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HANKEL DETERMINANTS FOR A NEW SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR

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Abstract. Using the operator $L(a,c)$ defined by Carlson and Shaffer, we defined a new subclass of analytic functions $ML(\lambda,a,c)$. The well known Fekete-Szegö problem, upper bound of Hankel determinant of order two, and coefficient bound of the fourth coefficient is determined. Our investigation generalises some previous results obtained in different articles.

1. Introduction

We denote by $\mathcal{H}(\mathbb{D})$ the class of functions which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k, \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{P}$ be the well-known class of *Carathéodory functions*, that is $P \in \mathcal{H}(\mathbb{D})$ with the power series expansion

$$P(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in \mathbb{D}, \tag{1.2}$$

and $\operatorname{Re} P(z) > 0$ for all $z \in \mathbb{D}$.

For two functions $f, g \in \mathcal{H}(\mathbb{D})$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function $\psi \in \mathcal{H}(\mathbb{D})$, with $|\psi(z)| < 1$, $z \in \mathbb{D}$ and $\psi(0) = 0$, such that $f(z) = g(\psi(z))$ for all $z \in \mathbb{D}$. In particular, if g is

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univalent in $\mathbb{D}$ then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $h_s(z) = \sum_{k=0}^{\infty} a_{k,s}\, z^k$, $s = 1, 2$, which are analytic in $\mathbb{D}$, then the well-known *Hadamard* (*or convolution*) *product* of h_1 and h_2 is given by

$$(h_1 * h_2)(z) := \sum_{k=0}^{\infty} a_{k,1}\, a_{k,2}\, z^k, \quad z \in \mathbb{D}.$$

The *Carlson-Shaffer operator* [2] $L(a, c) : \mathcal{A} \to \mathcal{A}$ is defined by

$$L(a, c)f(z) := \widetilde{\varphi}(a, c; z) * f(z), \quad z \in \mathbb{D}, \tag{1.3}$$

where

$$\widetilde{\varphi}(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad z \in \mathbb{D},\, a \in \mathbb{C},\, c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\, \mathbb{Z}_0^- := \{\dots, -2, -1, 0\},$$

is the incomplete beta function and $(t)_k$ denotes the *Pochhammer symbol* (or the *shifted factorial*) defined in terms of the *Gamma function* by

$$(t)_k := \frac{\Gamma(t+k)}{\Gamma(t)} = \begin{cases} t(t+1)(t+2)\cdots(t+k-1), & \text{if } k \in \mathbb{N} := \{1, 2, \dots\}, \\ 1, & \text{if } k = 0. \end{cases}$$

For $f \in \mathcal{A}$ is given by (1.1) one can see by using (1.3) that

$$L(a, c)f(z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+1}\, z^{k+1}, \quad z \in \mathbb{D},$$

and

$$zL'(a, c)f(z) = aL(a+1, c)f(z) - (a-1)L(a, c)f(z), \quad z \in \mathbb{D}.$$

Remark 1.1. Next we will emphasize a few special cases of the operator $L(a, c)$, as follows:

(i) $L(a, a)f(z) = f(z)$;
(ii) $L(2, 1)f(z) = zf'(z)$;
(iii) $L(3, 1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$;
(iv) $L(m+1, 1)f(z) =: \mathcal{D}^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z)$, $m \in \mathbb{Z}$, $m > -1$, is the well-known *Ruscheweyh derivative* of f [22];
(v) $L(2, 2-\mu)f(z) =: \Omega_z^{\mu} f(z)$, $0 \le \mu < 1$, is the well-known *Owa-Srivastava fractional differential operator* [18].

For the function $f \in \mathcal{A}$ of the form (1.1) Noonan and Thomas [16] defined q-th Hankel determinant as

$$\mathcal{H}_{q,k}(f) := \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \dots & a_{k+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & a_{k+q} & \dots & a_{k+2q-2} \end{vmatrix}, \quad a_1 = 1,\, q, k \in \mathbb{N}.$$

The above determinant $\mathcal{H}_{q,k}(f)$ has been studied by several authors, for example, Pommerenke [19], Noonan and Thomas [16], Ehrenborg [4] and Noor [17].

These authors studied the Hankel determinant in their own developed way: for instance Noor [17] studied the rate of growth of $\mathcal{H}_{q,k}$ as $k \to \infty$ for functions of the form (1.1) with bounded boundary rotation. Unlike to Noor, Ehrenborg [4] has studied different order Hankel determinants taking a family of exponential polynomials. Layman's article [11] gave some ideas on Hankel transform of an integer sequence, and the article discusses some properties of the transform for integer sequences.

For $k = 1$, $q = 2$, $a_1 = 1$ and $k = q = 2$ the Hankel determinant simplifies to the functionals $|a_3 - a_2^2|$ and $|a_2a_4 - a_3^2|$, called Hankel determinants of order two, denoted by $\wedge_1 := \mathcal{H}_{2,1}(f)$ and $\wedge_2 := \mathcal{H}_{2,2}(f)$, respectively. It is well-known (see Duren [3]) that if f is given by (1.1) and is univalent in $\mathbb{D}$, then $\wedge_1 \le 1$ occurs, and this result is sharp.

For $\mathcal{T} \subset \mathcal{A}$, to find a sharp (best possible) upper bound of $\widetilde{\wedge}_c := |a_3 - c\,a_2^2|$ for the subclass $\mathcal{T}$ is generally called *Fekete-Szegö problem* for the subclass $\mathcal{T}$, where c is a real or a complex number. There are some subclasses of univalent functions, such that the starlike functions, convex functions and close-to-convex functions, for which the problem of finding sharp upper bounds for the functional $\widetilde{\wedge}_c$ was completely solved (see [5, 8–10]). For the family of analytic functions $\mathcal{R}$, such that for $f \in \mathcal{R}$ we have $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$, Janteng et al. [6, 7] have found the sharp upper bound to the second Hankel determinant $\wedge_2$. For initial work on the class $\mathcal{R}$ one may refer to the article of MacGregor [15].

In our paper we have defined a subclass of $\mathcal{A}$ using the concept of subordination and the linear operator $L(a, c)$.

Definition 1.1. Let $ML(\lambda, a, c)$ denotes the subclass of $\mathcal{A}$, members of which are of the form (1.1) and satisfy the subordination condition

$$\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} \prec \sqrt{1+z}, \tag{1.4}$$

with $\sqrt{1+z}\Big|_{z=0} = 1$ or equivalently

$$\left| \left[\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} \right]^2 - 1 \right| < 1, \quad z \in \mathbb{D},$$

where $a \in \mathbb{C}$, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $0 \le \lambda \le 1$.

Remark 1.2. (i) We will discuss the geometrical significance of the class $ML(\lambda, a, c)$. If we set $h(z) = \sqrt{1+z}$, $z \in \mathbb{D}$, with $h(0) = 1$, and denote

$$\omega := h(e^{i\theta}) = \sqrt{1 + e^{i\theta}}, \quad \theta \in [0, 2\pi] \setminus \{\pi\},$$

this yields $\omega^2 - 1 = e^{i\theta}$ or $|\omega^2 - 1| = 1$. Letting $\omega = u + iv$, $u, v \in \mathbb{R}$, we deduce that

$$\left(u^2 + v^2\right)^2 = 2\left(u^2 - v^2\right).$$

Thus, $h(\mathbb{D})$ is the region bounded by the right-half of the *Bernoulli's lemniscate* given by $\left\{u+iv\in\mathbb{C}:\left(u^2+v^2\right)^2=2\left(u^2-v^2\right)\right\}$, which implies that the functions in $ML(\lambda,a,c)$ have a positive real part.

(ii) Using the point (i) of the Remark 1.1, for $a=c$ we denote $ML(\lambda):=ML(\lambda,a,a)$, and member of this class satisfies the subordination condition

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z}\prec\sqrt{1+z},$$

with $\left.\sqrt{1+z}\right|_{z=0}=1$ or equivalently

$$\left|\left[\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z}\right]^2-1\right|<1,\quad z\in\mathbb{D}.$$

(iii) Remark that the subclass

$$ML(0)=SL^*:=\left\{f\in\mathcal{A}:\left|\left[\frac{zf'(z)}{f(z)}\right]^2-1\right|<1,\ z\in\mathbb{D}\right\}$$

was introduced and studied by Sokól and Stankiewicz [25], and Raza and Mallik [21] determined the upper bound of third Hankel determinant for the class SL^*. Also, the subclass $ML(1):=\left\{f\in\mathcal{A}:\left|[f'(z)]^2-1\right|<1,\ z\in\mathbb{D}\right\}$ was studied by Sahoo and Patel [23].

In our work we have used the techniques of Libera and Zlotkiewicz [12] and Koepf [9], combined with the help of MAPLE™ software to find an upper bound of $\widetilde{\wedge}_\mu$ and $\wedge_2$, and of the coefficient a_4 for the functions belonging to the class $ML(\lambda,a,c)$.

2. Preliminaries

To establish our main results, we shall need the followings lemmas. The first lemma is the well-known *Carathéodory's lemma* (see also [20, Corollary 2.3.]).

Lemma 2.1 ([1]). *If $P\in\mathcal{P}$ and given by* (1.2)*, then $|p_k|\le 2$ for all $k\ge 1$ and the result is best possible for the function $P_*(z)=\frac{1+\rho z}{1-\rho z}$, $|\rho|=1$.*

The next lemma gives us a majorant for the coefficients of the functions of the class $\mathcal{P}$, and more details may be found in [14, Lemma 1].

Lemma 2.2 ([13]). *Let the function P given by* (1.2) *be a member of the class $\mathcal{P}$. Then*

$$\left|p_2-\nu p_1^2\right|\le 2\max\left\{1,|2\nu-1|\right\},\quad \text{where } \nu\in\mathbb{C}. \tag{2.1}$$

The result is sharp for the functions given by

$$P^*(z)=\frac{1+\rho^2z^2}{1-\rho^2z^2}\quad\text{and}\quad P_*(z)=\frac{1+\rho z}{1-\rho z},\quad |\rho|=1.$$

Lemma 2.3 ([13]). *Let the function P given by* (1.2) *be a member of the class $\mathcal{P}$. Then*

$$p_2 = \frac{1}{2}\left[p_1^2 + \left(4 - p_1^2\right)x\right] \tag{2.2}$$

and

$$p_3 = \frac{1}{4}\left[p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)(1 - |x|^2)z\right], \tag{2.3}$$

for some complex numbers x, z satisfying $|x| \le 1$ and $|z| \le 1$.

Other details regarding the above lemma my be found in [13], relations (3.9) and (3.10).

3. Main Results

In our first result we will determine an upper bound for $\widetilde{\Lambda}_\mu$, and this tends to solve the Fekete-Szegö problem for the subclass $ML(\lambda, a, c)$.

Theorem 3.1. *For $f \in ML(\lambda, a, c)$ and is in the form given by* (1.1) *then, for any $\mu \in \mathbb{C}$ we have*

$$\begin{aligned}\left|a_3 - \mu\, a_2^2\right| \le & \frac{|(c)_2|}{|(a)_2|} \cdot \frac{1}{2(2+\lambda)} \\ & \times \max\left\{1,\ \frac{|(3\lambda - 1)(1+\lambda)a(c+1) + 2\mu(2+\lambda)c(a+1)|}{4(1+\lambda)^2|a(c+1)|}\right\}.\end{aligned} \tag{3.1}$$

Proof. If $f \in ML(\lambda, a, c)$, from (1.4) it follows that there exists a function $\psi \in \mathcal{H}(\mathbb{D})$ satisfying the conditions $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathbb{D}$, such that

$$\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} = \sqrt{1 + \psi(z)}, \quad z \in \mathbb{D}. \tag{3.2}$$

Setting

$$P(z) := \frac{1 + \psi(z)}{1 - \psi(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in \mathbb{D},$$

then $P \in \mathcal{P}$. From the above relation, we get

$$\psi(z) = \frac{P(z) - 1}{P(z) + 1}, \quad z \in \mathbb{D},$$

and from (3.2) it follows that

$$\frac{zL'(a,c)f(z)}{(1-\lambda)L(a,c)f(z) + \lambda z} = \left(\frac{2P(z)}{1 + P(z)}\right)^{\frac{1}{2}}, \quad z \in \mathbb{D}. \tag{3.3}$$

It is easy to show that

$$\left(\frac{2P(z)}{1+P(z)}\right)^{\frac{1}{2}} = 1+\frac{1}{4}p_1 z+\left(\frac{1}{4}p_2-\frac{5}{32}p_1^2\right)z^2 + \left(\frac{1}{4}p_3-\frac{5}{16}p_1p_2+\frac{13}{128}p_1^3\right)z^3+\cdots, \quad z\in\mathbb{D},$$

and identifying the coefficients of z, z^2 and z^3 in (3.3) we deduce that

$$a_2=\frac{c}{a}\cdot\frac{p_1}{4(1+\lambda)}, \tag{3.4}$$

$$a_3=\frac{(c)_2}{(a)_2}\cdot\frac{1}{4(2+\lambda)}\left[p_2-\frac{(7\lambda+3)}{8(1+\lambda)}p_1^2\right], \tag{3.5}$$

$$a_4=\frac{(c)_3}{(a)_3}\cdot\frac{1}{4(3+\lambda)}\left[p_3-\frac{7\lambda^2+16\lambda+7}{4(1+\lambda)(2+\lambda)}p_1p_2+\frac{25\lambda^2+40\lambda+13}{32(1+\lambda)(2+\lambda)}p_1^3\right]. \tag{3.6}$$

Thus, from (3.4) and (3.5) we get

$$\left|a_3-\mu\, a_2^2\right|=\frac{1}{4(2+\lambda)}\cdot\frac{|(c)_2|}{|(a)_2|}\left|p_2-\left[\frac{(7\lambda+3)(\lambda+1)a(c+1)+2\mu(2+\lambda)c(a+1)}{8(1+\lambda)^2a(c+1)}\right]p_1^2\right|,$$

which with the aid of the inequality (2.1) of Lemma 2.2 yields the required estimate (3.1). □

For $a=c$ the above theorem reduces to the following special case.

Corollary 3.1. *If $f\in ML(\lambda)$ and is given by* (1.1)*, then for any $\mu\in\mathbb{C}$ we have*

$$\left|a_3-\mu\, a_2^2\right|\le\frac{1}{2(2+\lambda)}\max\left\{1,\ \frac{|(3\lambda-1)(1+\lambda)+2\mu(2+\lambda)|}{4(1+\lambda)^2}\right\}.$$

If we take $\mu\in\mathbb{R}$ in Theorem 3.1 we get the next special case.

Corollary 3.2. *If the function $f\in ML(\lambda,a,c)$ and is given by* (1.1)*, with $\mu\in\mathbb{R}$ and $a>c\ge 0$, then*

$$\left|a_3-\mu\, a_2^2\right|\le\begin{cases}\dfrac{a(c+1)(3\lambda-1)(\lambda+1)+2\mu c(a+1)(2+\lambda)}{8(\lambda+1)^2a(c+1)(2+\lambda)}\cdot\dfrac{(c)_2}{(a)_2}, & \text{if } \mu<\delta_1,\\[2mm] \dfrac{1}{2(2+\lambda)}\cdot\dfrac{(c)_2}{(a)_2}, & \text{if } \delta_1\le\mu\le\delta_2,\\[2mm] -\dfrac{a(c+1)(3\lambda-1)(\lambda+1)+2\mu c(a+1)(2+\lambda)}{8(\lambda+1)^2a(c+1)(2+\lambda)}\cdot\dfrac{(c)_2}{(a)_2}, & \text{if } \mu>\delta_2,\end{cases}$$

where

$$\delta_1:=-\frac{(7\lambda+3)(\lambda+1)}{2(2+\lambda)}\cdot\frac{a(c+1)}{c(a+1)}\quad\text{and}\quad\delta_2:=\frac{(\lambda+1)(\lambda+5)}{2(2+\lambda)}\cdot\frac{a(c+1)}{c(a+1)}.$$

Remark 3.1. (i) Putting $\lambda=1$ in Corollary 3.1 and Corollary 3.2 we get the recent results due to Sahoo and Patel [23, Theorem 2.1] and [23, Corollary 2.2], respectively.

(ii) For $\lambda = 0$, Corollary 3.1 and Corollary 3.2 reduce to the results of Raza and Malik [21, Theorem 2.1] and [21, Theorem 2.2], respectively.

The next result deals with an upper bound of $\wedge_2$ for the subclass $ML(\lambda, a, c)$.

Theorem 3.2. *For $a \geq c > 0$, if the function f given by* (1.1) *belongs to the class $ML(\lambda, a, c)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \left(\frac{(c)_2}{(a)_2}\right)^2 \frac{1}{4(2+\lambda)^2}. \tag{3.7}$$

Proof. If $f \in ML(\lambda, a, c)$, using a similar proof like in the proof of Theorem 3.1, from (3.4), (3.5) and (3.6) we get

$$a_2a_4 - a_3^2 = k_1p_1^4 + k_2p_1^2p_2 + k_3p_1p_3 + k_4p_2^2,$$

where

$$\begin{aligned} k_1 &= \frac{25\lambda^2 + 40\lambda + 13}{512(1+\lambda)^2(2+\lambda)(3+\lambda)} \cdot \frac{c}{a} \cdot \frac{(c)_3}{(a)_3} - \left(\frac{(c)_2}{(a)_2}\right)^2 \frac{1}{16(2+\lambda)^2} \left(\frac{7\lambda+3}{8(1+\lambda)}\right)^2, \\ k_2 &= \frac{7\lambda+3}{64(2+\lambda)^2(1+\lambda)} \left(\frac{(c)_2}{(a)_2}\right)^2 - \frac{c}{a} \cdot \frac{(c)_3}{(a)_3} \cdot \frac{7\lambda^2+16\lambda+7}{64(1+\lambda)^2(2+\lambda)(3+\lambda)}, \\ k_3 &= \frac{c}{a} \cdot \frac{(c)_3}{(a)_3} \cdot \frac{1}{16(1+\lambda)(3+\lambda)}, \\ k_4 &= -\left[\left(\frac{(c)_2}{(a)_2}\right)^2 \frac{1}{16(2+\lambda)^2}\right]. \end{aligned}$$

Using the relations (2.2) and (2.3) of Lemma 2.3, we get

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \qquad (3.8)\\ &= \left|Ap_1^4 + B\left(4-p_1^2\right)xp_1^2 + \left[\frac{k_4}{4}\left(4-p_1^2\right) - \frac{k_3}{4}p_1^2\right]\left(4-p_1^2\right)x^2\right. \\ &\left. + \frac{k_3}{2}p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\right|, \qquad (3.9) \end{aligned}$$

with $|x| \leq 1$, $|z| \leq 1$ and

$$\begin{aligned} A :=& \frac{1}{4}\left(4k_1 + 2k_2 + k_3 + k_4\right) = \frac{c(c)_2}{a(a)_2\left[1024(a+1)(a+2)(2+\lambda)^2(1+\lambda)^2(3+\lambda)\right]} \\ &\times \Big[(-4ac - 13c + a - 8)\lambda^3 + (-11ac - 11a - 40c - 22)\lambda^2 \\ &+ (19ac + 36c + 21a + 41)\lambda + (3ac + 3c + 5a + 9)\Big], \\ B :=& \frac{1}{2}\left(k_2 + k_3 + k_4\right) = \frac{c(c)_2\Big[3(c-a)\lambda^2 + (ac - 6a + 9c + 2)\lambda - 5ac - 7a\Big]}{a(a)_2\Big[128(1+\lambda)(2+\lambda)^2(3+\lambda)(a+1)(a+2)\Big]}. \end{aligned}$$

Since $P \in \mathcal{P}$ it follows that $P\left(e^{-i \arg p_1} z\right) \in \mathcal{P}$, hence we may assume without loss of generality that $p := p_1 \geq 0$, and according to Lemma 2.1 it follows that $p \in [0,2]$. Now, using the triangle's inequality in (3.8) and substituting $|x| = t$ we get

$$\left|a_2 a_4 - a_3^2\right| \leq |A|\, p^4 + |B| \left(4-p^2\right) p^2 t + \frac{|k_4|}{4} \left(4-p^2\right)^2 t^2 + \frac{|k_3|}{4} p^2 \left(4-p^2\right) t^2 + \frac{|k_3|}{2} p \left(4-p^2\right)(1-t^2) =: \mathcal{G}(p,t), \quad 0 \leq p \leq 2,\ 0 \leq t \leq 1.$$

Next, we will find maximum of $\mathcal{G}(p,t)$ on the closed rectangle $[0,2] \times [0,1]$. Using the MAPLE™ software for the following code, where we denoted $C := k_4$ and $D = E := k_3$,

```
[> G:= abs(A)*p^4+abs(B)*(-p^2+4)*p^2*t+(1/4)*abs(C)*(-p^2+4)^2*t^2
+(1/4)*abs(D)*p^2*(-p^2+4)*t^2+(1/2)*abs(\mathbb{D})*p
*(-p^2+4)*(-t^2+1);
[> maximize(G, p = 0 .. 2, t = 0 .. 1, location);
```

we get

```
max(16 |A|, 4 |C|), {[{p = 2}, 16 |A|], [{p = 0, t = 1}, 4 |C|]}
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that is

$$\max\left\{\mathcal{G}(p,t) : (p,t) \in [0,2] \times [0,1]\right\} = \max\{16|A|, 4|C|\}$$

and

$$16|A| = \mathcal{G}(2,t), \quad 4|C| = \mathcal{G}(0,1).$$

We will prove that under our assumption we have $4|C| \geq 16|A|$ and therefore

$$\max\left\{\mathcal{G}(p,t) : (p,t) \in [0,2] \times [0,1]\right\} = 4|C| = 4\,|k_4| = \mathcal{G}(0,1). \tag{3.10}$$

Letting $\alpha := \frac{c}{a} \cdot \frac{(c)_3}{(a)_3}$ and $\beta := \left(\frac{(c)_2}{(a)_2}\right)^2$, since $a \geq c > 0$ it follows that $\alpha \geq \beta > 0$, and first we will show that $A > 0$. A simple computation shows that

$$4A = 4k_1 + 2k_2 + k_3 + k_4 = \alpha \frac{5\lambda^2+1}{128(1+\lambda)^2(2+\lambda)(3+\lambda)} - \beta \frac{9\lambda^2 - 6\lambda + 1}{256(1+\lambda)^2(2+\lambda)^2},$$

and using the fact that

$$\frac{5\lambda^2+1}{128(1+\lambda)^2(2+\lambda)(3+\lambda)} - \frac{9\lambda^2-6\lambda+1}{256(1+\lambda)^2(2+\lambda)^2} = \frac{\lambda^3 + 19\lambda + (1-\lambda^2)}{256(1+\lambda)^2(2+\lambda)^2(3+\lambda)} > 0, \quad 0 \leq \lambda \leq 1,$$

it follows that $A > 0$. Hence,

$$16|A| - 4|C| = \alpha \left[\frac{5\lambda^2+1}{32(1+\lambda)^2(2+\lambda)(3+\lambda)}\right] - \beta \left[\frac{9\lambda^2-6\lambda+1}{64(1+\lambda)^2(2+\lambda)^2} + \frac{1}{4(2+\lambda)^2}\right]$$
$$= \frac{\lambda^3(10\alpha - 25\beta) + \lambda^2(20\alpha - 101\beta) + \lambda(2\alpha - 95\beta) + (4\alpha - 51\beta)}{64(1+\lambda)^2(2+\lambda)^2(3+\lambda)},$$

and since $0 \le \lambda \le 1$, each term of the numerator is not positive if

$$\frac{\alpha}{\beta} \le \min\left\{\frac{25}{10}, \frac{101}{20}, \frac{95}{2}, \frac{51}{4}\right\} = \frac{25}{10},$$

which is equivalent to $3ac + a + 8c + 6 \ge 0$. This last inequality holds for all $a > 0$ and $c \ge 0$, and therefore $16|A| \le 4|C|$. Since (3.10) was proved, the upper bound of $\mathcal{G}(p,t)$ on the closed rectangle $[0,2] \times [0,1]$ is attained at $p = 0$ and $t = 1$, which implies the inequality (3.7). □

For $a = c$ Theorem 3.2 reduces to the next special case.

Corollary 3.3. *If the function f given by* (1.1) *belongs to the class $ML(\lambda)$, then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{4(2+\lambda)^2}.$$

Remark 3.2. (i) For $\lambda = 1$, Corollary 3.3 reduces to the result due to Sahoo and Patel [23, Theorem 2.2].

(ii) Taking $\lambda = 0$ in Corollary 3.3 we obtain the recent result of Raza and Malik [21, Theorem 2.4].

In our last result we found an upper bound of the fourth coefficient for the functions of $ML(\lambda, a, c)$.

Theorem 3.3. *If $a \ge c > 0$ and the function f given by* (1.1) *belongs to the class $ML(\lambda, a, c)$, then*

$$|a_4| \le \frac{(c)_3}{(a)_3} \cdot \frac{1}{2(3+\lambda)}.$$

Proof. If $f \in ML(\lambda, a, c)$, using a similar proof like in the proof of Theorem 3.1, from (3.6) we obtain

$$a_4 = \frac{(c)_3}{(a)_3} \cdot \frac{1}{4(3+\lambda)}\left[p_3 - \frac{7\lambda^2 + 16\lambda + 7}{4(1+\lambda)(2+\lambda)}p_1p_2 + \frac{25\lambda^2 + 40\lambda + 13}{32(1+\lambda)(2+\lambda)}p_1^3\right]. \tag{3.11}$$

Replacing in (3.11) the values of p_2 and p_3 with those given by the relations (2.2) and (2.3), respectively, and denoting $p := p_1$ we get

$$\begin{aligned} a_4 =& \frac{(c)_3}{(a)_3} \cdot \frac{1}{4(3+\lambda)}\left[\frac{5\lambda^2 + 1}{32(1+\lambda)(2+\lambda)}p^3 - \frac{3\lambda^2 + 4\lambda - 1}{8(1+\lambda)(2+\lambda)}\left(4 - p^2\right)px\right. \\ &\left. - \frac{1}{4}\left(4 - p^2\right)px^2 + \frac{1}{2}\left(4 - p^2\right)(1 - |x|^2)z\right], \end{aligned}$$

for some complex numbers x and z, with $|x| < 1$ and $|z| \leq 1$. Using the triangle's inequality and substituting $|x| = y$ we get

$$\begin{aligned}|a_4| \leq & \frac{(c)_3}{(a)_3} \cdot \frac{1}{4(3+\lambda)} \times \left[\frac{5\lambda^2+1}{32(1+\lambda)(2+\lambda)} p^3 + \frac{|3\lambda^2+4\lambda-1|}{8(1+\lambda)(2+\lambda)} \left(4-p^2\right) py \right. \\ & \left. + \frac{1}{4}\left(4-p^2\right) py^2 + \frac{1}{2}\left(4-p^2\right)\left(1-y^2\right) \right] =: \mathcal{T}(p,y), \quad 0 \leq p \leq 2,\, 0 \leq y \leq 1.\end{aligned}$$

Now we will find the maximum of the function $\mathcal{T}(p,y)$ on the closed rectangle $[0,2] \times [0,1]$. Denoting

$$\begin{aligned}\mathcal{H}(p,y) := & \frac{5\lambda^2+1}{32(1+\lambda)(2+\lambda)} p^3 + \frac{|3\lambda^2+4\lambda-1|}{8(1+\lambda)(2+\lambda)} \left(4-p^2\right) py \\ & + \frac{1}{4}\left(4-p^2\right) py^2 + \frac{1}{2}\left(4-p^2\right)\left(1-y^2\right),\end{aligned}$$

and using the MAPLE$^{\text{TM}}$ software for the following code

```
[> H := (5*l^2+1)*p^3/((32*(1+l))*(2+l))
+abs(3*l^2+4*l-1)*(-p^2+4)*p*y/((8*(1+l))*(2+l))
+(1/4*(-p^2+4))*p*y^2+(1/2*(-p^2+4))*(-y^2+1);
[> maximize(H, p = 0 .. 2, y = 0 .. 1, location);
```

we get

```
max(2, (1/4)*(5*l^2+1)/((1+l)*(2+l))),
{[{p = 2}, (1/4)*(5*l^2+1)/((1+l)*(2+l))], [{p = 0, y = 0}, 2]}
```

that is

$$\max\left\{\mathcal{H}(p,y) : (p,y) \in [0,2] \times [0,1]\right\} = \max\left\{2, \frac{5\lambda^2+1}{4(1+\lambda)(2+\lambda)}\right\},$$

and

$$2 = \mathcal{H}(0,0), \quad \frac{5\lambda^2+1}{4(1+\lambda)(2+\lambda)} = \mathcal{H}(2,y).$$

A simple computation shows that $2 > \frac{5\lambda^2+1}{4(1+\lambda)(2+\lambda)}$, whenever $\lambda \geq 0$, therefore

$$\max\left\{\mathcal{H}(p,t) : (p,t) \in [0,2] \times [0,1]\right\} = 2 = \mathcal{H}(0,0),$$

which implies that

$$\max\left\{\mathcal{T}(p,y) : (p,y) \in [0,2] \times [0,1]\right\} = \frac{(c)_3}{(a)_3} \cdot \frac{1}{2(3+\lambda)} = \mathcal{T}(0,0),$$

and the proof of our theorem is complete. □

Putting $a = c$ in Theorem 3.3 we get the next special case.

Corollary 3.4. *If the function f given by* (1.1) *belongs to the class $ML(\lambda)$, then*

$$|a_4| \leq \frac{1}{2(3+\lambda)}.$$

Remark 3.3. (i) For $\lambda = 1$, Corollary 3.4 reduces to the recent result due to Sahoo and Patel [23, Theorem 2.3].

(ii) Taking $\lambda = 0$ in Corollary 3.4 we get the result due to Sokół [24, Theorem 2].

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