Kragujevac Journal of Mathematics
Volume 51(9) (2027), Pages 1481–1503.

POSITIVITY AND BOUNDEDNESS PRESERVING EULER-MARUYAMA METHOD FOR STOCHASTIC FOOT AND MOUTH DISEASE MODEL

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Abstract. In this paper, motivated by the need to control the spread of the foot and mouth disease, the stochastic model describing the dynamics of the disease within certain population is considered. The model is determined by the system of four stochastic differential equations with locally Lipschitz continuous coefficients, and for such system the existence, uniqueness and boundedness of the global solution is proven. Also, corresponding positivity and boundedness preserving Euler-Maruyama method is considered, with the emphasis on the L^θ-convergence of the approximate solutions to the exact solution of the system, where $\theta > 0$. The theoretical findings are followed by an example and numerical simulations.

1. Introduction

It is well-known that many real-life phenomena are modeled by stochastic differential equations. One of them is foot and mouth disease (FMD), which has been attracting the attention of numerous researchers. Some of the papers where different mathematical models (deterministic and stochastic) of the FMD are studied are [1, 5, 8, 10–18].

On the basis of the detailed review of the FMD in [1], it was in Venice in 1514 that an Italian monk Hieronymus Fracastorius first explained the possible FMD among cattle. The victims of this disease could not eat, and their mouths and feet had vesicles. These five hundred years-old descriptions of the FMD have close similarity

Key words and phrases. Foot and mouth disease, existence and uniqueness of the global solution, positivity and boundedness of the exact solution, positivity and boundedness preserving Euler-Maruyama method, L^θ-convergence.

2020 *Mathematics Subject Classification.* Primary: 60H10.

DOI

Received: January 26, 2026.

Accepted: August 28, 2026.

with the present state of the same disease. This disease is deemed to be the most common one among split hoof animals, as it affects 70 species of split hoof animals such as goats and sheep. The disease was ubiquitous in areas where livestock used to be kept. The disease still exists in more than 100 countries, and its spread reflects economic progress. Although most of the countries have eradicated this disease, its spread in new countries, which were previously disease-free, could result in huge financial losses. A single-stranded positive-sense RNA virus is responsible for this disease known as FMD virus. The FMD virus is known as aphthous virus from the family of picornaviridae. In that sense, the goal of the present paper is to describe the virus and to classify the FMD outbreaks, such that this information would help explain the present existence of the disease all over the world, bearing in mind that the FMD progressive control pathway helps in checking the spread of disease in affected countries. More data on the spread, controlling measures and financial consequences of the FMD can be found in [1].

The subject of this paper is stochastic FMD model, which is described by the system of four stochastic differential equations and can be found in [1]. First, we will prove the existence and uniqueness of the global positive and bounded solution of that system. However, since it could not be solved explicitly, we will propose the numerical approximate method, called positivity and boundedness preserving Euler-Maruyama method. The main result of that part of the paper will be the L^θ-convergence theorem for $\theta > 0$. It should be emphasized that positivity and boundedness preserving methods play an important role when we deal with positive and/or bounded exact solutions of stochastic differential equations. Some of the papers where different positivity and/or boundedness preserving methods are considered and applied to various models are [2, 4, 6, 7]. The results from the paper [2] are of great importance for obtaining the results in the present paper, while the structure of the stochastic FDM model requires certain modification of the approach from the cited paper.

The results of the present paper are organized in four sections. In Introduction some historical facts on the FMD are given, together with the motivation for consideration in other sections. In Section 2, we establish the existence and uniqueness of the global positive and bounded solution of this system. Then, in Section 3, we define the corresponding positivity and boundedness preserving Euler-Maruyama approximate solution and prove that it converges to the exact solution of the system in the L^θ-sense, where $\theta > 0$. Finally, in Section 4, we deal with an example and corresponding numerical simulations.

2. Existence and Uniqueness

In this section we consider the existence and uniqueness of the global positive and bounded solution of the system of stochastic differential equations describing the stochastic FMD model from [1]. In that sense, we first introduce the basic notation. Let at any arbitrary time $t \in [0, T]$, where $T > 0$, the variables be stated as follows: $S(t)$ represents the susceptible group, $V(t)$ represents the vaccinated group,

$L(t)$ represents the latently infected group, and $I(t)$ represents the infectious group, whereas $N(t)$ represents the population. Here, μ denotes the birth and death rate of susceptible animals, β denotes the bilinear mass action, ω denotes the rate of vaccine wanes, $\emptyset$ denotes the vaccinated susceptible animals, δ denotes the infected animals, γ denotes the rate of progression from the latent stage to the infection stage, α denotes the culled infectious animals, d denotes the additional disease-related mortality. We will consider the following system of stochastic differential equations which represents the stochastic FMD model, introduced in [1]:

$$dS(t) = (\mu N - \beta I(t)S(t) - (\mu + \omega)S(t) + \emptyset V(t))dt + \sigma_1 S(t)dB^1(t), \tag{2.1}$$

$$dV(t) = (\omega S(t) - (\mu + \emptyset)V(t))dt + \sigma_2 V(t)dB^2(t), \tag{2.2}$$

$$dL(t) = (\beta I(t)S(t) - (\mu + \delta + \gamma)L(t))dt + \sigma_3 L(t)dB^3(t), \tag{2.3}$$

$$dI(t) = (\gamma L(t) - (\mu + \alpha + d)I(t))dt + \sigma_4 I(t)dB^4(t), \tag{2.4}$$

satisfying the initial conditions $S(0) = S_0 > 0$, $V(0) = V_0 > 0$, $L(0) = L_0 > 0$, $I(0) = I_0 > 0$, where $S_0 + V_0 + L_0 + I_0 \leq N$. Moreover, $B(t) = (B^1(t), B^2(t), B^3(t), B^4(t))^T$ is a 4-dimensional standard Brownian motion with pairwise independent component processes on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, while $\sigma_i > 0$, $i = 1, 2, 3, 4$. Additionally, $\mu > 0$, $\beta > 0$, $\omega > 0$, $\emptyset > 0$, $\delta > 0$, $\gamma > 0$, $\alpha > 0$, $d > 0$.

Our goal in this section is to establish the existence, uniqueness and positivity of the global solution of the system (2.1)–(2.4). In that sense, let us define the set

$$S = \lim_{n\to+\infty} S_n = \bigcup_{n=1}^{+\infty} S_n = (0, +\infty) \times (0, +\infty) \times (0, +\infty) \times (0, +\infty),$$

where

$$S_n = \left\{(s, v, l, i) : \min\{s, v, l, i\} > \frac{1}{n}, \max\{s, v, l, i\} < n\right\}. \tag{2.5}$$

There exists large enough $n_0 \in N$, such that $(S_0, V_0, L_0, I_0) \in S_{n_0}$. Moreover, define the sequence of stopping times $\{\tau_n, n \geq n_0\}$, where

$$\tau_n = \inf\{t \in [0, \tau_e \wedge T) : (S(t), V(t), L(t), I(t)) \notin S_n\}, \quad n \geq n_0, \tag{2.6}$$

such that $\inf \emptyset = +\infty$, while τ_e is the explosion time. Define $\tau_\infty = \lim_{n\to+\infty} \tau_n$, while $\tau_\infty \leq \tau_e$ a.s.

Now, we are in a position to state the main result of this section.

Theorem 2.1. *There exists unique global solution of the system* (2.1)–(2.4) *in*

$$S \cap \{(s, v, l, i) : s + v + l + i \leq N\}.$$

Proof. First, we will prove the existence and uniqueness of the positive global solution of the stochastic system (2.1)–(2.4). In that sense, observe that the drift and diffusion

coefficients F and G of the stochastic system (2.1)–(2.4), given by

$$F(s,v,l,i) = \begin{pmatrix} \mu N - \beta is - (\mu+\omega)s + \emptyset \\ \omega s - (\mu+\emptyset)v \\ \beta is - (\mu+\delta+\gamma)l \\ \gamma l - (\mu+\alpha+d)i \end{pmatrix}, \tag{2.7}$$

$$G(s,v,l,i) = \begin{pmatrix} \sigma_1 s & 0 & 0 & 0 \\ 0 & \sigma_2 v & 0 & 0 \\ 0 & 0 & \sigma_3 l & 0 \\ 0 & 0 & 0 & \sigma_4 i \end{pmatrix}, \tag{2.8}$$

satisfy the local Lipschitz condition. So, there exists unique local maximal solution $\{(S(t),V(t),L(t),I(t)),\ t \in [0,\tau_e)\}$. Our next goal is to prove that there exists a unique S-valued global solution of the system under consideration.

Let $C^2(\mathbb{R}^4_+;\mathbb{R}_+)$ denote the family of all nonnegative functions $f(x)$ defined on $\mathbb{R}^4_+$ which are continuously twice differentiable in $x \in \mathbb{R}^4_+$. Then, define $U \in C^2(\mathbb{R}^4_+,\mathbb{R}_+)$ as

$$U(s,v,l,i) = \sum_{x\in\{s,v,l,i\}} (x - 1 - \ln x), \quad s,v,l,i > 0. \tag{2.9}$$

Using the fact that $x - 1 - \ln x \geq 0,\ \ x > 0$, we observe that

$$\lim_{x\to 0^+}(x - 1 - \ln x) = \lim_{x\to+\infty}(x - 1 - \ln x) = +\infty.$$

Thus, $U(s,v,l,i) \to +\infty$ whenever at least one of the components s,v,l,i approaches zero or infinity. Moreover,

$$U'(s,v,l,i) = \left(1 - \frac{1}{s}, 1 - \frac{1}{v}, 1 - \frac{1}{l}, 1 - \frac{1}{i}\right)^T, \quad U"(s,v,l,i) = \begin{pmatrix} \frac{1}{s^2} & 0 & 0 & 0 \\ 0 & \frac{1}{v^2} & 0 & 0 \\ 0 & 0 & \frac{1}{l^2} & 0 \\ 0 & 0 & 0 & \frac{1}{i^2} \end{pmatrix}.$$

Furtheron, define an operator $LU : S \to R$, as

$$LU(s,v,l,i) = U'(s,v,l,i)F(s,v,l,i) + \frac{1}{2}\text{trace}\left[G^T(s,v,l,i)U''(s,v,l,i)G(s,v,l,i)\right],$$

whenever $(s,v,l,i) \in S$, where F and G are of the form (2.7) and (2.8), respectively. Then, we have that

$$\begin{aligned} LU(s,v,l,i) =& \left(1 - \frac{1}{s}\right)(\mu N - \beta is - (\mu+\omega)s + \emptyset v) + \left(1 - \frac{1}{v}\right)(\omega s - (\mu+\emptyset)v) \\ &+ \left(1 - \frac{1}{l}\right)(\beta is - (\mu+\delta+\gamma)l) + \left(1 - \frac{1}{i}\right)(\gamma l - (\mu+\alpha+d)i) \\ &+ \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 \\ =& \mu N - \mu s - \frac{\mu N}{s} + \beta i + 4\mu + \omega - \frac{\emptyset v}{s} - \mu v - \frac{\omega s}{v} + \emptyset - (\mu+\delta)l - \frac{\beta is}{l} \end{aligned} \tag{2.10}$$

$$+\delta+\gamma-(\mu+\alpha+d)i-\frac{\gamma l}{i}+\alpha+d+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2).$$

Then, we obtain that

$$LU(s,v,l,i)\le \beta i+K, \tag{2.11}$$

where

$$K=\mu N+4\mu+\omega+\emptyset+\delta+\gamma+\alpha+d+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2).$$

Define

$$W(s,v,l,i,t)=e^{-2\beta t}U(s,v,l,i),\quad (s,v,l,i)\in S,$$

and observe that

$$W(s,v,l,i)\ge e^{-2\beta T}U(s,v,l,i)\to+\infty \tag{2.12}$$

whenever at list one of the components s,v,l,i approaches zero or infinity. Then, on the basis of (2.11), we have that

$$\begin{aligned}LW(s,v,l,i,t)&=-2\beta e^{-2\beta t}U(s,v,l,i)+e^{-2\beta t}LU(s,v,l,i)\\&\le -2\beta e^{-2\beta t}U(s,v,l,i)+e^{-2\beta t}(\beta i+K)\\&\le K-\beta e^{-2\beta t}(2U(s,v,l,i)-i).\end{aligned}$$

Observe that

$$\begin{aligned}2U(s,v,l,i)-i&=2(s-1-\ln s)+2(v-1-\ln v)+2(l-1-\ln l)+i-2-2\ln i\\&\ge i-2-2\ln i\ge -2\ln 2,\end{aligned}$$

such that

$$LW(s,v,l,i,t)\le K+2\beta\ln 2e^{-2\beta t}\le K_1, \tag{2.13}$$

where $K_1=K+2\beta\ln 2$.

Applying the Itô formula and taking the expectation, we obtain from (2.13) that, for any $t\in[0,T]$,

$$EW(S(t\wedge\tau_n),V(t\wedge\tau_n),L(t\wedge\tau_n),I(t\wedge\tau_n),t)\le W\Big(s_0,v_0,l_0,i_0,0\Big)+K_1T. \tag{2.14}$$

Particularly,

$$EW(S(T\wedge\tau_n),V(T\wedge\tau_n),L(T\wedge\tau_n),I(T\wedge\tau_n),t)\le W\Big(s_0,v_0,l_0,i_0,0\Big)+K_1T. \tag{2.15}$$

Let us define

$$\eta_n=\inf_{(s,v,l,i,t)\in S_n^C\times[0,T]}W(s,v,l,i,t),\quad n\ge n_0.$$

In view of (2.15), we have that

$$\begin{aligned}\eta_nP\{\tau_n\le T\}&\le EW(S(T\wedge\tau_n),V(T\wedge\tau_n),L(T\wedge\tau_n),I(T\wedge\tau_n),t)\\&\le W\Big(s_0,v_0,l_0,i_0,0\Big)+K_1T.\end{aligned}$$

Clearly, on the basis of (2.12) we have that $\lim_{n\to+\infty}\eta_n=+\infty$. Thus, letting $n\to+\infty$ in the last inequality, we obtain that $P\{\tau_\infty\le T\}=0$, that is

$$P\{\tau_\infty>T\}=1. \tag{2.16}$$

So, the solution of the system (2.1)–(2.4) is S-valued and does not explode on the interval $[0,T]$.

On the other hand, letting $n\to+\infty$ in (2.15), we get

$$EW(S(t),V(t),L(t),I(t),t)\le W\Big(S_0,V_0,L_0,I_0,0\Big)+K_1T,\quad t\in[0,T]. \tag{2.17}$$

Our next goal is to prove that, for $S_0+V_0+L_0+I_0\le N$, the solution

$$\{(S(t),V(t),L(t),I(t)),t\in[0,T]\}$$

has the property $S(t)+V(t)+L(t)+I(t)\le N$, for all $t\in[0,T]$, with probability 1. In that sense, let us choose a strictly decreasing sequence $\{c_k,k\in N_0\}$, such that $c_0=1,\ \lim_{k\to+\infty}c_k=0$ and

$$\int_{c_k}^{c_{k-1}}\frac{du}{(u+N)^2}=k. \tag{2.18}$$

Let $\zeta_k:(c_k,c_{k-1})\to\mathbb{R}$ be C^1-functions, that is continuously differentiable functions, such that

$$0\le\zeta_k(u)\le\frac{2}{k(u+N)^2} \tag{2.19}$$

and

$$\int_{c_k}^{c_{k-1}}\zeta_k(u)=1. \tag{2.20}$$

Furtheron, for $s,v,l,i>0$ and $k\in N_0$, let us introduce the C^2-approximation of the expression $[N-(s+v+l+i)]^-$ as

$$\rho_k(s,v,l,i)=\begin{cases}\int_0^{-(N-(s+v+l+i))}dy\int_0^y\zeta_k(u)du, & s+v+l+i>N,\\ 0, & \text{otherwise.}\end{cases} \tag{2.21}$$

In the sequel we will justify that $\rho_k\in C^2(\mathbb{R}^4_+,\mathbb{R}_+)$ for any $k\in N_0$, such that one can apply the Itô formula. First, observe that

$$0\le\frac{\partial\rho_k}{\partial s}=\int_{c_k}^{-(N-(s+v+l+i))}\zeta_k(u)du\le\int_{c_k}^{c_{k-1}}\zeta_k(u)du=1,\quad c_k<s+v+l+i-N,$$

while, on the basis of (2.20) and (2.21) we find that $\frac{\partial\rho_k}{\partial s}=0$, otherwise. In the same way we conclude that

$$0\le\frac{\partial\rho_k}{\partial v}=\frac{\partial\rho_k}{\partial l}=\frac{\partial\rho_k}{\partial i}=\int_{c_k}^{-(N-(s+v+l+i))}\zeta_k(u)du\le1,\quad c_k<s+v+l+i-N,$$

while $\frac{\partial \rho_k}{\partial v} = \frac{\partial \rho_k}{\partial l} = \frac{\partial \rho_k}{\partial i} = 0$ otherwise. Moreover, when $c_k < s + v + l + i - N < c_{k-1}$, $k \geq 1$, we have that

$$\begin{aligned} 0 \leq \frac{\partial^2 \rho_k}{\partial s^2} &= \frac{\partial^2 \rho_k}{\partial v^2} = \frac{\partial^2 \rho_k}{\partial l^2} = \frac{\partial^2 \rho_k}{\partial i^2} = \frac{\partial}{\partial s}\Big(\int_{c_k}^{-(N-(s+v+l+i))} \zeta_k(u)du \Big) \\ &= \zeta_k(-N + s + v + l + i) \\ &\leq \frac{2}{k(s+v+l+i)^2}. \end{aligned}$$

Otherwise, we obtain

$$\frac{\partial^2 \rho_k}{\partial s^2} = \frac{\partial^2 \rho_k}{\partial v^2} = \frac{\partial^2 \rho_k}{\partial l^2} = \frac{\partial^2 \rho_k}{\partial i^2} = 0.$$

Also, in a view of (2.20), we get

$$\begin{aligned} \rho_k(s,l,v,i) \leq & I_{\{c_k < s+v+l+i-N < c_{k-1}\}} \int_{c_k}^{-(N-(s+v+l+i))} dy \int_{c_k}^{y} \zeta_k(u)du \\ & + I_{\{c_{k-1} \leq s+v+l+i-N\}} \int_{c_k}^{c_{k-1}} dy \int_{c_k}^{y} \zeta_k(u)du \\ \leq & I_{\{c_k < s+v+l+i-N < c_{k-1}\}} \int_{c_k}^{-(N-(s+v+l+i))} dy \int_{c_k}^{c_{k-1}} \zeta_k(u)du \\ & + I_{\{c_{k-1} \leq s+v+l+i-N\}} \int_{c_k}^{-(N-(s+v+l+i))} dy \int_{c_k}^{c_{k-1}} \zeta_k(u)du \\ = & I_{\{s+v+l+i-N > c_k\}} \int_{c_k}^{-(N-(s+v+l+i))} dy \\ \leq & [N - (s+v+l+i)]^- \end{aligned} \tag{2.22}$$

and

$$\rho_k(s,l,v,i) \geq [N - (s+v+l+i)]^- - c_{k-1}, \quad k \geq 1, \tag{2.23}$$

where

$$[b]^- = \begin{cases} -b, & b < 0, \\ 0, & \text{otherwise.} \end{cases}$$

In order to justify the inequality (2.23), denote

$$h_k(x) = \int_0^{-(N-x)} dy \int_0^y \zeta_k(u)du - x + N + c_{k-1}, \quad x > N,\ k \geq 1,$$

where $h_k(N) = c_{k-1}$ and, on the basis of (2.20),

$$\begin{aligned} h_k'(x) &= 0 \cdot I_{\{c_{k-1} \leq x-N\}} + I_{\{c_k < x-N < c_{k-1}\}} \Big(\int_{c_k}^{-(N-x)} \zeta_k(u)du - 1 \Big) \\ &= I_{\{c_k < x-N < c_{k-1}\}} \Big(\int_{c_k}^{-(N-x)} \zeta_k(u)du - 1 \Big) \\ &\leq 0. \end{aligned} \tag{2.24}$$

Also, $h_k(c_k+N)=c_{k-1}-c_k>0$ and $h_k(c_{k-1}+N)=\int_{c_k}^{c_{k-1}}dy\int_{c_k}^{y}\zeta_k(u)du\geq 0$. Thus, $h_k(x)\geq h_k(N+c_{k-1})\geq 0$, $c_k+N<x<c_{k-1}+N$, that is, (2.23) holds for $c_k+N<x<c_{k-1}+N$. If $x\geq c_{k-1}+N$, we have that $h_k'(x)=0$ and $h_k(x)=h_k(c_{k-1}+N)>0$. When $N<x\leq c_k+N$,

$$h_k(x)=-x+N+c_{k-1}\geq -c_k+c_{k-1}>0,$$

such that (2.23) holds for all $s,l,v,i\in\mathbb{R}$ and $k\geq 1$. Then, we get

$$\begin{aligned}L\rho_k(s,v,l,i)\leq & I_{\{c_k<s+v+l+i-N\}}\int_{c_k}^{-(N-(s+v+l+i))}\zeta_k(u)du\\ &\times[\mu(N-s-v-l-i)-\delta l-(\alpha+d)i]\\ &+\frac{\sigma_1^2s^2+\sigma_2^2v^2+\sigma_3^2l^2+\sigma_4^2i^2}{k(s+l+v+i)^2}\\ <&\frac{\max_{1\leq j\leq 4}\sigma_j^2}{k}.\end{aligned}$$

Applying the Itô formula and taking the expectation in (2.23), we obtain

$$\begin{aligned}&E[N-(S(t\wedge\tau_m)+V(t\wedge\tau_m)+L(t\wedge\tau_m)+I(t\wedge\tau_m))]^- -c_{k-1}\\ \leq&E(\rho_k(S(t\wedge\tau_m),V(t\wedge\tau_m),L(t\wedge\tau_m),I(t\wedge\tau_m)))\\ \leq&\frac{\max_{1\leq j\leq 4}\sigma_j^2(t\wedge\tau_m)}{k}.\end{aligned}$$

Letting $m\to+\infty$, the previous estimate becomes

$$\begin{aligned}&E[N-(S(t\wedge\tau_\infty)+V(t\wedge\tau_\infty)+L(t\wedge\tau_\infty)+I(t\wedge\tau_\infty))]^- -c_{k-1}\\ \leq&\frac{\max_{1\leq j\leq 4}\sigma_j^2(t\wedge\tau_\infty)}{k}\leq\frac{\max_{1\leq j\leq 4}\sigma_j^2(T\wedge\tau_\infty)}{k}.\end{aligned}$$

Then, on the basis of (2.16), we find that

$$E[N-(S(t)+V(t)+L(t)+I(t))]^- -c_{k-1}\leq\frac{\max_{1\leq j\leq 4}\sigma_j^2T}{k}.$$

Letting $k\to+\infty$ in the previous inequality gives

$$E[N-(S(t)+V(t)+L(t)+I(t))]^-=0.$$

Finally, we obtain

$$P\{S(t)+V(t)+L(t)+I(t)>N\}=0,$$

for all $t\in[0,T]$. Since $\{(S(t),V(t),L(t),I(t)),t\in[0,T]\}$ is continuous, we must have $S(t)+V(t)+L(t)+I(t)\leq N$ for all $t\in[0,T]$ almost surely.

□

3. Positivity and boundedness preserving Euler–Maruyama method

In this section, we propose a positivity and boundedness preserving Euler–Maruyama method corresponding to the system (2.1)–(2.4) and prove the L^θ-convergence theorem of the sequence of the approximate solutions to the exact solution of the system, where $\theta > 0$.

On the basis of the previous section, we conclude that

$$F : R^4_+ \to R^4, \qquad G : R^4_+ \to R^{4\times 4}.$$

Moreover, we will normalize the variables to $N = 1$ such that $S(t)+V(t)+L(t)+I(t) \le 1$, for all $t \in [0, T]$. Then, define the mappings $f : R^4 \to R^4$ and $g : R^4 \to R^{4\times 4}$ by

$$f(s, v, l, i) = F(\pi_{01}(s, v, l, i)), \quad g(s, v, l, i) = G(\pi_{01}(s, v, l, i)),$$

where

$$\pi_{01}(s^0, v^0, l^0, i^0) = \begin{cases} (s^0, v^0, l^0, i^0)^T, & s^0 + v^0 + l^0 + i^0 \le 1, \\ (\bar{s}^0, \bar{v}^0, \bar{l}^0, \bar{i}^0)^T, & \text{otherwise}, \end{cases}$$

and

$$\begin{aligned} &s^0 = s \vee 0, \;\; v^0 = v \vee 0, \;\; l^0 = l \vee 0, \;\; i^0 = i \vee 0, \\ &\bar{s}^0 = \frac{s^0}{s^0 + v^0 + l^0 + i^0}, \;\; \bar{v}^0 = \frac{v^0}{s^0 + v^0 + l^0 + i^0}, \;\; \bar{l}^0 = \frac{l^0}{s^0 + v^0 + l^0 + i^0}, \\ &\bar{i}^0 = \frac{i^0}{s^0 + v^0 + l^0 + i^0}, \end{aligned}$$

with $a \vee b = \max\{a, b\}$. Then, we can rewrite the system (2.1)–(2.4) in the form

$$dZ(t) = f(Z(t))dt + g(Z(t))dB(t), \tag{3.1}$$

where $Z(t) = (S(t), V(t), L(t), I(t))^T$, and for $z = (s, v, l, i)^T$,

$$f(z) = \begin{cases} \begin{pmatrix} \mu - \beta i^0 s^0 - (\mu + \omega)s^0 + \emptyset v^0 \\ \omega s^0 - (\mu + \emptyset)v^0 \\ \beta i^0 s^0 - (\mu + \delta + \gamma)l^0 \\ \gamma l^0 - (\mu + \alpha + d)i^0 \end{pmatrix}, & s^0 + v^0 + l^0 + i^0 \le 1, \\ \begin{pmatrix} \mu - \frac{\beta i^0 s^0}{(s^0+v^0+l^0+i^0)^2} - \frac{(\mu+\omega)s^0}{s^0+v^0+l^0+i^0} + \frac{\emptyset v^0}{s^0+v^0+l^0+i^0} \\ \frac{\omega s^0}{s^0+v^0+l^0+i^0} - \frac{(\mu+\emptyset)v^0}{s^0+v^0+l^0+i^0} \\ \frac{\beta i^0 s^0}{(s^0+v^0+l^0+i^0)^2} - \frac{(\mu+\delta+\gamma)l^0}{s^0+v^0+l^0+i^0} \\ \frac{\gamma l^0}{s^0+v^0+l^0+i^0} - \frac{(\mu+\alpha+d)i^0}{s^0+v^0+l^0+i^0} \end{pmatrix}, & \text{otherwise}, \end{cases}$$

$$g(z)=\begin{cases}\begin{pmatrix}\sigma_1 s^0 & 0 & 0 & 0\\ 0 & \sigma_2 v^0 & 0 & 0\\ 0 & 0 & \sigma_3 l^0 & 0\\ 0 & 0 & 0 & \sigma_4 i^0\end{pmatrix}, & s^0+v^0+l^0+i^0\le 1,\\ \begin{pmatrix}\frac{\sigma_1 s^0}{s^0+v^0+l^0+i^0} & 0 & 0 & 0\\ 0 & \frac{\sigma_2 v^0}{s^0+v^0+l^0+i^0} & 0 & 0\\ 0 & 0 & \frac{\sigma_3 l^0}{s^0+v^0+l^0+i^0} & 0\\ 0 & 0 & 0 & \frac{\sigma_4 i^0}{s^0+v^0+l^0+i^0}\end{pmatrix}, & \text{otherwise.}\end{cases}$$

One can easily observe that partial derivatives of the functions f and g are bounded, such that f and g satisfy the global Lipschitz condition.

Without loss of generality, we may assume that T/Δ is a rational number; otherwise we may replace T by a larger number. Let the step size $\Delta\in(0,1/5)$ be $\Delta=T/M$ for some integer $M>5T$ and let $s^\Delta+v^\Delta+l^\Delta+i^\Delta=N^\Delta$. We next introduce a projection operator π_Δ which maps an arbitrary state into a Δ-dependent subset of the positive state space S, while preserving the prescribed upper bound on the total population. Therefore, we have

$$\pi_\Delta(s,v,l,i)$$
$$=\begin{cases}(s^\Delta,v^\Delta,l^\Delta,i^\Delta)^T, & N^\Delta\le 1-\Delta,\\ (1-\Delta)(\bar s^0,\bar v^0,\bar l^0,\bar i^0)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar v^0,\bar l^0,\bar i^0\ge\frac{\Delta}{1-\Delta},\\ \Big(\Delta,\Delta,\Delta,1-4\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar v^0,\bar l^0<\frac{\Delta\bar i^0}{1-4\Delta},\\ \Big(\Delta,\Delta,1-4\Delta,\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar v^0,\bar i^0<\frac{\Delta\bar l^0}{1-4\Delta},\\ \Big(\Delta,1-4\Delta,\Delta,\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar l^0,\bar i^0<\frac{\Delta\bar v^0}{1-4\Delta},\\ \Big(1-4\Delta,\Delta,\Delta,\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar v^0,\bar l^0,\bar i^0<\frac{\Delta\bar s^0}{1-4\Delta},\\ \Big(\Delta,\Delta,\frac{(1-3\Delta)l^0}{l^0+i^0},\frac{(1-3\Delta)i^0}{l^0+i^0}\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar v^0<\frac{\Delta(\bar l^0+\bar i^0)}{1-3\Delta},\ \bar l^0,\bar i^0\ge\frac{\Delta(\bar l^0+\bar i^0)}{1-3\Delta},\\ \Big(\Delta,\frac{(1-3\Delta)v^0}{v^0+i^0},\Delta,\frac{(1-3\Delta)i^0}{v^0+i^0}\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar l^0<\frac{\Delta(\bar v^0+\bar i^0)}{1-3\Delta},\ \bar v^0,\bar i^0\ge\frac{\Delta(\bar v^0+\bar i^0)}{1-3\Delta},\\ \Big(\Delta,\frac{(1-3\Delta)v^0}{v^0+l^0},\frac{(1-3\Delta)l^0}{v^0+l^0},\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0,\bar i^0<\frac{\Delta(\bar v^0+\bar l^0)}{1-3\Delta},\ \bar v^0,\bar l^0\ge\frac{\Delta(\bar v^0+\bar l^0)}{1-3\Delta},\\ \Big(\frac{(1-3\Delta)s^0}{s^0+i^0},\Delta,\Delta,\frac{(1-3\Delta)i^0}{s^0+i^0}\Big)^T, & N^\Delta>1-\Delta,\ \bar v^0,\bar l^0<\frac{\Delta(\bar s^0+\bar i^0)}{1-3\Delta},\ \bar s^0,\bar i^0\ge\frac{\Delta(\bar s^0+\bar i^0)}{1-3\Delta},\\ \Big(\frac{(1-3\Delta)s^0}{s^0+l^0},\Delta,\frac{(1-3\Delta)l^0}{s^0+l^0},\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar v^0,\bar i^0<\frac{\Delta(\bar s^0+\bar l^0)}{1-3\Delta},\ \bar s^0,\bar l^0\ge\frac{\Delta(\bar s^0+\bar l^0)}{1-3\Delta},\\ \Big(\frac{(1-3\Delta)s^0}{s^0+v^0},\frac{(1-3\Delta)v^0}{s^0+v^0},\Delta,\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar l^0,\bar i^0<\frac{\Delta(\bar s^0+\bar v^0)}{1-3\Delta},\ \bar s^0,\bar v^0\ge\frac{\Delta(\bar s^0+\bar v^0)}{1-3\Delta},\\ \Big(\Delta,\frac{(1-2\Delta)v^0}{v^0+l^0+i^0},\frac{(1-2\Delta)l^0}{v^0+l^0+i^0},\frac{(1-2\Delta)i^0}{v^0+l^0+i^0}\Big)^T, & N^\Delta>1-\Delta,\ \bar s^0<\frac{\Delta(\bar v^0+\bar l^0+\bar i^0)}{1-2\Delta},\ \bar v^0,\bar l^0,\bar i^0\ge\frac{\Delta(\bar v^0+\bar l^0+\bar i^0)}{1-2\Delta},\\ \Big(\frac{(1-2\Delta)s^0}{s^0+l^0+i^0},\Delta,\frac{(1-2\Delta)l^0}{s^0+l^0+i^0},\frac{(1-2\Delta)i^0}{s^0+l^0+i^0}\Big)^T, & N^\Delta>1-\Delta,\ \bar v^0<\frac{\Delta(\bar s^0+\bar l^0+\bar i^0)}{1-2\Delta},\ \bar s^0,\bar l^0,\bar i^0\ge\frac{\Delta(\bar s^0+\bar l^0+\bar i^0)}{1-2\Delta},\\ \Big(\frac{(1-2\Delta)s^0}{s^0+v^0+i^0},\frac{(1-2\Delta)v^0}{s^0+v^0+i^0},\Delta,\frac{(1-2\Delta)i^0}{s^0+v^0+i^0}\Big)^T, & N^\Delta>1-\Delta,\ \bar l^0<\frac{\Delta(\bar s^0+\bar v^0+\bar i^0)}{1-2\Delta},\ \bar s^0,\bar v^0,\bar i^0\ge\frac{\Delta(\bar s^0+\bar v^0+\bar i^0)}{1-2\Delta},\\ \Big(\frac{(1-2\Delta)s^0}{s^0+v^0+l^0},\frac{(1-2\Delta)v^0}{s^0+v^0+l^0},\frac{(1-2\Delta)l^0}{s^0+v^0+l^0},\Delta\Big)^T, & N^\Delta>1-\Delta,\ \bar i^0<\frac{\Delta(\bar s^0+\bar v^0+\bar l^0)}{1-2\Delta},\ \bar s^0,\bar v^0,\bar l^0\ge\frac{\Delta(\bar s^0+\bar v^0+\bar l^0)}{1-2\Delta},\end{cases}$$

where

$$s^{\Delta} = s \vee \Delta, \quad v^{\Delta} = v \vee \Delta, \quad l^{\Delta} = l \vee \Delta, \quad i^{\Delta} = i \vee \Delta.$$

Let

$$S_{\Delta} = \{(s, v, l, i) \in S : \ s, v, l, i \geq \Delta, \ s + v + l + i \leq 1 - \Delta\}.$$

Since $\Delta \in (0, 1/5)$, the set S_{Δ} is nonempty and compact. By the definition of π_{Δ}, for every $x \in R^4_+$, $\pi_{\Delta}(x) \in S_{\Delta}$. In particular, π_{Δ} is globally bounded and

$$|\pi_{\Delta}(x)| \leq 1 - \Delta, \quad x \in R^4. \tag{3.2}$$

Furthermore, for every fixed $\Delta \in (0, 1/5)$, the mapping π_{Δ} is globally Lipschitz continuous, i.e., there exists a constant $L_{\Delta} > 0$ such that

$$|\pi_{\Delta}(x) - \pi_{\Delta}(y)| \leq L_{\Delta}|x - y|, \qquad x, y \in \mathbb{R}^4. \tag{3.3}$$

Indeed, the coordinatewise mappings $u \to u \vee \Delta$ and $u \to 0 \vee \Delta$ are Lipschitz continuous. Moreover, on each of the finitely many regions occurring in the definition of π_{Δ}, the corresponding expression is continuously differentiable. The denominators appearing in these expressions are uniformly bounded away from zero by positive constants depending only on Δ, while the normalized components involved in the numerators are bounded. Hence, the first derivatives of the corresponding expressions are bounded on each region. Since the expressions coincide on the common boundaries of the regions, it follows that π_{Δ} is globally Lipschitz continuous.
Therefore, the projection operator ensures that all components of the projected numerical approximation remain strictly positive and that their sum does not exceed $1 - \Delta$.

Then, we will define the discrete-time positivity and boundedness preserving Euler–Maruyama solution on the equidistant partition $t_k = k\Delta$, for $k = 0, 1, 2, \ldots, M$, such that $\bar{Z}_{\Delta}(0) = Z_{\Delta}(0) = (S_0, V_0, L_0, I_0)$ while, for any $k \in \{0, 1, 2, \ldots, M - 1\}$,

$$\bar{Z}_{\Delta}(t_{k+1}) = \bar{Z}_{\Delta}(t_k) + f(Z_{\Delta}(t_k))\Delta + g(Z_{\Delta}(t_k))\Delta B_k, \tag{3.4}$$

$$Z_{\Delta}(t_{k+1}) = \pi_{\Delta}(\bar{Z}_{\Delta}(t_{k+1})), \tag{3.5}$$

where

$$\Delta B_k = (\Delta B_k^1, \Delta B_k^2, \Delta B_k^3, \Delta B_k^4)^T.$$

and $\Delta B_k^i = B_i((k+1)\Delta) - B_i(k\Delta)$ for $i \in \{1, 2, 3, 4\}$.

We then extend the definition of $Z_{\Delta}(\cdot)$ to the whole $t \in [0, T]$ by

$$Z_{\Delta}(t) = Z_{\Delta}(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots, M - 1, \tag{3.6}$$

and $Z_{\Delta}(T) = Z_{\Delta}(t_M)$. For further analysis, we define the continuous-time approximate solution $\{z_{\Delta}(t), t \in [0, T]\}$ as

$$z_{\Delta}(t) = z(0) + \int_0^t f(Z_{\Delta}(s))ds + \int_0^t g(Z_{\Delta}(s))dB(s), \quad t \in [0, T]. \tag{3.7}$$

Clearly, for $t \in [t_k, t_{k+1})$, Eq. (3.7) can be written as

$$z_{\Delta}(t) = \bar{Z}_{\Delta}(t_k) + \int_{t_k}^t f(Z_{\Delta}(s))ds + \int_{t_k}^t g(Z_{\Delta}(s))dB(s). \tag{3.8}$$

Here, $\bar{Z}_\Delta(t_k)$ denotes the auxiliary unprojected approximation obtained before applying the projection, whereas $Z_\Delta(t_k) = \pi_\Delta(\bar{Z}_\Delta(t_k))$ is the projected discrete-time approximation. The projection ensures that the numerical state remains in the prescribed positive and bounded region. The piecewise-constant process $Z_\Delta(t)$ defined in (3.6) is used in the drift and diffusion coefficients, while $z_\Delta(t)$, defined in (3.7), is the corresponding continuous-time interpolation.

It is useful to observe that $z_\Delta(t_k) = \bar{Z}_\Delta(t_k)$ for every $k = 0, 1, 2, \dots M$, namely the auxiliary discrete-time approximation $\bar{Z}_\Delta$ and the continuous-time approximation z_Δ coincide at the grid points t_k.

Lemma 3.1. *For any $\theta > 0$, we have that*

$$\sup_{0<\Delta<1/5} E\left(\sup_{t\in[0,T]} |z_\Delta(t)|^\theta\right) \le H, \tag{3.9}$$

where H is constant independent of Δ.

Proof. Fix any $\theta > 0$. By the definitions of the drift and diffusion coefficients f and g of system (2.1)–(2.4), together with the property (3.2) of the projection operator π_Δ, we can conclude that there exists a constant $C > 0$, independent of Δ, such that

$$\sup_{t\in[0,T]} |f(Z_\Delta(t))| \le C, \qquad \sup_{t\in[0,T]} |g(Z_\Delta(t))| \le C. \tag{3.10}$$

Now, using the elementary inequality

$$\left(\sum_{k=1}^n a_k\right)^p \le (n^{(p-1)} \vee 1) \sum_{k=1}^n a_k^p, \quad n \in \mathbb{N},\ a_k \ge 0,\ k = 1, 2, \dots, n,\ p > 0, \tag{3.11}$$

as well as the definition (3.7), we obtain

$$E\left(\sup_{t\in[0,T]} |z_\Delta(t)|^\theta\right) \le (3^{\theta-1} \vee 1)\Bigg\{|z(0)|^\theta + E\left[\sup_{t\in[0,T]} \left|\int_0^t f(Z_\Delta(s))\, ds\right|^\theta\right] \tag{3.12}$$

$$+ E\left[\sup_{t\in[0,T]} \left|\int_0^t g(Z_\Delta(s))\, dB(s)\right|^\theta\right]\Bigg\}. \tag{3.13}$$

From the uniform boundedness (3.10) of f, we have

$$E\left[\sup_{t\in[0,T]} \left|\int_0^t f(Z_\Delta(s))\, ds\right|^\theta\right] \le E\left(\int_0^T |f(Z_\Delta(s))|\, ds\right)^\theta \le C^\theta T^\theta.$$

The Burkholder–Davis–Gundy inequality and (3.10) yield

$$E\left[\sup_{t\in[0,T]} \left|\int_0^t g(Z_\Delta(s))\, dB(s)\right|^\theta\right] \le C_\theta E\left(\int_0^T |g(Z_\Delta(s))|^2\, ds\right)^{\theta/2} \le C_\theta (C^2 T)^{\theta/2}.$$

Therefore, we obtain that

$$E\left(\sup_{t\in[0,T]} |z_\Delta(t)|^\theta\right) \le (3^{\theta-1} \vee 1)\left(|z(0)|^\theta + C^\theta T^\theta + C_\theta C^\theta T^{\theta/2}\right) = H.$$

Since the right-hand side is independent of Δ, it follows that (3.9) hold. □

Now, we present the following three lemmas that are essential for the proof of the main results.

Lemma 3.2. *Let $\theta > 0$ and let $l \in N$ such that $l > \frac{\theta}{2}$ and $l \geq 3$. Then, we have that*

$$E\left(\sup_{t\in[0,T]}\left|\bar{Z}_\Delta(t) - z_\Delta(t)\right|^\theta\right) \leq C_{\theta,l,T}\Delta^{\frac{\theta}{3}},$$

where $C_{\theta,l,T} > 0$ is a generic constant independent of Δ.

Proof. Using the elementary inequality (3.11), definitions (3.4) and (3.8), as well as Lemma 3.1, we obtain that

$$\begin{aligned}
&E\left(\sup_{t\in[0,T]}\left|\bar{Z}_\Delta(t) - z_\Delta(t)\right|^\theta\right)\\
\leq& E\left(\max_{0\leq k\leq M-1}\sup_{t_k\leq t\leq t_{k+1}}|f(Z_\Delta(t_k))(t-t_k) + g(Z_\Delta(t_k))(B(t)-B(t_k))|^\theta\right)\\
\leq&(2^{\theta-1}\vee 1)E\Big(\max_{0\leq k\leq M-1}\sup_{t_k\leq t\leq t_{k+1}}\Big(|f(Z_\Delta(t_k))|^\theta|t-t_k|^\theta\\
&\qquad +|g(Z_\Delta(t_k))|^\theta|B(t)-B(t_k)|^\theta\Big)\Big)\\
\leq&H^\theta(2^{\theta-1}\vee 1)\left(\Delta^\theta + E\left(\max_{0\leq k\leq M-1}\sup_{t_k\leq t\leq t_{k+1}}|B(t)-B(t_k)|^\theta\right)\right). \qquad (3.14)
\end{aligned}$$

Now, we choose an integer $l \in N$ such that $l > \frac{\theta}{2}$ and $l \geq 3$. Since $2l > \theta$, Hölder's inequality yields

$$\begin{aligned}
E\left(\max_{0\leq k\leq M-1}\sup_{t_k\leq t\leq t_{k+1}}|B(t)-B(t_k)|^\theta\right) &\leq E\left(\max_{0\leq k\leq M-1}\sup_{t_k\leq t\leq t_{k+1}}|B(t)-B(t_k)|^{2l}\right)^{\frac{\theta}{2l}}\\
&\leq\left(\sum_{k=0}^{M-1}E\left(\sup_{t_k\leq t\leq t_{k+1}}|B(t)-B(t_k)|^{2l}\right)\right)^{\frac{\theta}{2l}}.
\end{aligned}$$

Applying Doob's martingale inequality, we obtain

$$\begin{aligned}
&E\left(\max_{0\leq k\leq M-1}\sup_{t_k\leq t\leq t_{k+1}}|B(t)-B(t_k)|^\theta\right)\\
&\leq\left(\sum_{k=0}^{M-1}\left(\frac{2l}{2l-1}\right)^{2l}E|B(t_{k+1})-B(t_k)|^{2l}\right)^{\frac{\theta}{2l}}. \qquad (3.15)
\end{aligned}$$

Because $B(t) = (B^1(t), B^2(t), B^3(t), B^4(t))^T$, $t \in [0,T]$ is an 4-dimensional standard Brownian motion, we have

$$E|\Delta B_k|^{2l} = E|B(t_{k+1}) - B(t_k)|^{2l}$$

$$
\begin{aligned}
&= E\left(|\Delta B_k^1|^2 + |\Delta B_k^2|^2 + |\Delta B_k^3|^2 + |\Delta B_k^4|^2\right)^l \\
&\leq (4^{l-1} \vee 1) \sum_{i=1}^{4} E|\Delta B_k^i|^{2l} \\
&= 4(4^{l-1} \vee 1)(2l-1)!!\Delta^l, \qquad (3.16)
\end{aligned}
$$

where $(2l-1)!! = 1 \times 3 \times \cdots \times (2l-1)$. Substituting (3.16) into (3.15), we obtain

$$
\begin{aligned}
&E\left(\max_{0\leq k\leq M-1} \sup_{t_k\leq t\leq t_{k+1}} |B(t) - B(t_k)|^\theta\right) \\
&\leq \left(\sum_{k=0}^{M-1} \left(\frac{2l}{2l-1}\right)^{2l} 4(4^{l-1} \vee 1)(2l-1)!!\Delta^l\right)^{\frac{\theta}{2l}} \\
&= (4(4^{l-1} \vee 1))^{\frac{\theta}{2l}} \left(\frac{2l}{2l-1}\right)^{\theta} \left((2l-1)!!T\Delta^{l-1}\right)^{\frac{\theta}{2l}} \\
&= (4(4^{l-1} \vee 1))^{\frac{\theta}{2l}} \left(\frac{2l}{2l-1}\right)^{\theta} \left(((2l-1)!!)^{\frac{1}{l}}\right)^{\frac{\theta}{2}} T^{\frac{\theta}{2l}} \Delta^{\frac{(l-1)\theta}{2l}} \\
&= (4(4^{l-1} \vee 1))^{\frac{\theta}{2l}} \left(\frac{2l}{2l-1}\right)^{\theta} l^{\frac{\theta}{2}} T^{\frac{\theta}{2l}} \Delta^{\frac{(l-1)\theta}{2l}},
\end{aligned}
$$

where we use the fact that $((2l-1)!!)^{\frac{1}{l}} \leq \frac{1}{l}\sum_{k=1}^{l}(2k-1) = l$. Since $l \geq 3$ we have that $\frac{l-1}{2l} \geq \frac{1}{3}$. On the other hand, since $0 < \Delta < 1$, we obtain that

$$\Delta^{\frac{l-1}{2l}} \leq \Delta^{\frac{1}{3}}.$$

Hence, we can conclude that

$$E\left(\max_{0\leq k\leq M} \sup_{t_k\leq t\leq t_{k+1}} |B(t) - B(t_k)|^\theta\right) \leq (4(4^{l-1} \vee 1))^{\frac{\theta}{2l}} \left(\frac{2l}{2l-1}\right)^{\theta} l^{\frac{\theta}{2}} T^{\frac{\theta}{2l}} \Delta^{\frac{\theta}{3}}.$$

Substituting the previous estimate into (3.14) we get

$$
\begin{aligned}
E\left(\sup_{t\in[0,T]} \left|\bar{Z}_\Delta(t) - z_\Delta(t)\right|^\theta\right) &\leq H^\theta(2^{\theta-1} \vee 1)\left(\Delta^\theta + (4(4^{l-1} \vee 1))^{\frac{\theta}{2l}} \left(\frac{2l}{2l-1}\right)^{\theta} l^{\frac{\theta}{2}} T^{\frac{\theta}{2l}} \Delta^{\frac{\theta}{3}}\right) \\
&\leq C_{\theta,l,T}\Delta^{\frac{\theta}{3}},
\end{aligned}
$$

where $C_{\theta,l,T} > 0$ is independent of Δ. This proves the desired assertion. □

Lemma 3.3. *For any $\theta > 0$, we have that*

$$\lim_{\Delta\to 0} E\left(\sup_{t\in[0,T]} |z_\Delta(t) - Z(t)|^\theta\right) = 0. \qquad (3.17)$$

and

$$\lim_{\Delta\to 0} E\left(\sup_{t\in[0,T]} \left|\bar{Z}_\Delta(t) - Z_\Delta(t)\right|^\theta\right) = 0. \qquad (3.18)$$

Proof. Let $\epsilon \in (0,1)$ be arbitrary. One can find a $\delta = \delta(\epsilon) \in (0, 2/9)$ so small that

$$P(\Omega_1) \geq 1 - \frac{\epsilon}{2}, \tag{3.19}$$

where

$$\begin{aligned} \Omega_1 = \Big\{ &\inf_{t\in[0,T]} S(t) > \delta, \inf_{t\in[0,T]} V(t) > \delta, \inf_{t\in[0,T]} L(t) > \delta, \inf_{t\in[0,T]} I(t) > \delta, \\ &S(t)+V(t)+L(t)+I(t) < 1-\frac{\delta}{2}\Big\}. \end{aligned} \tag{3.20}$$

With the chosen δ, we define

$$f_\delta(z) = F(\pi_\delta(z)), \quad g_\delta(z) = G(\pi_\delta(z)),$$

where

$$\pi_\delta(s,v,l,i) = \begin{cases} (s^\delta, v^\delta, l^\delta, i^\delta)^T, & N^\delta \leq 1-\frac{\delta}{2}, \\ (1-\frac{\delta}{2})(\bar{s}^0, \bar{v}^0, \bar{l}^0, \bar{i}^0)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{v}^0, \bar{l}^0, \bar{i}^0 \geq \frac{\delta}{2-\delta}, \\ \left(\frac{\delta}{2}, \frac{\delta}{2}, \frac{\delta}{2}, 1-2\delta\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{v}^0, \bar{l}^0 < \frac{\delta\bar{i}^0}{2-4\delta}, \\ \left(\frac{\delta}{2}, \frac{\delta}{2}, 1-2\delta, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{v}^0, \bar{i}^0 < \frac{\delta\bar{l}^0}{2-4\delta}, \\ \left(\frac{\delta}{2}, 1-2\delta, \frac{\delta}{2}, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{l}^0, \bar{i}^0 < \frac{\delta\bar{v}^0}{2-4\delta}, \\ \left(1-2\delta, \frac{\delta}{2}, \frac{\delta}{2}, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{v}^0, \bar{l}^0, \bar{i}^0 < \frac{\delta\bar{s}^0}{2-4\delta}, \\ \left(\frac{\delta}{2}, \frac{\delta}{2}, \frac{(1-\frac{3}{2}\delta)l^0}{l^0+i^0}, \frac{(1-\frac{3}{2}\delta)i^0}{l^0+i^0}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{v}^0 < \frac{\delta(\bar{l}^0+\bar{i}^0)}{2-3\delta}, \bar{l}^0, \bar{i}^0 \geq \frac{\delta(\bar{l}^0+\bar{i}^0)}{2-3\delta}, \\ \left(\frac{\delta}{2}, \frac{(1-\frac{3}{2}\delta)v^0}{v^0+i^0}, \frac{\delta}{2}, \frac{(1-\frac{3}{2}\delta)i^0}{v^0+i^0}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{l}^0 < \frac{\delta(\bar{v}^0+\bar{i}^0)}{2-3\delta}, \bar{v}^0, \bar{i}^0 \geq \frac{\delta(\bar{v}^0+\bar{i}^0)}{2-3\delta}, \\ \left(\frac{\delta}{2}, \frac{(1-\frac{3}{2}\delta)v^0}{v^0+l^0}, \frac{(1-\frac{3}{2}\delta)l^0}{v^0+l^0}, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0, \bar{i}^0 < \frac{\delta(\bar{v}^0+\bar{l}^0)}{2-3\delta}, \bar{v}^0, \bar{l}^0 \geq \frac{\delta(\bar{v}^0+\bar{l}^0)}{2-3\delta}, \\ \left(\frac{(1-\frac{3}{2}\delta)s^0}{s^0+i^0}, \frac{\delta}{2}, \frac{\delta}{2}, \frac{(1-\frac{3}{2}\delta)i^0}{s^0+i^0}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{v}^0, \bar{l}^0 < \frac{\delta(\bar{s}^0+\bar{i}^0)}{2-3\delta}, \bar{s}^0, \bar{i}^0 \geq \frac{\delta(\bar{s}^0+\bar{i}^0)}{2-3\delta}, \\ \left(\frac{(1-\frac{3}{2}\delta)s^0}{s^0+l^0}, \frac{\delta}{2}, \frac{(1-\frac{3}{2}\delta)l^0}{s^0+l^0}, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{v}^0, \bar{i}^0 < \frac{\delta(\bar{s}^0+\bar{l}^0)}{2-3\delta}, \bar{s}^0, \bar{l}^0 \geq \frac{\delta(\bar{s}^0+\bar{l}^0)}{2-3\delta}, \\ \left(\frac{(1-\frac{3}{2}\delta)s^0}{s^0+v^0}, \frac{(1-\frac{3}{2}\delta)v^0}{s^0+v^0}, \frac{\delta}{2}, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{l}^0, \bar{i}^0 < \frac{\delta(\bar{s}^0+\bar{v}^0)}{2-3\delta}, \bar{s}^0, \bar{v}^0 \geq \frac{\delta(\bar{s}^0+\bar{v}^0)}{2-3\delta}, \\ \left(\frac{\delta}{2}, \frac{(1-\delta)v^0}{v^0+l^0+i^0}, \frac{(1-\delta)l^0}{v^0+l^0+i^0}, \frac{(1-\delta)i^0}{v^0+l^0+i^0}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{s}^0 < \frac{\delta(\bar{v}^0+\bar{l}^0+\bar{i}^0)}{2(1-\delta)}, \bar{v}^0, \bar{l}^0, \bar{i}^0 \geq \frac{\delta(\bar{v}^0+\bar{l}^0+\bar{i}^0)}{2(1-\delta)}, \\ \left(\frac{(1-\delta)s^0}{s^0+l^0+i^0}, \frac{\delta}{2}, \frac{(1-\delta)l^0}{s^0+l^0+i^0}, \frac{(1-\delta)i^0}{s^0+l^0+i^0}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{v}^0 < \frac{\delta(\bar{s}^0+\bar{l}^0+\bar{i}^0)}{2(1-\delta)}, \bar{s}^0, \bar{l}^0, \bar{i}^0 \geq \frac{\delta(\bar{s}^0+\bar{l}^0+\bar{i}^0)}{2(1-\delta)}, \\ \left(\frac{(1-\delta)s^0}{s^0+v^0+i^0}, \frac{(1-\delta)v^0}{s^0+v^0+i^0}, \frac{\delta}{2}, \frac{(1-\delta)i^0}{s^0+v^0+i^0}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{l}^0 < \frac{\delta(\bar{s}^0+\bar{v}^0+\bar{i}^0)}{2(1-\delta)}, \bar{s}^0, \bar{v}^0, \bar{i}^0 \geq \frac{\delta(\bar{s}^0+\bar{v}^0+\bar{i}^0)}{2(1-\delta)}, \\ \left(\frac{(1-\delta)s^0}{s^0+v^0+l^0}, \frac{(1-\delta)v^0}{s^0+v^0+l^0}, \frac{(1-\delta)l^0}{s^0+v^0+l^0}, \frac{\delta}{2}\right)^T, & N^\delta > 1-\frac{\delta}{2}, \bar{i}^0 < \frac{\delta(\bar{s}^0+\bar{v}^0+\bar{l}^0)}{2(1-\delta)}, \bar{s}^0, \bar{v}^0, \bar{l}^0 \geq \frac{\delta(\bar{s}^0+\bar{v}^0+\bar{l}^0)}{2(1-\delta)}, \end{cases}$$

and

$$N^\delta = s^\delta + v^\delta + l^\delta + i^\delta, \quad s^\delta = s^0 \vee \frac{\delta}{2}, \quad v^\delta = v^0 \vee \frac{\delta}{2}, \quad l^\delta = l^0 \vee \frac{\delta}{2}, \quad i^\delta = i^0 \vee \frac{\delta}{2}.$$

Observe that, for all $x \in R^4_+$, $\pi_\delta(x) \in S_\delta$, where

$$S_\delta = \left\{(s,v,l,i) \in S : \ s,v,l,i \geq \frac{\delta}{2}, \ s+v+l+i \leq 1 - \frac{\delta}{2}\right\}.$$

The projection operator π_δ satisfies (3.3), that is, it is globally Lipschitz continuous. Moreover, since F and G are locally Lipschitz continuous, their restrictions to the compact set S_δ are global Lipschitz continuous. Hence, the compositions

$$f_\delta = F \circ \pi_\delta, \quad g_\delta = G \circ \pi_\delta,$$

are globally Lipschitz continuous. Furthermore, since $\pi_\delta(R^4_+) \subseteq S_\delta$ and F and G are continuous on the compact set S_δ, $f_\delta(\cdot)$ and $g_\delta(\cdot)$ are globally bounded.
Because of that, the stochastic differential equation

$$dy(t) = f_\delta(y(t))dt + g_\delta(y(t))dB(t), \quad t \in [0,T], \tag{3.21}$$

with the initial value $y(0) = z(0)$, has a unique global solution $y(t)$ on $t \in [0,T]$ (see [9]). For each step size $\Delta \in (0,1]$, we can apply the Euler–Maryama method to Eq. (3.21). More precise, we form the discrete-time Euler–Maryama solution $Y_k = y(t_k)$ for $t_k = k\Delta$, by setting $Y_0 = y(0)$ and computing

$$Y_{k+1} = Y_k + f_\delta(Y_k)\Delta + g_\delta(Y_k)\Delta B_k, \quad k = 0,1,2,\cdots,M-1. \tag{3.22}$$

Then, we define

$$Y_\Delta(t) = Y_k, \quad t \in [t_k, t_{k+1}), \quad k = 0,1,2,\ldots,M-1,$$

and $Y_\Delta(T) = Y_M$. Hence the Itô process is defined by

$$y_\Delta(t) = z(0) + \int_0^t f_\delta(Y_\Delta(s))ds + \int_0^t g_\delta(Y_\Delta(s))dB(s), \quad t \in [0,T]. \tag{3.23}$$

Let the ith component of $y_\Delta(t)$ be $y_{\Delta,i}(t)$, for $i \in \{1,2,3,4\}$. By the standard strong convergence theorem for the Eulerâ€“Maruyama method applied to stochastic differential equations with globally Lipschitz coefficients (see [9]), we have, for every $\theta > 0$,

$$E\left(\sup_{t\in[0,T]} |y(t) - y_\Delta(t)|^\theta\right) \leq C(\delta)\Delta^{\frac{\theta}{2}}, \tag{3.24}$$

for any $\Delta \in (0,1]$, where $C(\delta) > 0$ depends on δ, θ, T, the initial value and the model parameters, but is independent of the step-size Δ. Therefore, we have that

$$E\left(\sup_{t\in[0,T]} |y(t) - y_\Delta(t)|^\theta I_{\Omega_1}(\omega)\right) \leq C(\delta)\Delta^{\frac{\theta}{2}}. \tag{3.25}$$

Now, from the Chebyshev's inequality we get

$$P\left(\sup_{t\in[0,T]}|y(t)-y_\Delta(t)|\,I_{\Omega_1}(\omega)\geq\frac{\delta}{8}\right)\leq C(\delta)\Delta^{\frac{\theta}{2}}\left(\frac{8}{\delta}\right)^{\theta}.$$

Defining

$$\Omega_2=\left\{\sup_{t\in[0,T]}|y(t)-y_\Delta(t)|<\frac{\delta}{8}\right\}\cap\Omega_1,$$

one can find a step size $\Delta_1\in(0,\delta/2]$ such that

$$P(\Omega_2)\geq 1-\epsilon,$$

for any $\Delta\in(0,\Delta_1]$. For every $\omega\in\Omega_1$, the exact solution Z of the system remains in the region S_δ where the projection π_δ acts as the identity. Hence,

$$f_\delta(Z(t))=F(Z(t)),\quad g_\delta(Z(t))=G(Z(t)),\quad t\in[0,T].$$

Therefore, processes $\{Z(t),t\in[0,T]\}$ and $\{y(t),t\in[0,T]\}$ satisfy the same stochastic differential equation with the same initial condition and are driven by the same Brownian motion. By pathwise uniqueness, we have that

$$y(t)=Z(t),\quad t\in[0,T], \tag{3.26}$$

almost surely on Ω_1, and hence, in particular, on $\Omega_2\subset\Omega_1$. So, for any $\omega\in\Omega_2$, we have that

$$\sup_{t\in[0,T]}|Z(t)-y_\Delta(t)|<\frac{\delta}{8}. \tag{3.27}$$

Moreover, for any $\Delta\in(0,\Delta_1]$ and $\omega\in\Omega_2$, (3.20) and (3.27) give

$$\inf_{t\in[0,T]}y_{\Delta,1}(t)\geq\inf_{t\in[0,T]}S(t)-\sup_{t\in[0,T]}|Z(t)-y_\Delta(t)|>\delta-\frac{\delta}{8}=\frac{7\delta}{8}.$$

Similarly, we obtain that

$$\inf_{t\in[0,T]}y_{\Delta,2}(t)>\frac{7\delta}{8},\ \inf_{t\in[0,T]}y_{\Delta,3}(t)>\frac{7\delta}{8}\ \inf_{t\in[0,T]}y_{\Delta,4}(t)>\frac{7\delta}{8}.$$

Also, from (3.20) and (3.27) we have that

$$\begin{aligned}
&\sup_{t\in[0,T]}(y_{\Delta,1}(t)+y_{\Delta,2}(t)+y_{\Delta,3}(t)+y_{\Delta,4}(t))\\
&\quad\leq\sup_{t\in[0,T]}S(t)+\sup_{t\in[0,T]}|S(t)-y_{\Delta,1}(t)|+\sup_{t\in[0,T]}V(t)+\sup_{t\in[0,T]}|V(t)-y_{\Delta,2}(t)|\\
&\qquad+\sup_{t\in[0,T]}L(t)+\sup_{t\in[0,T]}|L(t)-y_{\Delta,3}(t)|+\sup_{t\in[0,T]}I(t)+\sup_{t\in[0,T]}|I(t)-y_{\Delta,4}(t)|\\
&\quad\leq\sup_{t\in[0,T]}(S(t)+V(t)+L(t)+I(t))+4\sup_{t\in[0,T]}|Z(t)-y_\Delta(t)|\\
&\quad\leq 1-\delta+4\frac{\delta}{8}\\
&\quad=1-\frac{\delta}{2}.
\end{aligned}$$

Hence, for every $\omega \in \Omega_2$ and $\Delta \in (0, \Delta_1]$, the numerical approximation y_Δ defined by (3.23) satisfies

$$y_\Delta(t) \in S_\delta \cap S_\Delta, \quad t \in [0,T],$$

and therefore both relevant projection operators act as the identity. Consequently, the coefficients in the Euler–Maruyama scheme defining y_Δ coincide with those in the scheme defining z_Δ. Since the two approximations have the same initial value and use the same Brownian increments, an induction over the grid points t_k shows that their discrete values coincide. By the definitions of the corresponding continuous-time interpolations (3.7) and (3.23), it follows that

$$y_\Delta(t) = z_\Delta(t), \quad t \in [0,T],\ \Delta \in (0, \Delta_1]. \tag{3.28}$$

Combining (3.26) and (3.28) with (3.24) we have that for any $\Delta \in (0, \Delta_1]$,

$$E\left(\sup_{t\in[0,T]} |Z(t) - z_\Delta(t)|^\theta I_{\Omega_2}(\omega)\right) \le C(\delta)\Delta^{\frac{\theta}{2}}. \tag{3.29}$$

Furthermore, substituting (3.28) into (3.27) we can conclude that for any $\omega \in \Omega_2$ and $\Delta \in (0, \Delta_1]$,

$$\sup_{t\in[0,T]} |Z(t) - z_\Delta(t)| < \frac{\delta}{8}. \tag{3.30}$$

We are now ready to show the required two assertions (3.17) and (3.18). It follows from (3.29) that for any $\Delta \in (0, \Delta_1]$

$$\begin{aligned}
E\left(\sup_{t\in[0,T]} |Z(t) - z_\Delta(t)|^\theta\right) =& E\left(\sup_{t\in[0,T]} |Z(t) - z_\Delta(t)|^\theta I_{\Omega_2}(\omega)\right) \\
&+ E\left(\sup_{t\in[0,T]} |Z(t) - z_\Delta(t)|^\theta I_{\Omega_2^c}(\omega)\right) \\
\le& C(\delta)\Delta^{\frac{\theta}{2}} + (P(\Omega_2^c))^{\frac{1}{2}} \left(E\left(\sup_{t\in[0,T]} |Z(t) - z_\Delta(t)|^{2\theta}\right)\right)^{\frac{1}{2}} \\
\le& C(\delta)\Delta^{\frac{\theta}{2}} + 2^\theta \epsilon^{\frac{1}{2}} \left(E\left(\sup_{t\in[0,T]} |Z(t)|^{2\theta}\right) + E\left(\sup_{t\in[0,T]} |z_\Delta(t)|^{2\theta}\right)\right)^{\frac{1}{2}} \\
\le& C(\delta)\Delta^{\frac{\theta}{2}} + 2^\theta \epsilon^{\frac{1}{2}} \left(E\left(\sup_{t\in[0,T]} |Z(t)|^{2\theta}\right) + H\right)^{\frac{1}{2}} \\
\le& C(\delta)\Delta^{\frac{\theta}{2}} + C\epsilon^{\frac{1}{2}},
\end{aligned} \tag{3.31}$$

where we use Lemma 3.1 and the boundedness of the exact solution Z of the system. Let $\Delta \to 0$ and then assertion (3.17) follows as ϵ is arbitrary.

Now, let the ith components of $\bar{Z}(t)$ be $\bar{Z}_{\Delta,i}(t)$ for $i \in \{1,2,3,4\}$. Then, using (3.30) we obtain that, for any $\Delta \in (0, \Delta_1]$ and $\omega \in \Omega_2$,

$$\sup_{t\in[0,T]} (\bar{Z}_{\Delta,1}(t) + \bar{Z}_{\Delta,2}(t) + \bar{Z}_{\Delta,3}(t) + \bar{Z}_{\Delta,4}(t))$$

$$
\begin{aligned}
&\leq \sup_{t\in[0,T]} (\bar{z}_{\Delta,1}(t) + \bar{z}_{\Delta,2}(t) + \bar{z}_{\Delta,3}(t) + \bar{z}_{\Delta,4}(t)) \\
&\leq \sup_{t\in[0,T]} (S(t) + V(t) + L(t) + I(t) + |S(t) - \bar{z}_{\Delta,1}(t)| + |V(t) - \bar{z}_{\Delta,2}(t)| \\
&\quad + |L(t) - \bar{z}_{\Delta,3}(t)| + |I(t) - \bar{z}_{\Delta,4}(t)|) \\
&\leq \sup_{t\in[0,T]} (S(t) + V(t) + L(t) + I(t)) + 4 \sup_{t\in[0,T]} |Z(t) - \bar{z}_{\Delta}(t)|) \\
&< 1 - \frac{\delta}{2} + 4\frac{\delta}{8} \\
&= 1 - \delta,
\end{aligned}
$$

and

$$
\begin{aligned}
\inf_{t\in[0,T]} \bar{Z}_{\Delta,1}(t) &\geq \inf_{t\in[0,T]} z_{\Delta,1}(t) \\
&\geq \inf_{t\in[0,T]} S(t) - \sup_{t\in[0,T]} |S(t) - z_{\Delta,1}(t)| \\
&\geq \inf_{t\in[0,T]} S(t) - \sup_{t\in[0,T]} |Z(t) - z_{\Delta}(t)| \\
&> \delta - \frac{\delta}{8} \\
&= \frac{7\delta}{8}.
\end{aligned}
$$

Similarly, we have

$$
\inf_{t\in[0,T]} \bar{Z}_{\Delta,2}(t) > \frac{7\pi}{8}, \quad \inf_{t\in[0,T]} \bar{Z}_{\Delta,3}(t) > \frac{7\pi}{8}, \quad \inf_{t\in[0,T]} \bar{Z}_{\Delta,4}(t) > \frac{7\pi}{8}.
$$

Thus, we can conclude that for any $\omega \in \Omega_2$

$$
Z_{\Delta}(t) = \bar{Z}_{\Delta}(t), \quad t \in [0,T]. \tag{3.32}
$$

Fix any $\Delta \in (0, \Delta_1]$. Recalling (3.32) we see that

$$
\begin{aligned}
E\left(\sup_{t\in[0,T]} |Z_{\Delta}(t) - \bar{Z}_{\Delta}(t)|^{\theta}\right) &= E\left(\sup_{t\in[0,T]} |Z_{\Delta}(t) - \bar{Z}_{\Delta}(t)|^{\theta} I_{\Omega_2^c}\right) \\
&\leq (P(\Omega_2^c))^{\frac{1}{2}} \left(E\left(\sup_{t\in[0,T]} |Z_{\Delta}(t) - \bar{Z}_{\Delta}(t)|^{2\theta}\right)\right)^{\frac{1}{2}} \\
&\leq 2^{\theta}\epsilon^{\frac{1}{2}} \left(E\left(\sup_{t\in[0,T]} |Z_{\Delta}(t)|^{2\theta}\right) + E\left(\sup_{t\in[0,T]} |\bar{Z}_{\Delta}(t)|^{2\theta}\right)\right)^{\frac{1}{2}} \\
&\leq C\epsilon^{\frac{1}{2}},
\end{aligned}
$$

where we use Lemma 3.1 and the boundedness of the exact solution Z of the system. Since $\epsilon \in (0,1)$ is arbitrary, assertion (3.18) follows immediately. □

Theorem 3.1. *For any $\theta > 0$, we have that*

$$\lim_{\Delta \to 0} E\left(\sup_{t\in[0,T]} |Z_\Delta(t) - Z(t)|^\theta\right) = 0. \tag{3.33}$$

Proof. For every $t \in [0,T]$, by the triangle inequality, we have that

$$|Z_\Delta(t) - Z(t)| \le |Z_\Delta(t) - \bar{Z}_\Delta(t)| + |\bar{Z}_\Delta(t) - z_\Delta(t)| + |z_\Delta(t) - Z(t)|.$$

Therefore, using the elementary inequality (3.11), we obtain that for every $\theta > 0$

$$\begin{aligned} &E\left[\sup_{t\in[0,T]} |Z_\Delta(t) - Z(t)|^\theta\right] \\ &\le (3^{\theta-1} \vee 1)\Bigg\{E\left(\sup_{t\in[0,T]} |Z_\Delta(t) - \bar{Z}_\Delta(t)|^\theta\right) + E\left(\sup_{t\in[0,T]} |\bar{Z}_\Delta(t) - z_\Delta(t)|^\theta\right) \\ &\qquad + E\left(\sup_{t\in[0,T]} |z_\Delta(t) - Z(t)|^\theta\right)\Bigg\}. \end{aligned}$$

By the estimates established in Lemmas 3.2 and 3.3, we have that

$$\lim_{\Delta \to 0} E\left(\sup_{t\in[0,T]} |Z_\Delta(t) - Z(t)|^\theta\right) = 0.$$

This completes the proof. □

4. Example and Numerical Simulations

In this section we will illustrate the previous theoretical findings by an example. In that sense, in the sequel we provide values of the parameters of the stochastic FMD model described by (2.1)–(2.4).

On the basis of the birth rate $r_h = 200\text{day}^{-1}$ and assumed natural death rate 20day^{-1} from [11], we obtain the birth and death rate $\mu = 180\text{day}^{-1}$. Moreover, the following values of the parameters of the model are taken from the cited paper: $\omega = 0.001\text{day}^{-1}$, $\emptyset = 0.5$, $d = 0.001\text{day}^{-1}$, $\gamma = 0.26\text{day}^{-1}$, $\beta = 0.004$, $\alpha = 0.2$. Additionally, we will assume that $\delta = 0.05$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.05$. In Figures 1–4, for $T = 1$ and $\Delta = 10^{-6}$, we plotted trajectories corresponding to the positivity and boundedness preserving Euler-Maruyama method and to Euler-Maruyama method for each coordinate process V, S, L and I, with the initial conditions $V_0 = 0.2, S_0 = 0.1, L_0 = 0.15$ and $I_0 = 0.1$. As one can observe, the trajectories corresponding to the Euler-Maruyama method may cross the lines $z = 0$ or $z = 1$, while those corresponding to the positivity and boundedness preserving Euler-Maruyama method stay within the interval $(0, 1)$.

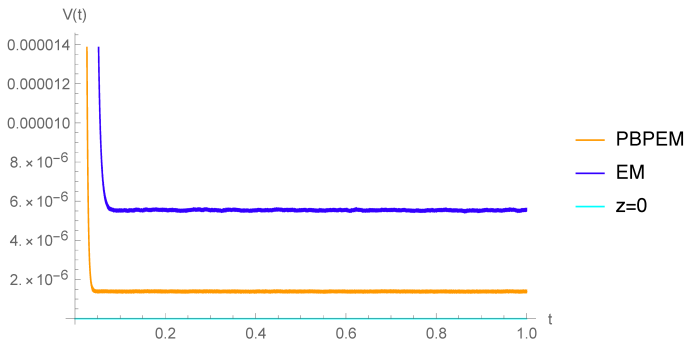


FIGURE 1. Trajectories of the PBPEM and EM solutions corresponding to V with $\Delta = 10^{-6}$

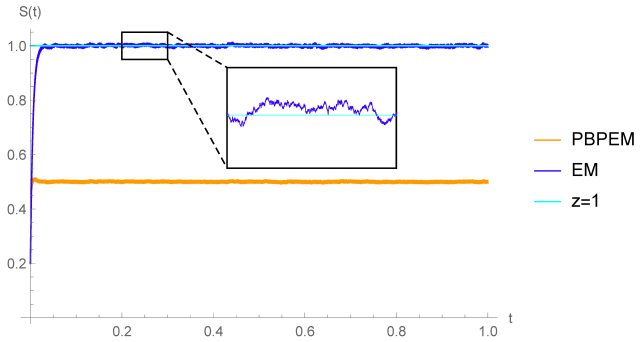


FIGURE 2. Trajectories of the PBPEM and EM solutions corresponding to S with $\Delta = 10^{-6}$

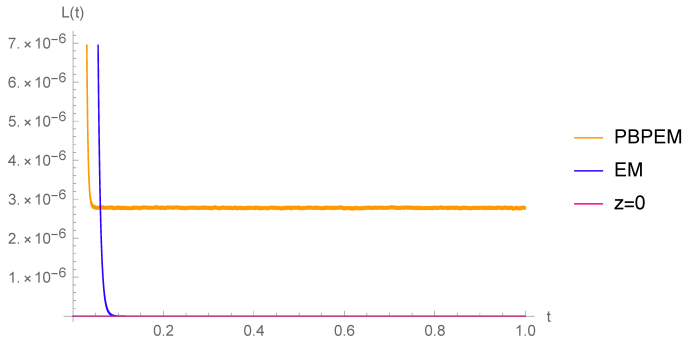


FIGURE 3. Trajectories of the PBPEM and EM solutions corresponding to L with $\Delta = 10^{-6}$

Acknowledgements. We would like to express our thankfulness to the anonymous reviewer for his valuable suggestions which improved the manuscript.

This work is supported by Ministry of Education, Science and Technological Development of the Republic of Serbia, Contract Numbers: 451-03-34/2026-03/200124 for the first author and 451-03-33/2026-03/200124 for second author.

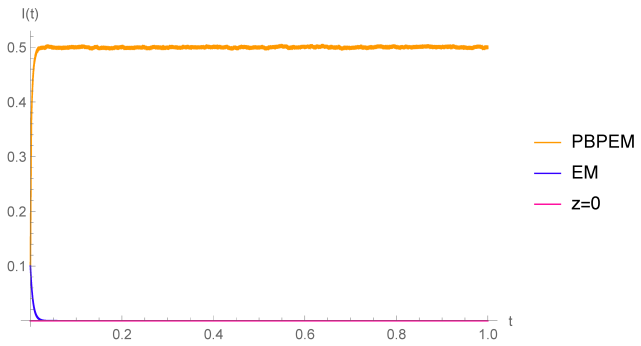


FIGURE 4. Trajectories of the PBPEM and EM solutions corresponding to I with $\Delta = 10^{-6}$

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