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APPROXIMATION BY AN EXPONENTIAL-TYPE COMPLEX OPERATORS

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Abstract. In the present paper, we discuss the approximation properties of a complex exponential kind operator. Upper estimate, Voronovskaya-type formula and exact estimate are obtained.

1. Introduction

In the year 1978, Ismail [10] and Ismail and May [11] introduced and studied some exponential type operators. A type of the operators constructed in [11, (3.11)] is the following sequence

$$Q_n(f,x)=\int_0^\infty W(n,x,t)f(t)dt, \quad x\in(0,\infty),\, n\in\mathbb{N}, \tag{1.1}$$

where the kernel is given by

$$W(n,x,t)=\left(\frac{n}{2\pi}\right)^{1/2}\exp\left(n/x\right)t^{-3/2}\exp\left(-\frac{nt}{2x^2}-\frac{n}{2t}\right).$$

The kernel of these operators satisfies the partial differential equation

$$\frac{\partial}{\partial x}W(n,x,t)=\frac{n(t-x)}{x^3}W(n,x,t). \tag{1.2}$$

Due to its complicated behavior in integration, these operators were not previously much studied by researchers. Recently in case of real variables these operators were studied by Gupta [8], who established some direct results. The asymptotic formula for certain exponential type operators are discussed in [1].

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Also, in the recent years, the study of approximation by complex operators on compact disks is an active area of research, see for instance [2–4, 6, 7, 9] and [12] etc.

In this paper, we study the approximation properties of the complex variant in (1.1), obtained by replacing x with z in the formula (1.1). Section 2 contains some auxiliary results used in the next sections. Section 3 deals with upper estimate, while in Section 4, we study a Voronovskaya-type result and the exact estimate in approximation.

2. Auxiliary Results

The proofs of our main results require three additional lemmas, as follows.

Lemma 2.1. *If we denote $T_{n,m}(x) = Q_n(e_m, x)$, $e_m(t) = t^m$, then using Mapple, we find that $T_{n,0}(x) = 1$ and there holds the following recurrence relation:*

$$nT_{n,m+1}(x) = x^3[T_{n,m}(x)]' + nxT_{n,m}(x), \quad n, m \in \mathbb{N}.$$

In particular

$$\begin{aligned}
T_{n,0}(x) =&1,\\
T_{n,1}(x) =&x,\\
T_{n,2}(x) =&x^2 + \frac{x^3}{n},\\
T_{n,3}(x) =&x^3 + \frac{3x^4}{n} + \frac{3x^5}{n^2},\\
T_{n,4}(x) =&x^4 + \frac{6x^5}{n} + \frac{15x^6}{n^2},\\
T_{n,5}(x) =&x^5 + \frac{10x^6}{n} + \frac{45x^7}{n^2} + \frac{105x^8}{n^3} + \frac{105x^9}{n^4},\\
T_{n,6}(x) =&x^6 + \frac{15x^7}{n} + \frac{105x^8}{n^2} + \frac{420x^9}{n^3} + \frac{945x^{10}}{n^4} + \frac{945x^{11}}{n^5}.
\end{aligned}$$

Proof. By definition

$$T_{n,m}(x) \quad = \quad \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/x) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2x^2} - \frac{n}{2t}\right) t^m dt.$$

Thus, differentiating w.r.t x both the sides and using (1.2), we have

$$\begin{aligned}
x^3[T_{n,m}(x)]' =& \int_0^\infty x^3[W(n,x,t)]' t^m dt\\
=& \int_0^\infty n(t-x)W(n,x,t)t^m dt\\
=& nT_{n,m+1}(x) - nxT_{n,m}(x).
\end{aligned}$$

This completes the proof of lemma, other consequences follow from the recurrence relation. ∎

Lemma 2.2. *Suppose that $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, is an entire function satisfying the condition $|c_k| \leq M\frac{A^k}{k!}$, $k = 0, 1, \dots$, with $M > 0$ and $A \in (0, 1/2)$ (which implies that f is of exponential growth since $|f(z)| \leq M \exp(A|z|)$ for all $z \in \mathbb{C}$). Then $Q_n(f, z)$ is well defined for any $n \in \mathbb{N}$ and any $z \in \mathbb{C}$ satisfying*

$$\operatorname{Re}(z^2) > 0 \quad \textit{and} \quad \frac{|z|^2}{\operatorname{Re}(z^2)} < \frac{1}{2A}. \tag{2.1}$$

Proof. Since $|\exp(z)| = \exp(\operatorname{Re}(z))$, $\operatorname{Re}(1/z) = \operatorname{Re}(z)/|z|$ and $\operatorname{Re}(1/z^2) = \operatorname{Re}(z^2)/|z|^2$, we get

$$\begin{aligned}
&|Q_n(f, z)| \\
\leq &M\left(\frac{n}{2\pi}\right)^{1/2} |e(n/z)| \int_0^{\infty} t^{-3/2} \exp(-n/(2t) + At)| \exp(-nt/(2z^2))|dt \\
= &M \exp(n\operatorname{Re}(z)/|z|) \int_0^{\infty} t^{-3/2} \exp(-n/(2t)) \exp(-t[n\operatorname{Re}(z^2)/(2|z|^2) - A])dt.
\end{aligned}$$

By the hypothesis on z, we easily seen that $n\operatorname{Re}(z^2)/(2|z|^2) - A > 0$ for all $n \geq 1$. Therefore, for fixed z as in the hypothesis and denoting $n\operatorname{Re}(z^2)/(2|z|^2) - A$ with $C > 0$, we have to deal with the existence of the integral

$$I := \int_0^{\infty} t^{-3/2} \exp(-n/(2t)) \exp(-Ct)dt.$$

Changing the variable $t = \frac{1}{v}$, we easily obtain

$$I = \int_0^{\infty} v^{-1/2} \exp(-nv/2) \exp(-C/v)dv < \infty.$$

Indeed, for $K > 0$ an arbitrary fixed constant, we have

$$\begin{aligned}
I = &\int_0^{K} v^{-1/2} \exp(-nv/2) \exp(-C/v)dv + \int_K^{\infty} v^{-1/2} \exp(-nv/2) \exp(-C/v)dv \\
:= &I_1 + I_2,
\end{aligned}$$

where

$$I_1 \leq \int_0^K \exp(-nv/2)v^{-1/2}\frac{v}{C}dv \leq \frac{1}{C}\int_0^K v^{1/2} \exp(-nv/2)dv < \infty$$

and $I_2 \leq \frac{1}{\sqrt{K}} \int_K^{\infty} e(-nv/2)dv < \infty$. □

Lemma 2.3. *Suppose that f is an entire function, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{C}$ such that there exist $M > 0$ and $A \in (0, 1)$, with the property $|c_k| \leq M\frac{A^k}{k!}$ for all $k = 0, 1, \dots$ (which implies $|f(z)| \leq M \exp(A|z|)$ for all $z \in \mathbb{C}$).*

Then for all $n \in \mathbb{N}$ and z satisfying (2.1), *we have*

$$Q_n(f, z) = \sum_{k=0}^{\infty} c_k Q_n(e_k, z).$$

Proof. Since we can write

$$Q_n(f;z) = \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/z) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right) \left(\sum_{k=0}^\infty c_k t^k\right) dt,$$

if above the integral would commute with the infinite sum, then we would obtain

$$\begin{aligned} Q_n(f,z) =& \sum_{k=0}^\infty c_k \left(\frac{n}{2\pi}\right)^{1/2} \exp(n/z) \int_0^\infty t^{-3/2} \exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right) t^k dt \\ =& \sum_{k=0}^\infty c_k Q_n(e_k, z). \end{aligned}$$

It is well-known by the Fubini type result that a sufficient condition for the commutativity is that

$$\int_0^\infty t^{-3/2} \left|\exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right)\right| \left(\sum_{k=0}^\infty |c_k| t^k\right) dt < \infty.$$

Applied to our case, for $n \in \mathbb{N}$ and z satisfying (2.1), we get

$$\begin{aligned} &\int_0^\infty t^{-3/2} \left|\exp\left(-\frac{nt}{2z^2} - \frac{n}{2t}\right)\right| \left(\sum_{k=0}^\infty |c_k| t^k\right) dt \\ \leq& M \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t}\right) \exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)\right) \left(\sum_{k=0}^\infty \frac{A^k t^k}{k!}\right) dt \\ =& M \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t}\right) \exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2)\right) e^{At} dt \\ =& M \int_0^\infty t^{-3/2} \exp\left(-\frac{n}{2t}\right) \exp\left(-nt\mathrm{Re}\,(z^2)/(2|z|^2) + At\right) dt < \infty, \end{aligned}$$

by the proof of Lemma 2.2. □

Remark 2.1. It is easy to see that from geometric point of view, the conditions on z in (2.1) means that z belongs to two symmetric cones with respect to origin (but without containing the origin) containing the x axis, which are included in the two symmetric cones with respect to origin between the first and second bisectrix, containing the x axis. Indeed, since $|z|^2 = x^2 + y^2$ and $\mathrm{Re}\,(z^2) = x^2 - y^2$, simple calculations show that the condition (2.1) satisfied by $z = x + iy$ can easily be written under the form

$$\sqrt{\left(1 + \frac{1}{2A}\right)}|y| < \sqrt{\left(\frac{1}{2A} - 1\right)}|x|,$$

that is

$$\frac{|y|}{|x|} < \frac{\sqrt{1/(2A) - 1}}{\sqrt{1/(2A) + 1}} < 1.$$

3. Upper Estimate

The first main result concerns an upper estimate in approximation by $Q_n(f,z)$.

Theorem 3.1. *Suppose that f is an entire function, i.e., $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{C}$ such that there exist $M>0$ and $A\in(0,1/2)$, with the property $|c_k|\le M\frac{A^k}{k!}$, for all $k=0,1,\dots$ (which implies $|f(z)|\le Me^{A|z|}$ for all $z\in\mathbb{C}$). Consider $1\le r<\frac{1}{A}$.*

Then for all $n\ge r^2$, $|z|\le r$ and z satisfying (2.1), the following estimate hold:

$$|Q_n(f,z)-f(z)|\le\frac{C_{r,M,A}}{n},$$

where $C_{r,M,A}=Mr\sum_{k=2}^{\infty}(k+1)(Ar)^k<\infty$.

Proof. By Lemma 2.1 written with x replaced by z, we easily obtain

$$n[T_{n,m+1}(z)-z^{m+1}]=z^3[T_{n,m}(z)-z^m]'+nz[T_{m,n}(z)-z^m]+mz^{m+2}.$$

Applying the Bernstein's inequality on $|z|\le r$ to the polynomial of degree m, $T_{n,m}(z)-z^m$, we get $\|[T_{n,m}(z)-z^m]'\|_r\le\frac{m}{r}\|T_{n,m}(z)-z^m\|_r$, where $\|P\|_r=\sup_{|z|\le r}|P(z)|$. Then, denoting $e_m=z^m$, from the above recurrence we immediately obtain

$$\|T_{n,m+1}-e_{m+1}\|_r\le\left(r+\frac{mr^2}{n}\right)\|T_{m,n}-e_m\|_r+\frac{mr^{m+2}}{n}.$$

In what follows we prove by mathematical induction with respect to m that for $n\ge r^2$, this recurrence implies

$$||T_{n,m}-e_m||_r\le\frac{(m+1)!}{n}r^{m+1},\quad\text{for all } m\ge 0.$$

Indeed for $m=0$ and $m=1$ it is trivial, as the left-hand side is zero. Suppose that it is valid for m, the above recurrence relation implies that

$$||T_{n,m+1}-e_{m+1}||_r\le\left(r+\frac{r^2m}{n}\right)\frac{(m+1)!}{n}r^{m+1}+\frac{m}{n}r^{m+2}.$$

It remains to prove that

$$\left(r+\frac{r^2m}{n}\right)\frac{(m+1)!}{n}r^{m+1}+\frac{m}{n}r^{m+2}\le\frac{(m+2)!}{n}r^{m+2},$$

or after simplifications, equivalently to

$$\left(r+\frac{r^2m}{n}\right)(m+1)!+rm\le(m+2)!r,$$

for all $m\in\mathbb{N}$ and $r\ge 1$.

Since $n\ge r^2$, we get

$$\left(r+\frac{r^2m}{n}\right)(m+1)!+rm\le(r+m)\,(m+1)!+rm,$$

it is good enough if we prove that

$$(r+m)\,(m+1)!+rm \leq (m+2)!r.$$

But this last inequality is obviously equivalent with

$$m(m+1)!+rm \leq rm(m+1)!+r(m+1)!,$$

which is clearly valid for all $m \geq 1$ (and fixed $r \geq 1$).

Finally, taking into account Lemma 2.3, for all $n \geq r^2$, we obtain

$$\begin{aligned}|Q_n(f,z)-f(z)| &\leq \sum_{k=0}^{\infty}|c_k|\cdot|Q_n(e_k,z)-e_k(z)|\\ &\leq \frac{M}{n}\cdot\sum_{k=2}^{\infty}\frac{A^k}{k!}\cdot(k+1)!r^{k+1}=\frac{C_{r,M,A}}{n},\end{aligned}$$

where $C_{r,M,A}=Mr\sum_{k=2}^{\infty}(k+1)(Ar)^k<\infty$. □

Remark 3.1. The smaller A is, the larger is the portion of the symmetrical cones where the estimation in Theorem 3.1 takes place. This happens because of the intersection between the symmetrical cones and the disk $\{|z|\leq r\}$ with $1\leq r<\frac{1}{A}$, where if $A\searrow 0$ then $r\nearrow\infty$.

4. Voronovskaya Type Formula and Exact Estimate

The following estimate is a Voronovskaja-kind quantitative result.

Theorem 4.1. *Suppose that f is an entire function, i.e., $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{C}$ such that there exist $M>0$ and $A\in(0,1/2)$, with the property $|c_k|\leq M\frac{A^k}{k!}$, for all $k=0,1,\dots$ (which implies $|f(z)|\leq M\exp(A|z|)$ for all $z\in\mathbb{C}$). Consider $1\leq r<\frac{1}{A}$.*

Then for all $n\geq r^2$, $|z|\leq r$ and z satisfying (2.1)*, the following estimate holds:*

$$\left|Q_n(f,z)-f(z)-\frac{z^3f''(z)}{2n}\right|\leq\frac{E_{r,M,A}(f)}{n^2},$$

where

$$E_{r,M,A}(f)=3Mr^2\sum_{k=2}^{\infty}(k+1)^2(Ar)^k<\infty.$$

Proof. Everywhere in the proof consider z and n as in hypothesis.

By the proof of Lemma 2.3, we can write $Q_n(f,z)=\sum_{k=0}^{\infty}c_k\,Q_n(e_k,z)$. Also, since

$$\frac{z^3f''(z)}{2n}=\frac{z^3}{2n}\sum_{k=2}^{\infty}c_kk(k-1)z^{k-2}=\frac{1}{2n}\sum_{k=2}^{\infty}c_k\,k(k-1)\,z^{k+1},$$

we get

$$\left|Q_n(f,z)-f(z)-\frac{z^3\,f''(z)}{2n}\right|\leq\sum_{k=2}^{\infty}|c_k|\left|T_{n,k}(z)-e_k(z)-\frac{k(k-1)z^{k+1}}{2n}\right|.$$

By Lemma 2.1, we have

$$T_{n,k}(z) = \frac{z^3}{n}\, T'_{n,k-1}(z) + z\, T_{n,k-1}(z).$$

If we denote

$$J_{n,k}(z) = T_{n,k}(z) - e_k(z) - \frac{k(k-1)z^{k+1}}{2n},$$

then it is obvious that $J_{n,k}(z)$ is a polynomial of degree less than or equal to $k+2$ and by simple computation and the use of above recurrence relation, we are led to

$$J_{n,k}(z) \;=\; \frac{z^3}{n} J'_{n,k-1}(z) + zJ_{n,k-1}(z) + X_{n,k}(z),$$

where after simple computation, we have

$$X_{n,k}(z) \;=\; \frac{k(k-1)(k-2)z^{k+2}}{2n^2}.$$

Using the estimate in the proof of Theorem 3.1, we have

$$|T_{n,k}(z) - e_k(z)| \le \frac{(k+1)!}{n} \cdot r^{k+1}.$$

It follows

$$|J_{n,k}(z)| \le \frac{r^3}{n}\, |J'_{n,k-1}(z)| + r\, |J_{n,k-1}(z)| + |X_{n,k}(z)|,$$

where

$$|X_{n,k}(z)| \le \frac{k(k-1)(k-2)r^{k+2}}{2n^2}.$$

Now we shall find the estimation of $|J'_{n,k-1}(z)|$. Taking into account the fact that $J_{n,k-1}(z)$ is a polynomial of degree $\le k+1$, we have

$$\begin{aligned}
|J'_{n,k-1}(z)| \le& \frac{k}{r} ||J_{n,k-1}(z)||_r \\
\le& \frac{k}{r}\left[||T_{n,k-1}(z) - e_{k-1}(z)||_r + \frac{(k-1)(k-2)r^k}{2n} \right] \\
\le& \frac{(k+1)!}{n} \cdot r^{k-1} + \frac{k(k-1)(k-2)r^{k-1}}{2n}.
\end{aligned}$$

Thus,

$$\frac{r^3}{n} |J'_{n,k-1}(z)| \le \frac{1}{n}\left[\frac{(k+1)!}{n} r^{k+2} + \frac{k(k-1)(k-2)r^{k+2}}{2n} \right]$$

and

$$\begin{aligned}
|J_{n,k}(z)| \le& r|J_{n,k-1}(z)| + \frac{1}{n}\left[\frac{(k+1)!}{n} r^{k+2} + \frac{k(k-1)(k-2)r^{k+2}}{2n} \right] \\
&+ \frac{k(k-1)(k-2)r^{k+2}}{2n^2}.
\end{aligned}$$

This immediately implies

$$|J_{n,k}(z)| \leq r|J_{n,k-1}(z)| + \frac{3}{n^2}(k+1)!r^{k+2}.$$

By writing this inequality for $k = 1, 2, 3, \ldots$, we easily obtain step by step the following

$$|J_{n,k}(z)| \leq \frac{3}{n^2} r^{k+2} \left[\sum_{j=1}^{k+1} j!\right] \leq \frac{3}{n^2} r^{k+2}(k+1)!(k+1).$$

In conclusion,

$$\begin{aligned}\left|Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n}\right| \leq & \frac{3}{n^2} \cdot \sum_{k=2}^{\infty} |c_k| r^{k+2} \cdot (k+1)!(k+1) \\ \leq & \frac{3Mr^2}{n^2} \cdot \sum_{k=2}^{\infty} (k+1)^2 (Ar)^k.\end{aligned}$$

This completes the proof of theorem. □

Using the above Voronovskaja's theorem, we obtain the following lower order in approximation.

Theorem 4.2. *Under the hypothesis in Theorem* 4.1*, if f is not a polynomial of degree ≤ 1, then for all $n \geq r^2$ we have*

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{K_{r,M,A}(f)}{n},$$

where $\|F\|_r^* = \sup\{|F(z)| : |z| \leq r$ and z satisfies (2.1)$\}$ and $K_{r,M,A}(f)$ is a constant which depends only on f, M, A and r.

Proof. For all $n \geq r^2$, $|z| \leq r$ and z satisfying (2.1), we have

$$Q_n(f,z) - f(z) = \frac{1}{n}\left[0.5\, z^3 f''(z) + \frac{1}{n}\left\{n^2\left(Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n}\right)\right\}\right].$$

Also, we have

$$||F + G||_r^* \geq |||F||_r^* - ||G||_r^*| \geq ||F||_r^* - ||G||_r^*.$$

It follows

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{1}{n}\left[||\, 0.5\, e_3 f''\, ||_r^* - \frac{1}{n}\left\{n^2 \left\|Q_n(f,\cdot) - f - \frac{e_3 f''}{2n}\right\|_r^*\right\}\right].$$

Taking into account that by hypothesis, f is not a polynomial of degree ≤ 1, we get $||0.5\, e_3 f''||_r^* > 0$. Indeed, supposing the contrary it follows that $z^3 f''(z) = 0$, which by the fact that f is entire function, clearly implies $f''(z) = 0$, i.e., f is a polynomial of degree ≤ 1, a contradiction with the hypothesis.

Now by Theorem 4.1, we have

$$n^2 \left\|Q_n(f,z) - f(z) - \frac{z^3 f''(z)}{2n}\right\|_r^* \leq E_{r,M,A}(f).$$

Therefore, there exists an index n_0 depending only on f and r, such that for all $n \geq n_0$, we have

$$|| 0.5\, e_3\, f'' ||_r^* - \frac{1}{n}\left\{n^2 \left\|Q_n(f,z) - f(z) - \frac{0.5\, z^3\, f''(z)}{n}\right\|_r^*\right\} \geq \frac{1}{2}\; || 0.5 e_3\, f'' ||_r^*\,,$$

which immediately implies

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{1}{2n} || 0.5\, e_3\, f'' ||_r^*, \quad \text{for all } n \geq n_0.$$

For $n \in \{1, 2, \ldots, n_0 - 1\}$ we obviously have

$$||Q_n(f,\cdot) - f||_r^* \geq \frac{M_{r,n}(f)}{n},$$

with $M_{r,n}(f) = n||Q_n(f,\cdot) - f||_r^* > 0$. Indeed, if we would have $||Q_n(f,\cdot) - f||_r^* = 0$, then would follow $Q_n(f,z) = f(z)$ for all $|z| \leq r$, z satisfying (2.1), which is valid only for f a polynomial of degree ≤ 1, contradicting the hypothesis on f. Hence, we obtain $||Q_n(f,\cdot) - f||_r^* \geq \frac{K_{r,M,A}(f)}{n}$ for all n, where

$$K_{r,M,A}(f) = \min\left\{M_{r,1}(f), M_{r,2}(f), \ldots, M_{r,n_0-1}(f),\ \frac{1}{2}\; || 0.5\, e_3\, f'' ||_r^*\right\},$$

which completes the proof. □

Combining Theorem 3.1 with Theorem 4.2, we immediately get the following exact estimate.

Corollary 4.1. *Under the hypothesis in Theorem* 4.1*, if f is not a polynomial of degree ≤ 1, then we have*

$$||Q_n(f,\cdot) - f||_r^* \sim \frac{1}{n}, \quad n \in \mathbb{N},$$

where the symbol $\sim$ represents the well-known equivalence between the orders of approximation.

Remark 4.1. Particular cases of the exponential-type operators studied in the real case in [11], are the Bernstein polynomials, the operators of Szász, of Post-Widder, of Gauss-Weierstrass, of Baskakov, to mention only a few. In the complex variable case, only the approximation properties of the operators of Bernstein, Szász, Baskakov and Post-Widder were already studied, see, e.g., [5, 7, 9]. It remains as open question to use the method in this paper for other complex exponential-type operators, too.

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