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THE WEIGHTED HOLOMORPHIC INTERPOLATIVE IDEAL

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Abstract. Building upon the interpolative ideal procedure of Matter [13] for linear operators, we introduce the concept of interpolative ideals of a weighted holomorphic ideal $\mathcal{I}^{\mathcal{H}_v^\infty}$. For $\sigma \in [0,1)$, we prove that the resulting ideal $(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma$ is an injective weighted holomorphic ideal which is located between the injective hull and the closed injective hull of $\mathcal{I}^{\mathcal{H}_v^\infty}$. We apply this interpolation procedure to weighted holomorphic ideals generated by the methods of composition and dual. In particular, a domination theorem of Pietsch type is established for weighted holomorphic ideals induced by composition with p-summing operators ideals.

1. Introduction

Let E and F be complex Banach spaces and let U be an open subset of E. Let $\mathcal{H}(U,F)$ be the space of all holomorphic mappings from U into F. A weight v on U is a (strictly) positive continuous function. The space of weighted holomorphic mappings, denoted by $\mathcal{H}_v^\infty(U,F)$, is the Banach space of all mappings $f \in \mathcal{H}(U,F)$ such that

$$\|f\|_v := \sup\{v(x)\,\|f(x)\| : x \in U\} < +\infty,$$

under the weighted supremum norm $\|\cdot\|_v$. For simplicity, we write $\mathcal{H}_v^\infty(U)$ instead of $\mathcal{H}_v^\infty(U,\mathbb{C})$. For complete information about these function spaces, the interested reader can consult the survey by Bonet [2] and the references therein.

Recently, the study of weighted holomorphic ideals was initiated in [4] (see also [5] for the special case of bounded holomorphic ideals). This research was continued in [9] with the introduction of the so-called injective procedure to generate new weighted

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holomorphic ideals, embodied in the concepts of the injective hull and the closed injective hull of an ideal of weighted holomorphic mappings.

These studies find their motivation in the extensive literature on operator ideals, the starting point of which is the famous monograph [16] by Pietsch. Another procedure for generating operator ideals, closely related to the previous one, is the so-called interpolative procedure, which was introduced by Matter [13, 14] and later further developed by Jarchow and Matter [8]. This method has also proven to be very productive in other contexts to generate new function ideals, such as in [17] by Saleh and [18] by Achour, Dahia and Yahi for spaces of Lipschitz functions, in [10] by Manzano, Rueda and Sánchez-Pérez for spaces of multilinear operators and in [11] by Mastyło and [12] by Mastyło and Szwedek for Banach operator ideals.

Our aim in this paper is to extend Matter's interpolative procedure to the setting of $\mathcal{H}_v^\infty$-spaces. We now describe the content of this paper. Given a weighted holomorphic ideal $\mathcal{I}^{\mathcal{H}_v^\infty}$ and $\sigma \in [0, 1)$, we find that the interpolative ideal $(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma$ is an injective weighted holomorphic ideal that lies between the injective hull and the closed injective hull of $\mathcal{I}^{\mathcal{H}_v^\infty}$. This fact requires the application of a version for weighted holomorphic mappings, stated in [9], of a well-known description of the closed injective hull of an operator ideal in terms of an Ehrling-type inequality [6], given by Jarchow and Pelczyński in [7].

We describe the form of interpolative ideals of weighted holomorphic ideals generated by composition with an operator ideal. Using this description, we can also determine the form of the interpolative ideals of the dual weighted holomorphic ideal of an operator ideal $\mathcal{I}$. Interpolative ideals of weighted holomorphic ideals, induced by composition with the classical ideal of p-summing operators [15, 16], deserve special attention, since such interpolative ideals are characterized by a Pietsch-type domination property.

2. Results

By [4, Definition 2.4], a Banach weighted holomorphic ideal is an assignment $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ associating every pair (U, F), where E is a complex Banach space, U is an open subset of E and F is a complex Banach space, with a set $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F) \subseteq \mathcal{H}_v^\infty(U, F)$ equipped with a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} : \mathcal{I}^{\mathcal{H}_v^\infty}(U, F) \to \mathbb{R}_0^+$ that satisfies the following conditions.

- (P1) $(\mathcal{I}^{\mathcal{H}_v^\infty}(U, F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}})$ is a Banach space with $\|f\|_v \leq \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ for all $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$.
- (P2) Given $h \in \mathcal{H}_v^\infty(U)$ and $y \in F$, the map $h \cdot y \colon x \in U \mapsto h(x)y \in F$ is in $\mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$, with $\|h \cdot y\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h\|_v \|y\|$.
- (P3) [The ideal property] If V is an open subset of E such that $V \subseteq U$, $h \in \mathcal{H}(V, U)$, with $c_v(h) := \sup_{x \in V}(v(x)/v(h(x))) < +\infty$, $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, F)$ and $T \in \mathcal{L}(F, G)$, where G is a complex Banach space, then $T \circ f \circ h \in \mathcal{I}^{\mathcal{H}_v^\infty}(V, G)$, with $\|T \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|T\| \, \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} c_v(h)$.

A Banach weighted holomorphic ideal $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ is said to be:

(I) injective if for any map $f \in \mathcal{H}_v^\infty(U,F)$, any complex Banach space G and any into linear isometry $\iota\colon F \to G$, one has $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$, with $\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|\iota \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ whenever $\iota \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$.

We now introduce the method of interpolation in the setting of weighted holomorphic mappings.

Definition 2.1. Let U be an open subset of a complex Banach space E, let v be a weight on U and let F be a complex Banach space. For a Banach weighted holomorphic ideal $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ and $0 \leq \sigma < 1$, a map $f \in \mathcal{H}_v^\infty(U,F)$ belongs to $(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$ if there exists a complex Banach space G and a map $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$ such that

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma,$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. For each $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$, denote

$$\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} = \inf\left\{\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}\right\},$$

where the infimum is taken over all complex Banach spaces G and all maps $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$ satisfying the inequality above.

In the following, unless otherwise specified, E denotes a complex Banach space, U an open subset of E, v a weight on U and F a complex Banach space.

In this paper we will use the standard Banach space notation. The symbol $\mathcal{L}(E,F)$ denotes the Banach space of all bounded linear operators from E to F, equipped with the canonical norm of the operator. E^* and B_E represent the dual space and the closed unit ball of E, respectively.

Given Banach weighted holomorphic ideals $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ and $\left[\mathcal{J}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}\right]$, we write

$$\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right] \leq \left[\mathcal{J}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}_v^\infty}}\right]$$

to indicate that for any complex Banach space E, any open set $U \subseteq E$ and any complex Banach space F, we have $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F) \subseteq \mathcal{J}^{\mathcal{H}_v^\infty}(U,F)$ with $\|f\|_{\mathcal{J}^{\mathcal{H}_v^\infty}} \leq \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ for $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$.

Proposition 2.1. *Let* $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ *be a Banach weighted holomorphic ideal and* $0 \leq \sigma < 1$. *Then,* $\left[(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}\right]$ *is an injective Banach weighted holomorphic ideal such that*

$$\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right] \leq \left[(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}\right].$$

Proof. Through this proof, let $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$. Then, we can take a complex Banach space G and a map $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$ such that

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma,$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$.

(P1) In particular, we have

$$v(x) \|f(x)\| \leq (v(x) \|g(x)\|)^{1-\sigma} \leq \|g\|_v^{1-\sigma} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma},$$

for all $x \in U$. Hence, $f \in \mathcal{H}_v^\infty(U, F)$ with $\|f\|_v \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}$, and taking the infimum over all such G's and g's, we conclude that $\|f\|_v \leq \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$.

If $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} = 0$, then $\|f\|_v = 0$ by the preceding inequality, and thus $f = 0$. We will now prove the triangle inequality for $\|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. For $j = 1, 2$, take a map $f_j \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U, F)$, a complex Banach space G_j, and a map $g_j \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G_j)$ satisfying

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f_j(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g_j(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma,$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. Let $G = (G_1 \oplus G_2)_{\ell_1}$, and let $I_j \colon G_j \to G$ be the canonical injection. Clearly, $g = \sum_{j=1}^2 \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{-\sigma} (I_j \circ g_j)$ is in $\mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$ with $\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \sum_{j=1}^2 \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}$. An application of Hölder's inequality yields

$$\begin{aligned}
\left\|\sum_{i=1}^n \lambda_i v(x_i) \left(\sum_{j=1}^2 f_j\right)(x_i)\right\| &= \left\|\sum_{j=1}^2 \sum_{i=1}^n \lambda_i v(x_i) f_j(x_i)\right\| \\
&\leq \sum_{j=1}^2 \left\|\sum_{i=1}^n \lambda_i v(x_i) f_j(x_i)\right\| \\
&\leq \sum_{j=1}^2 \left\|\sum_{i=1}^n \lambda_i v(x_i) g_j(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma \\
&= \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma \sum_{j=1}^2 \left\|\sum_{i=1}^n \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{-\sigma} \lambda_i v(x_i) g_j(x_i)\right\|_{G_j}^{1-\sigma} \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{\sigma(1-\sigma)} \\
&\leq \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma \left(\sum_{j=1}^2 \left\|\sum_{i=1}^n \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{-\sigma} \lambda_i v(x_i) g_j(x_i)\right\|_{G_j}\right)^{1-\sigma} \\
&\quad \times \left(\sum_{j=1}^2 \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}\right)^\sigma \\
&= \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|_G^{1-\sigma} \left(\sum_{j=1}^2 \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}\right)^\sigma,
\end{aligned}$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. Thus, $\sum_{j=1}^2 f_j \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U, F)$ with

$$\left\|\sum_{j=1}^2 f_j\right\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \left(\sum_{j=1}^2 \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}\right)^\sigma \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \leq \sum_{j=1}^2 \|g_j\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}.$$

Passing to the infimum over all such G_1, G_2 and g_1, g_2 gives

$$\left\|\sum_{j=1}^{2} f_j\right\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le \sum_{j=1}^{2} \|f_j\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}.$$

Let $\lambda \in \mathbb{C}$. Clearly,

$$\left\|\sum_{i=1}^{n} \lambda_i v(x_i)(\lambda f)(x_i)\right\| \le \left\|\sum_{i=1}^{n} \lambda_i v(x_i)(\lambda^{\frac{1}{1-\sigma}} g)(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^{n} |\lambda_i|\right)^{\sigma}$$

and since $\lambda^{\frac{1}{1-\sigma}} g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$, we have $\lambda f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U, F)$, with

$$\|\lambda f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le \left\|\lambda^{\frac{1}{1-\sigma}} g\right\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} = |\lambda| \cdot \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}.$$

For $\lambda = 0$, we obtain $\|\lambda f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} = 0 = |\lambda| \, \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. For $\lambda \neq 0$, we deduce that $\|\lambda f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le |\lambda| \, \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. Hence, $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le |\lambda|^{-1} \|\lambda f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$, so, $|\lambda| \, \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le \|\lambda f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$, and thus, $\|\lambda f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} = |\lambda| \, \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. Therefore, $((\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U, F), \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma})$ is a normed space.

To prove its completeness, let $(f_m)_{m\ge1}$ be a sequence in $(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U, F)$ for which

$$\sum_{m=1}^{+\infty} \|f_m\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} < +\infty.$$

Since $\|\cdot\|_v \le \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$ on $(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U, F)$ and $(\mathcal{H}_v^\infty(U, F), \|\cdot\|_v)$ is a Banach space, there exists $f_0 \in \mathcal{H}_v^\infty(U, F)$ such that $\sum_{m=1}^{+\infty} f_m = f_0$ for $\|\cdot\|_v$. We will prove that $\sum_{m=1}^{+\infty} f_m = f_0$ for $\|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. Let $\varepsilon > 0$, and for each $m \in \mathbb{N}$, we can take a complex Banach space G_m and a map $g_m \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G_m)$ for which

$$\left\|\sum_{i=1}^{n} \lambda_i v(x_i) f_m(x_i)\right\| \le \left\|\sum_{i=1}^{n} \lambda_i v(x_i) g_m(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^{n} |\lambda_i|\right)^{\sigma},$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$, with

$$\|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \le \|f_m\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} + \frac{\varepsilon}{2^m}.$$

Then,

$$\sum_{m=1}^{+\infty} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \le \sum_{m=1}^{+\infty} \|f_m\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} + \varepsilon.$$

Let $g = \sum_{m=1}^{+\infty} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{-\sigma} (I_m \circ g_m) \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, G)$, where $G = (\oplus_{m=1}^{+\infty} G_m)_{\ell_1}$ and $I_m \colon G_m \to G$ is the natural inclusion. Hence, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \lambda_i v(x_i) \left(\sum_{m=1}^{+\infty} f_m \right)(x_i) \right\| &= \left\| \sum_{m=1}^{+\infty} \sum_{i=1}^{n} \lambda_i v(x_i) f_m(x_i) \right\| \\
&\le \sum_{m=1}^{+\infty} \left\| \sum_{i=1}^{n} \lambda_i v(x_i) f_m(x_i) \right\| \\
&\le \sum_{m=1}^{+\infty} \left\| \sum_{i=1}^{n} \lambda_i v(x_i) g_m(x_i) \right\|^{1-\sigma} \left(\sum_{i=1}^{n} |\lambda_i| \right)^{\sigma} \\
&= \left(\sum_{i=1}^{n} |\lambda_i| \right)^{\sigma} \sum_{m=1}^{+\infty} \left\| \sum_{i=1}^{n} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{-\sigma} \lambda_i v(x_i) g_m(x_i) \right\|_{G_m}^{1-\sigma} \\
&\quad \times \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{\sigma(1-\sigma)} \\
&\le \left(\sum_{i=1}^{n} |\lambda_i| \right)^{\sigma} \left(\sum_{m=1}^{+\infty} \left\| \sum_{i=1}^{n} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{-\sigma} \lambda_i v(x_i) g_m(x_i) \right\|_{G_m} \right)^{1-\sigma} \\
&\quad \times \left(\sum_{m=1}^{+\infty} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \right)^{\sigma} \\
&= \left(\sum_{i=1}^{n} |\lambda_i| \right)^{\sigma} \left\| \sum_{i=1}^{n} \lambda_i v(x_i) g(x_i) \right\|_{G}^{1-\sigma} \left(\sum_{m=1}^{\infty} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \right)^{\sigma},
\end{aligned}$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. This implies that $f_0 = \sum_{m=1}^{+\infty} f_m \in (\mathcal{I}^{\mathcal{H}_v^{+\infty}})_\sigma(U, F)$ with

$$\left\| \sum_{m=1}^{+\infty} f_m \right\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le \left(\sum_{m=1}^{+\infty} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \right)^{\sigma} \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \le \sum_{m=1}^{+\infty} \|g_m\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \le \sum_{m=1}^{+\infty} \|f_m\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} + \varepsilon,$$

and by letting ε approach zero, we obtain $\left\| \sum_{m=1}^{+\infty} f_m \right\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le \sum_{m=1}^{+\infty} \|f_m\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$.

Moreover, we have

$$\left\| f_0 - \sum_{k=1}^{m} f_k \right\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} = \left\| \sum_{k=m+1}^{+\infty} f_k \right\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \le \sum_{k=m+1}^{+\infty} \|f_k\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma},$$

for all $m \in \mathbb{N}$, and thus $\sum_{m=1}^{+\infty} f_m = f_0$ for $\|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$.

(P2) Let $h \in \mathcal{H}_v^\infty(U)$ and $y \in F$. Clearly, $h \cdot y \in \mathcal{H}_v^\infty(U, F)$ with $\|h \cdot y\|_v = \|h\|_v \|y\|$. Assume $h \neq 0$. By the ideal property of $\mathcal{I}^{\mathcal{H}_v^\infty}$, the function $g := h/\|h\|_v = h \cdot (1/\|h\|_v)$

is in $\mathcal{I}^{\mathcal{H}_v^\infty}(U,\mathbb{C})$. For all $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ and $x_1,\ldots,x_n \in U$, it holds that

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i)(h\cdot y)(x_i)\right\| &= \|h\|_v \|y\| \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma} \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{\sigma} \\ &\leq \|h\|_v \|y\| \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^{\sigma},\end{aligned}$$

and so $h\cdot y \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$ with $\|h\cdot y\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \|h\|_v||y||$. Since $\|h\|_v||y|| = \|h\cdot y\|_v \leq \|h\cdot y\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$, by (P1), we have that $\|h\cdot y\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} = \|h\|_v||y||$.

(P3) Let V be an open subset of E such that $V \subseteq U$, $h \in \mathcal{H}(V,U)$ with $c_v(h) < +\infty$ and $T \in \mathcal{L}(F,H)$, where H is a complex Banach space. We have

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i)(T\circ f\circ h)(x_i)\right\| &= \left\|T\left(\sum_{i=1}^n \lambda_i v(x_i)(f\circ h)(x_i)\right)\right\| \\ &\leq \|T\| \left\|\sum_{i=1}^n \lambda_i v(x_i)(f\circ h)(x_i)\right\| \\ &= \|T\| \left\|\sum_{i=1}^n \lambda_i \frac{v(x_i)}{v(h(x_i))} v(h(x_i)) f(h(x_i))\right\| \\ &\leq \|T\| \left\|\sum_{i=1}^n \lambda_i \frac{v(x_i)}{v(h(x_i))} v(h(x_i)) g(h(x_i))\right\|^{1-\sigma} \\ &\quad \times \left(\sum_{i=1}^n |\lambda_i| \frac{v(x_i)}{v(h(x_i))}\right)^{\sigma} \\ &\leq \|T\|\, c_v(h)^\sigma \left\|\sum_{i=1}^n \lambda_i v(x_i)(g\circ h)(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^{\sigma},\end{aligned}$$

for all $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ and $x_1,\ldots,x_n \in V$. Note that $g\circ h \in \mathcal{I}^{\mathcal{H}_v^\infty}(V,G)$, with $\|g\circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} c_v(h)$ due to the ideal property of $\mathcal{I}^{\mathcal{H}_v^\infty}$. Therefore, $T\circ f\circ h \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(V,H)$, with

$$\begin{aligned}\|T\circ f\circ h\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} &\leq \|T\|\, c_v(h)^\sigma \|g\circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \leq \|T\|\, c_v(h)^\sigma \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} c_v(h)^{1-\sigma} \\ &= \|T\|\ \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} c_v(h).\end{aligned}$$

Passing to the infimum over all such G's and g's, we deduce that $\|T\circ f\circ h\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \|T\|\cdot\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} c_v(h)$.

(I) Let $\iota\colon F \to G$ be an into linear isometry. Assume that $\iota \circ f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,G)$. Hence, there is a complex Banach space H and a map $h \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,H)$ such that

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| &= \left\|\iota\left(\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right)\right\| \\ &= \left\|\sum_{i=1}^n \lambda_i v(x_i)(\iota \circ f)(x_i)\right\| \\ &\leq \left\|\sum_{i=1}^n \lambda_i v(x_i) h(x_i)\right\|^{1-\sigma}\left(\sum_{i=1}^n |\lambda_i|\right)^\sigma,\end{aligned}$$

for all $n \in \mathbb{N}$, $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and $x_1,\dots,x_n \in U$, and thus $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$ with $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \|h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}$. By taking the infimum over all such H's and h's yields $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \|\iota \circ f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. The reverse inequality can be obtained by applying (P3).

Finally, we prove the last inequality in the statement. Let $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$. Given $n \in \mathbb{N}$, $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and $x_1,\dots,x_n \in U$, we have

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| &= \left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\|^{1-\sigma}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\|^{\sigma} \\ &\leq \left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\|^{1-\sigma}\|f\|_v^\sigma\left(\sum_{i=1}^n |\lambda_i|\right)^\sigma.\end{aligned}$$

Hence, $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$ with $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \|f\|_v^\sigma \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}$, and thus $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma} \leq \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$. □

Let $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ be a Banach weighted holomorphic ideal. By [9, Proposition 2.1], there exists a unique smallest injective Banach weighted holomorphic ideal, called the injective hull of $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ and denoted by $\left[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}\right]$, satisfying that

$$\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right] \leq \left[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}\right].$$

Note that for $\sigma = 0$, we have $[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}] = [(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}]$ by [9, Theorem 2.1].

Given a weighted holomorphic ideal $\mathcal{I}^{\mathcal{H}_v^\infty}$, a map $f \in \mathcal{H}_v^\infty(U,F)$ is said to belong to the closure of $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ in $(\mathcal{H}_v^\infty(U,F), \|\cdot\|_v)$, and it is denoted by $f \in \overline{\mathcal{I}^{\mathcal{H}_v^\infty}}(U,F)$ if there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ such that $\lim_{n\to+\infty}\|f_n - f\|_v = 0$. By [9, Proposition 2.4], $\overline{\mathcal{I}^{\mathcal{H}_v^\infty}}$ is a weighted holomorphic ideal containing $\mathcal{I}^{\mathcal{H}_v^\infty}$. The injective hull of the ideal $\overline{\mathcal{I}^{\mathcal{H}_v^\infty}}$ is denoted by $(\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}$ and is called the closed injective hull of $\mathcal{I}^{\mathcal{H}_v^\infty}$.

We now present a variant of a well-known result in the linear setting, attributed to Matter (see [13, Proposition 3.3]), adapted for weighted holomorphic mappings. We need to recall some facts about the linearization of weighted holomorphic maps.

Following [1, 3], $\mathcal{G}_v^\infty(U)$ is the space of all linear functionals on $\mathcal{H}_v^\infty(U)$ whose restriction to $B_{\mathcal{H}_v^\infty(U)}$ is continuous for the compact-open topology.

Theorem 2.1 ([1,3])**.** *Let U be an open set of a complex Banach space E and v be a weight on U.*

(*i*) *$\mathcal{G}_v^\infty(U)$ is a closed subspace of $\mathcal{H}_v^\infty(U)^*$ and the mapping $J_v\colon \mathcal{H}_v^\infty(U) \to \mathcal{G}_v^\infty(U)^*$, given by $J_v(f)(\phi) = \phi(f)$ for $\phi \in \mathcal{G}_v^\infty(U)$ and $f \in \mathcal{H}_v^\infty(U)$, is an isometric isomorphism.*

(*ii*) *For each $x \in U$, the functional $\delta_x\colon \mathcal{H}_v^\infty(U) \to \mathbb{C}$, defined by $\delta_x(f) = f(x)$ for $f \in \mathcal{H}_v^\infty(U)$, is in $\mathcal{G}_v^\infty(U)$.*

(*iii*) *The mapping $\Delta_v\colon U \to \mathcal{G}_v^\infty(U)$ given by $\Delta_v(x) = \delta_x$ is in $\mathcal{H}_v^\infty(U, \mathcal{G}_v^\infty(U))$ with $\|\Delta_v\|_v \le 1$.*

(*iv*) *$\mathcal{G}_v^\infty(U) = \overline{\mathrm{lin}}(\mathrm{At}_{\mathcal{G}_v^\infty(U)}) \subseteq \mathcal{H}_v^\infty(U)^*$, where $\mathrm{At}_{\mathcal{G}_v^\infty(U)} = \{v(x)\delta_x\colon x \in U\}$.*

(*v*) *For each $\phi \in \mathrm{lin}(\mathrm{At}_{\mathcal{G}_v^\infty(U)})$, we have*

$$\|\phi\| = \inf\left\{\sum_{i=1}^n |\lambda_i| : \phi = \sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i}\right\}.$$

(*vi*) *For every complex Banach space F and every mapping $f \in \mathcal{H}_v^\infty(U,F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}_v^\infty(U), F)$ such that $T_f \circ \Delta_v = f$. Furthermore, $\|T_f\| = \|f\|_v$.*

(*vii*) *For each $f \in \mathcal{H}_v^\infty(U,F)$, the mapping $f^t\colon F^* \to \mathcal{H}_v^\infty(U)$, defined by $f^t(y^*) = y^* \circ f$ for all $y^* \in F^*$, is in $\mathcal{L}(F^*, \mathcal{H}_v^\infty(U))$ with $||f^t|| = \|f\|_v$.* □

Theorem 2.2. *Let $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ be a Banach weighted holomorphic ideal.*

(*i*) *If $0 \le \sigma_1 \le \sigma_2 < 1$, then*

$$\left[(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1}}\right] \le \left[(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_2}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_2}}\right].$$

(*ii*) *If $0 \le \sigma < 1$, then*

$$\left[(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})^{inj}}\right] \le \left[(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma}}\right] \le \left[(\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}, \|\cdot\|_{(\overline{\mathcal{I}^{\mathcal{H}_v^\infty}})^{inj}}\right].$$

(*iii*) *If $0 \le \sigma_1, \sigma_2 < 1$, then*

$$\left[((\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1})_{\sigma_2}, \|\cdot\|_{((\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1})_{\sigma_2}}\right] \le \left[(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1+\sigma_2-\sigma_1\sigma_2}, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1+\sigma_2-\sigma_1\sigma_2}}\right].$$

Proof. (*i*) Let $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1}(U,F)$. Take a complex Banach space G and a map $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$ so that

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \le \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma_1} \left(\sum_{i=1}^n |\lambda_i|\right)^{\sigma_1},$$

for all $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $x_1, \dots, x_n \in U$. Since

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma_1} &= \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma_2+\sigma_2-\sigma_1} \\ &\le \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma_2} \|g\|_v^{\sigma_2-\sigma_1} \left(\sum_{i=1}^n |\lambda_i|\right)^{\sigma_2-\sigma_1},\end{aligned}$$

it follows that

$$\left\|\sum_{i=1}^{n}\lambda_i v(x_i)f(x_i)\right\| \leq \left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\|^{1-\sigma_2} \|g\|_v^{\sigma_2-\sigma_1}\left(\sum_{i=1}^{n}|\lambda_i|\right)^{\sigma_2}.$$

Hence, $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_2}(U,F)$ and $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_2}} \leq \|g\|_v^{\sigma_2-\sigma_1}\|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma_2} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma_1}$. Taking infimum over all such G's and g's, we conclude that $\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_2}} \leq \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1}}$.

(ii) The first inequality follows from Proposition 2.1 and [9, Proposition 2.1]. For the second, we will develop an argument used in the proof of [13, Proposition 3.3]. Firstly, if $\sigma = 0$, then the first inequality is in fact an equality by [9, Theorem 2.1], and the second follows from [9, Proposition 2.2 (2)].

Secondly, assume $0 < \sigma < 1$ and let $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma(U,F)$. Hence there is a complex Banach space G and a map $g \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$ such that

$$\left\|\sum_{i=1}^{n}\lambda_i v(x_i)f(x_i)\right\| \leq \left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\|^{1-\sigma}\left(\sum_{i=1}^{n}|\lambda_i|\right)^{\sigma},$$

for all $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ and $x_1,\ldots,x_n \in U$. Fix $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ and $x_1,\ldots,x_n \in U$ and suppose that $\lambda_i \neq 0$ for some $i \in \{1,\ldots,n\}$ (otherwise, there is nothing to prove as we will see at once), and define the function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\phi(\varepsilon) = \varepsilon^{-\frac{\sigma}{1-\sigma}}\left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\| + \varepsilon\sum_{i=1}^{n}|\lambda_i|, \quad \varepsilon > 0.$$

Clearly, ϕ is derivable and

$$\phi'(\varepsilon) = -\frac{\sigma}{1-\sigma}\varepsilon^{-\frac{1}{1-\sigma}}\left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\| + \sum_{i=1}^{n}|\lambda_i|, \quad \varepsilon > 0.$$

Moreover, $\phi'(\varepsilon_1) = 0$, and given $\varepsilon > 0$, it is easy to show that $\phi'(\varepsilon) \leq 0$ if and only if $\varepsilon \leq \varepsilon_1$, where

$$\varepsilon_1 = \left(\frac{\sigma}{1-\sigma}\frac{\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\|}{\sum_{i=1}^{n}|\lambda_i|}\right)^{1-\sigma}.$$

Therefore, $\phi(\varepsilon_1) \leq \phi(\varepsilon)$ for all $\varepsilon > 0$. An easy calculation gives

$$\phi(\varepsilon_1) = \frac{1}{\sigma^\sigma(1-\sigma)^{1-\sigma}}\left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\|^{1-\sigma}\left(\sum_{i=1}^{n}|\lambda_i|\right)^{\sigma},$$

and thus,

$$\frac{1}{\sigma^\sigma(1-\sigma)^{1-\sigma}}\left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\|^{1-\sigma}\left(\sum_{i=1}^{n}|\lambda_i|\right)^{\sigma} \leq \varepsilon^{-\frac{\sigma}{1-\sigma}}\left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\| + \varepsilon\sum_{i=1}^{n}|\lambda_i|,$$

for all $\varepsilon > 0$. Since $\sigma^\sigma(1-\sigma)^{1-\sigma} < 1$, it follows that

$$\left\|\sum_{i=1}^{n}\lambda_i v(x_i)f(x_i)\right\| \leq \varepsilon^{-\frac{\sigma}{1-\sigma}}\left\|\sum_{i=1}^{n}\lambda_i v(x_i)g(x_i)\right\| + \varepsilon\sum_{i=1}^{n}|\lambda_i|,$$

for all $\varepsilon > 0$. Now, by applying [9, Corollary 2.11], we deduce that $f \in \overline{(\mathcal{I}^{\mathcal{H}_v^\infty})}^{inj}$. Moreover, we claim that $\|f\|_{\overline{(\mathcal{I}^{\mathcal{H}_v^\infty})}^{inj}} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}$, and taking the infimum over all such G's and g's, we conclude that $\|f\|_{\overline{(\mathcal{I}^{\mathcal{H}_v^\infty})}^{inj}} \leq \|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_\sigma}$. In order to prove our claim, note that

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| &\leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma \\ &\leq \|g\|_v^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma \\ &\leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \sum_{i=1}^n |\lambda_i| .\end{aligned}$$

Taking the infimum over all representations of $\sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i}$, Theorem 2.1 (v) yields

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \left\|\sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i}\right\|.$$

In light of Theorem 2.1 (i) and Property (P2) in the definition of a weighted holomorphic ideal, the Hahn–Banach theorem now provides a function $h \in \mathcal{H}_v^\infty(U) = \mathcal{I}^{\mathcal{H}_v^\infty}(U, \mathbb{C})$ with $\|h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h\|_v \leq 1$ such that

$$\left\|\sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i}\right\| = \left|J_v(h)\left(\sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i}\right)\right| = \left|\sum_{i=1}^n \lambda_i v(x_i) h(x_i)\right|,$$

hence

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \left|\sum_{i=1}^n \lambda_i v(x_i) h(x_i)\right|.$$

Thus, [9, Theorem 2.1] gives $\|f\|_{\overline{(\mathcal{I}^{\mathcal{H}_v^\infty})}^{inj}} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma} \|h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|g\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-\sigma}$, as required.

(iii) Let $f \in ((\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1})_{\sigma_2}(U, F)$. Hence, there is a complex Banach space G and a map $g \in (\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1}(U, G)$ so that

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma_2} \left(\sum_{i=1}^n |\lambda_i|\right)^{\sigma_2},$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. Choose a complex Banach space H and a map $h \in \mathcal{I}^{\mathcal{H}_v^\infty}(U, H)$ such that

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\| \leq \left\|\sum_{i=1}^n \lambda_i v(x_i) h(x_i)\right\|^{1-\sigma_1} \left(\sum_{i=1}^n |\lambda_i|\right)^{\sigma_1},$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. We have

$$\begin{aligned}\left\|\sum_{i=1}^{n} \lambda_i v(x_i) f(x_i)\right\| &\leq \left\|\sum_{i=1}^{n} \lambda_i v(x_i) h(x_i)\right\|^{(1-\sigma_1)(1-\sigma_2)} \left(\sum_{i=1}^{n} |\lambda_i|\right)^{(1-\sigma_2)\sigma_1} \left(\sum_{i=1}^{n} |\lambda_i|\right)^{\sigma_2} \\ &= \left\|\sum_{i=1}^{n} \lambda_i v(x_i) h(x_i)\right\|^{1-(\sigma_1+\sigma_2-\sigma_1\sigma_2)} \left(\sum_{i=1}^{n} |\lambda_i|\right)^{\sigma_1+\sigma_2-\sigma_1\sigma_2}.\end{aligned}$$

Hence, $f \in (\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1+\sigma_2-\sigma_1\sigma_2}(U,F)$ and

$$\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1+\sigma_2-\sigma_1\sigma_2}} \leq \|h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{1-(\sigma_1+\sigma_2-\sigma_1\sigma_2)} = \|h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}^{(1-\sigma_1)(1-\sigma_2)}.$$

First, taking the infimum over all H's and h's, we obtain

$$\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1+\sigma_2-\sigma_1\sigma_2}} \leq \|g\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1}}^{1-\sigma_2}.$$

Next, passing to the infimum over all G's and g's, we conclude that

$$\|f\|_{(\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1+\sigma_2-\sigma_1\sigma_2}} \leq \|f\|_{((\mathcal{I}^{\mathcal{H}_v^\infty})_{\sigma_1})_{\sigma_2}}.$$

□

According to [4, Definition 2.5], given a Banach operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, a map $f \in \mathcal{H}_v^\infty(U,F)$ belongs to the composition ideal $\mathcal{I} \circ \mathcal{H}_v^\infty(U,F)$ if there exist a complex Banach space G, an operator $T \in \mathcal{I}(G,F)$ and a map $g \in \mathcal{H}_v^\infty(U,G)$ such that $f = T \circ g$. For $f \in \mathcal{I} \circ \mathcal{H}_v^\infty(U,F)$, we set

$$\|f\|_{\mathcal{I}\circ\mathcal{H}_v^\infty} = \inf\left\{\|T\|_{\mathcal{I}} \|g\|_v\right\},$$

by taking the infimum over all factorizations of f. By [4, Corollary 2.8], $\left[\mathcal{I} \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}\right]$ is a Banach weighted holomorphic ideal.

Our next purpose is to apply the interpolative procedure to the weighted holomorphic ideals of composition type.

Theorem 2.3. *Let* $\left[\mathcal{I}, \|\cdot\|_{\mathcal{I}}\right]$ *be a Banach operator ideal and* $0 \leq \sigma < 1$. *Then,*

$$\left[(\mathcal{I} \circ \mathcal{H}_v^\infty)_\sigma, \|\cdot\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)_\sigma}\right] = \left[\mathcal{I}_\sigma \circ \mathcal{H}_v^\infty, \|\cdot\|_{\mathcal{I}_\sigma\circ\mathcal{H}_v^\infty}\right].$$

Proof. Let $f \in (\mathcal{I} \circ \mathcal{H}_v^\infty)_\sigma(U,F)$. Take a complex Banach space G and a map $g \in \mathcal{I} \circ \mathcal{H}_v^\infty(U,G)$ such that

$$\left\|\sum_{i=1}^{n} \lambda_i v(x_i) f(x_i)\right\| \leq \left\|\sum_{i=1}^{n} \lambda_i v(x_i) g(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^{n} |\lambda_i|\right)^{\sigma},$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. In the light of Theorem 2.1 (*vi*), this means that

$$\left\|T_f\left(\sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i}\right)\right\| \leq \left\|T_g\left(\sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i}\right)\right\|^{1-\sigma} \left\|\sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i}\right\|^{\sigma},$$

for all $\sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i} \in \mathrm{lin}(\mathrm{At}_{\mathcal{G}_v^\infty(U)})$. Since $\mathcal{G}_v^\infty(U) = \overline{\mathrm{lin}}(\mathrm{At}_{\mathcal{G}_v^\infty(U)}) \subseteq \mathcal{H}_v^\infty(U)^*$, by Theorem 2.1 (*iv*), we deduce that $\|T_f(\phi)\| \leq \|T_g(\phi)\|^{1-\sigma} \|\phi\|^{\sigma}$ for all $\phi \in \mathcal{G}_v^\infty(U)$.

Taking into account that $T_g \in \mathcal{I}(\mathcal{G}_v^\infty(U), G)$ with $\|T_g\|_{\mathcal{I}} = \|g\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}$ by [4, Theorem 2.7], we obtain that $T_f \in \mathcal{I}_\sigma(\mathcal{G}_v^\infty(U), F)$ with $\|T_f\|_{\mathcal{I}_\sigma} \leq \|T_g\|_{\mathcal{I}}^{1-\sigma} = \|g\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}^{1-\sigma}$. Hence, $f \in \mathcal{I}_\sigma \circ \mathcal{H}_v^\infty(U, F)$ and $\|f\|_{\mathcal{I}_\sigma\circ\mathcal{H}_v^\infty} = \|T_f\|_{\mathcal{I}_\sigma} \leq \|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)_\sigma}$ by taking infimum over all such G's and g's.

Conversely, let $f \in \mathcal{I}_\sigma \circ \mathcal{H}_v^\infty(U, F)$. Hence, there exists a complex Banach space G, a map $g \in \mathcal{H}_v^\infty(U, G)$ and an operator $T \in \mathcal{I}_\sigma(G, F)$ such that $f = T \circ g$. Moreover, by [13, Definition 3.1], we can find a complex Banach space H and an operator $S \in \mathcal{I}(G, H)$ such that $\|T(y)\| \leq \|S(y)\|^{1-\sigma} \|y\|^\sigma$ for all $y \in G$. We have

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| &= \left\|T\left(\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right)\right\| \\ &\leq \left\|S\left(\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right)\right\|^{1-\sigma} \left\|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right\|^\sigma \\ &\leq \|g\|_v^\sigma \left\|\sum_{i=1}^n \lambda_i v(x_i)(S\circ g)(x_i)\right\|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma,\end{aligned}$$

for all $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $x_1, \dots, x_n \in U$. Since $S \circ g \in \mathcal{I} \circ \mathcal{H}_v^\infty(U, H)$, it follows that $f \in (\mathcal{I} \circ \mathcal{H}_v^\infty)_\sigma(U, F)$ and

$$\|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)_\sigma} \leq \|g\|_v^\sigma \|S \circ g\|_{\mathcal{I}\circ\mathcal{H}_v^\infty}^{1-\sigma} \leq \|g\|_v \|S\|_{\mathcal{I}}^{1-\sigma}.$$

Now, taking the infimum over all such H's and S's, we deduce that $\|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)_\sigma} \leq \|g\|_v \|T\|_{\mathcal{I}_\sigma}$. Finally, $\|f\|_{(\mathcal{I}\circ\mathcal{H}_v^\infty)_\sigma} \leq \|f\|_{\mathcal{I}_\sigma\circ\mathcal{H}_v^\infty}$ by taking the infimum over all factorizations of f as above. □

In view of Sections 4.4 and 8.2 in [16], given a Banach operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, the components

$$\mathcal{I}^{\mathrm{dual}}(E, F) := \{T \in \mathcal{L}(E, F) \colon T^* \in \mathcal{I}(F^*, E^*)\},$$

for any Banach spaces E and F, under the norm

$$\|T\|_{\mathcal{I}^{\mathrm{dual}}} = \|T^*\|_{\mathcal{I}}, \quad T \in \mathcal{I}^{\mathrm{dual}}(E, F),$$

define a Banach operator ideal denoted by $[\mathcal{I}^{\mathrm{dual}}, \|\cdot\|_{\mathcal{I}^d}]$ and called dual ideal of $\mathcal{I}$.

Similarly, according to [9, Definition 2.2] and using Theorem 2.1 (*vii*), the components

$$\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}(U, F) = \{f \in \mathcal{H}_v^\infty(U, F) \colon f^t \in \mathcal{I}(F^*, \mathcal{H}_v^\infty(U))\},$$

where E and F are complex Banach spaces, U is an open subset of E and v is a weight on U, under the norm

$$\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}} = \|f^t\|_{\mathcal{I}}, \quad f \in \mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}(U, F),$$

generate a Banach weighted holomorphic ideal, $\left[\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}}}\right]$, known as the dual weighted holomorphic ideal of $\left[\mathcal{I}, \|\cdot\|_{\mathcal{I}}\right]$.

Corollary 2.1. *Let* $\left[\mathcal{I}, \|\cdot\|_{\mathcal{I}}\right]$ *be a Banach operator ideal and* $0 \leq \sigma < 1$. *Assume that* $(\mathcal{I}^{\mathrm{dual}})_\sigma = (\mathcal{I}_\sigma)^{\mathrm{dual}}$ *(for sufficient conditions under which this equality holds, see* [8, p. 48]*). Then,*

$$\left[(\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}})_\sigma, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}})_\sigma}\right] = \left[(\mathcal{I}_\sigma)^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{(\mathcal{I}_\sigma)^{\mathcal{H}_v^\infty\text{-dual}}}\right].$$

Proof. Applying [9, Theorem 2.4], Theorem 2.3, and [9, Theorem 2.4] again, we obtain, respectively, that

$$\begin{aligned}
\left[(\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}})_\sigma, \|\cdot\|_{(\mathcal{I}^{\mathcal{H}_v^\infty\text{-dual}})_\sigma}\right] &= \left[(\mathcal{I}^{\mathrm{dual}} \circ \mathcal{H}_v^\infty)_\sigma, \|\cdot\|_{(\mathcal{I}^{\mathrm{dual}} \circ \mathcal{H}_v^\infty)_\sigma}\right] \\
&= \left[(\mathcal{I}^{\mathrm{dual}})_\sigma \circ \mathcal{H}_v^\infty, \|\cdot\|_{(\mathcal{I}^{\mathrm{dual}})_\sigma \circ \mathcal{H}_v^\infty}\right] \\
&= \left[(\mathcal{I}_\sigma)^{\mathrm{dual}} \circ \mathcal{H}_v^\infty, \|\cdot\|_{(\mathcal{I}_\sigma)^{\mathrm{dual}} \circ \mathcal{H}_v^\infty}\right] \\
&= \left[(\mathcal{I}_\sigma)^{\mathcal{H}_v^\infty\text{-dual}}, \|\cdot\|_{(\mathcal{I}_\sigma)^{\mathcal{H}_v^\infty\text{-dual}}}\right].
\end{aligned}$$

□

Let us recall [15,16] that an operator $T \in \mathcal{L}(E, F)$ is p-summing with $1 \leq p < +\infty$ if there exists a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^n \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{\frac{1}{p}},$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. The infimum of such constants C, denoted by $\pi_p(T)$ induces a complete norm on the linear space, $\Pi_p(E, F)$, of all p-summing operators from E into F.

Since $\mathcal{H}_v^\infty(U)$ is a dual Banach space by Theorem 2.1 (i), it can be equipped with its weak* topology. We denote by $\mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ the set of all Borel regular probability measures μ on $(B_{\mathcal{H}_v^\infty(U)}, w^*)$.

We now characterize the members of the interpolative hull of the composition ideal $\Pi_p \circ \mathcal{H}_v^\infty$ in terms of a domination inequality of Pietsch type and an inequality involved summability.

Corollary 2.2. *Let* $p \in [1, +\infty)$ *and* $\sigma \in [0, 1)$. *For* $f \in \mathcal{H}_v^\infty(U, F)$*, the following are equivalent.*

(i) $f \in (\Pi_p \circ \mathcal{H}_v^\infty)_\sigma(U, F)$.

(ii) *(Pietsch domination). There are a constant* $C \geq 0$ *and a measure* $\mu \in \mathcal{P}\left(B_{\mathcal{H}_v^\infty(U)}\right)$ *such that*

$$\left\|\sum_{i=1}^n \lambda_i v(x_i) f(x_i)\right\| \leq C \left(\int_{B_{\mathcal{H}_v^\infty(U)}} \left(\left|\sum_{i=1}^n \lambda_i v(x_i) g(x_i)\right|^{1-\sigma} \left(\sum_{i=1}^n |\lambda_i|\right)^\sigma\right)^{\frac{p}{1-\sigma}} d\mu(g)\right)^{\frac{1-\sigma}{p}},$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *and* $x_1, \ldots, x_n \in U$.

(iii) There exists a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}\lambda_i v(x_{ij})f(x_{ij})\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$
$$\leq C\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{j=1}^{m}\left|\sum_{i=1}^{n}\lambda_{ij}v(x_{ij})g(x_{ij})\right|^{1-\sigma}\left(\sum_{i=1}^{n}|\lambda_{ij}|\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}},$$

for all $n \in \mathbb{N}$, $1 \leq i \leq n$, $1 \leq j \leq m$, $\lambda_{ij} \in \mathbb{C}$ and $x_{ij} \in U$.

Furthermore, $\|f\|_{(\Pi_p\circ\mathcal{H}_v^\infty)_\sigma}$ is the infimum of the constants $C \geq 0$ for which, respectively, (ii) and (iii) hold.

Proof. An application of Theorem 2.3 and [4, Theorem 2.7] yields, respectively, that

$$f \in (\Pi_p \circ \mathcal{H}_v^\infty)_\sigma(U,F) \Leftrightarrow f \in (\Pi_p)_\sigma \circ \mathcal{H}_v^\infty(U,F) \Leftrightarrow T_f \in (\Pi_p)_\sigma(\mathcal{G}_v^\infty(U),F),$$

with $\|f\|_{(\Pi_p\circ\mathcal{H}_v^\infty)_\sigma} = \|f\|_{(\Pi_p)_\sigma\circ\mathcal{H}_v^\infty} = \|T_f\|_{(\Pi_p)_\sigma}$. Now, [13, Theorem 4.1] gives

$$T_f \in (\Pi_p)_\sigma(\mathcal{G}_v^\infty(U),F) \Leftrightarrow (ii) \Leftrightarrow (iii),$$

with $\|T_f\|_{(\Pi_p)_\sigma} = \inf\{C \geq 0 \text{ satisfying } (ii)\} = \inf\{C \geq 0 \text{ satisfying } (iii)\}$. □

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