)\,[2]_{p,q}^{2k}\, a_2^2}{\gamma} = \frac{D_1 C_1 q_1}{2} + \frac{D_0 C_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{D_0 C_2 q_1^2}{4}. \tag{2.12}$$

Now, (2.9) and (2.11) give

$$t_1 = -q_1 \tag{2.13}$$

and

$$8(1-\zeta)^2\,[2]_{p,q}^{2k}\, a_2^2 = \gamma^2 D_0^2 C_1^2 \left(t_1^2 + q_1^2\right). \tag{2.14}$$

Adding (2.10) and (2.12), we get

$$\frac{4(1-\zeta)\,[3]_{p,q}^{k}-2(1-\zeta^{2})\,[2]_{p,q}^{2k}}{\gamma}a_2^2=\frac{D_0C_1\,(t_2+q_2)}{2}+\frac{D_0\,(C_2-C_1)\,(t_1^2+q_1^2)}{4}. \tag{2.15}$$

By using (2.13), (2.14) and Lemma 2.1 in (2.15), we obtain

$$|a_2|\leq\frac{|\gamma|\,|D_0|\,C_1\sqrt{C_1}}{\sqrt{(1-\zeta)\left|2\,[3]_{p,q}^{k}\,\gamma C_1^2D_0-[2]_{p,q}^{2k}\,[(1-\zeta)(C_2-C_1)+(1+\zeta)\gamma C_1^2D_0]\right|}}.$$

Next, to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we have

$$\frac{4(1-\zeta)\,[3]_{p,q}^{k}}{\gamma}\left(a_3-a_2^2\right)=\frac{D_0C_1\,(t_2-q_2)}{2}+\frac{D_1C_1\,(t_1-q_1)}{2}. \tag{2.16}$$

It follows from (2.13), (2.14) and (2.16) that

$$a_3=\frac{\gamma^2D_0^2C_1^2\,(t_1^2+q_1^2)}{8(1-\zeta^2)\,[2]_{p,q}^{2k}}+\frac{\gamma D_1C_1\,(t_1-q_1)}{8(1-\zeta)\,[3]_{p,q}^{k}}+\frac{\gamma D_0C_1\,(t_2-q_2)}{8(1-\zeta)\,[3]_{p,q}^{k}}.$$

Applying Lemma 2.1 once again for the coefficients t_1, t_2, q_1 and q_2, we readily get

$$|a_3|\leq\frac{|\gamma D_0|^2\,C_1^2}{(1-\zeta)^2\,[2]_{p,q}^{2k}}+\frac{|\gamma D_1|\,C_1}{2(1-\zeta)\,[3]_{p,q}^{k}}+\frac{|\gamma D_0|\,C_1}{2(1-\zeta)\,[3]_{p,q}^{k}}.$$

This completes the proof of Theorem 2.1. □

Corollary 2.1. *Let f of the form* (1.1) *be in the class* $\boldsymbol{D}^k(\gamma,\zeta,\Psi)$. *Then*

$$|a_2|\leq\frac{|\gamma|\,|D_0|\,C_1\sqrt{C_1}}{\sqrt{(1-\zeta)\,|2\gamma C_1^2D_03^k-2^{2k}\,[(1-\zeta)(C_2-C_1)+(1+\zeta)\gamma C_1^2D_0]|}}$$

and

$$|a_3|\leq\frac{|\gamma D_0|^2\,C_1^2}{(1-\zeta)^22^{2k}}+\frac{|\gamma D_1|\,C_1}{2(1-\zeta)3^k}+\frac{|\gamma D_0|\,C_1}{2(1-\zeta)3^k}.$$

Corollary 2.2. *Let f of the form* (1.1) *be in the class* $\boldsymbol{D}_{p,q}^k(\gamma,\Psi)$. *Then*

$$|a_2|\leq\frac{|\gamma|\,|D_0|\,C_1\sqrt{C_1}}{\sqrt{\left|2\,[3]_{p,q}^{k}\,\gamma C_1^2D_0-[2]_{p,q}^{2k}\,[(C_2-C_1)+\gamma C_1^2D_0]\right|}}$$

and

$$|a_3|\leq\frac{|\gamma D_0|^2\,C_1^2}{[2]_{p,q}^{2k}}+\frac{|\gamma D_1|\,C_1}{2\,[3]_{p,q}^{k}}+\frac{|\gamma D_0|\,C_1}{2\,[3]_{p,q}^{k}}.$$

Corollary 2.3. *Let f of the form* (1.1) *be in the class* $S_\Sigma(\gamma,\Psi)$. *Then*

$$|a_2|\leq\frac{|\gamma D_0|\,C_1\sqrt{C_1}}{\sqrt{|C_1-C_2+\gamma C_1^2D_0|}}$$

and

$$|a_3| \leq |\gamma D_0|^2 C_1^2 + \frac{(|D_1| + |D_0|)\,|\gamma|\,C_1}{2}.$$

3. Concluding Remark

Various choices of Ψ as mentioned above and suitably choosing the values of C_1 and C_2, we state some interesting results analogous to Theorem 2.1 and the Corollaries 2.1 to 2.3. For example, the function Ψ is given by

$$\Psi(z) = \left(\frac{1+z}{1-z}\right)^{\theta} = 1 + 2\theta z + 2\theta^2 z^2 + \cdots, \quad 0 < \theta \leq 1,$$

which gives

$$C_1 = 2\theta \text{ and } C_2 = 2\theta^2.$$

By taking

$$\Psi(z) = \frac{1 + (1-2\mu)\,z}{1-z} = 1 + 2\,(1-\mu)\,z + 2\,(1-\mu)\,z^2 + \cdots, \quad 0 \leq \mu < 1,$$

we have

$$C_1 = C_2 = 2\,(1-\mu)\,.$$

On the other hand, for $-1 \leq B \leq A < 1$, if we let

$$\Psi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + \cdots, \quad 0 < \theta \leq 1,$$

then we have

$$C_1 = (A-B) \text{ and } C_2 = -B(A-B).$$

The details involved may be left as an exercise for the interested reader.

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[1,2]Department of Mathematics,
Bursa Uludag University,
16059, Bursa, Turkey
Email address: sahsenealtinkaya@gmail.com
Email address: syalcin@uludag.edu.tr