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SOME REMARKS ON FIXED POINT THEOREMS FOR GENERALIZED NON-EXPANSIVE MAPPINGS

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Abstract. The results of this paper concern the existence of fixed points for generalized non-expansive mappings, that is,

$$\sigma(Tx,Ty) \leq \alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \delta\sigma(x,Ty) + L\sigma(y,Tx),$$

where $\alpha+\beta+\gamma+2s\min\{\delta,L\}=1$, in the setting of complete b-metric spaces. In particular, we obtain interesting results in complete metric spaces that generalize those of: J. Bogin, *A generalization of a fixed point theorem of Goebel, Kirk and Shimi*, Canad. Math. Bull **19**(1) (1976), 7–12 and J. S. Bae, *Fixed point theorems of generalized non-expansive maps*, J. Korean Math. Soc. **21**(2) (1984), 233–248, without assuming the compactness of the space. Moreover, the provided examples demonstrate the applicability of our results, while various well-known contraction conditions fail to apply.

1. Introduction

In [16], the author obtained a fixed point theorem concerning a class of generalized contraction mappings, that is, mappings satisfying the following inequality for all $x, y \in E$:

$$d(Tx,Ty) \leq \alpha\, d(x,y) + \beta\, d(x,Tx) + \gamma\, d(y,Ty) + \delta\, d(x,Ty) + L\, d(y,Tx), \tag{1.1}$$

under the conditions $\alpha+\beta+\gamma+\delta+L<1$, where (E,d) is a complete metric space. Such mappings are also known as Hardy-Rogers type contraction mappings.

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In the case $\alpha+\beta+\gamma+\delta+L=1$, we say that T is a generalized non-expansive mapping. For such mappings, we cannot deduce the existence or uniqueness of a fixed point without imposing additional conditions.

One of the celebrated theorems concerning generalized non-expansive mappings is due to [7, Theorem 1]. We recall this theorem here for comparison purposes.

Theorem 1.1. *Let (E,d) be a complete metric space and a mapping $T: E \to E$ satisfying for all $x, y \in E$:*

$$d(Tx,Ty) \le \alpha d(x,y) + b\left[d(x,Tx)+d(y,Ty)\right] + c\left[d(x,Ty)+d(y,Tx)\right]. \tag{1.2}$$

If the following conditions hold

$$\alpha + 2b + 2c = 1, \quad b > 0, \quad c > 0, \tag{B}$$

then T has a unique fixed point.

We observe that Bogin assumed all coefficients are nonzero. From this observation, one may naturally ask whether some results can still be obtained if one of these coefficients vanishes. In [4], the author provided a partial positive answer to this question in the particular case $b=0$, under the stronger assumption that the space is compact.

Theorem 1.2. ([4, Theorem 2.3]). *Let (E,d) be a compact metric space and $T: E \to E$ a self-mapping satisfying*

$$d(Tx,Ty) \le \alpha d(x,y) + c\left[d(x,Ty)+d(y,Tx)\right], \tag{1.3}$$

where

$$\alpha + 2c = 1, \quad c \neq 0. \tag{1.4}$$

Then, T has a unique fixed point.

The aim of this work is to investigate the existence of fixed points for generalized non-expansive mappings under weaker conditions imposed on the associated coefficients, within the framework of complete b-metric spaces. The notion of a b-metric space (referred to in some references as a quasi-metric space) was introduced by Bakhtin [5], and later studied by Czerwik in the particular case $s=2$ in [11], and in full generality in [12]. This concept represents a natural extension of classical metric spaces, obtained by relaxing the triangle inequality by a constant factor $s \ge 1$, thereby offering greater flexibility. For a concise historical account of this class of spaces, we refer the reader to [6]. For completeness, we recall below the formal definition of these spaces.

Definition 1.1. Let E be a nonempty set, and $s \ge 1$ be a given real number. A mapping $\sigma: E \times E \to [0,+\infty)$ is called a b-metric if the following conditions hold for every $x, y, z \in E$:

(b_1) $\sigma(x,y)=0$ if and only if $x=y$;
(b_2) $\sigma(x,y)=\sigma(y,x)$;
(b_3) $\sigma(x,y) \le s[\sigma(x,z)+\sigma(z,y)]$.

The pair (E, σ) is called a b-metric space with constant s.

Some basic properties and examples of b-metric spaces may be found in several papers, see for example [1–3].

Recently, M. Cvetković in [10] proved that any Hardy-Rogers contraction in a metric space can be viewed as a Banach contraction by modifying the metric as follows.

Theorem 1.3. *Let (E, d) be a metric space, and let $T : E \to E$ be a mapping satisfying the Hardy-Rogers contraction:*

$$d(Tx, Ty) \leq a\, d(x, y) + b\Big(d(x, Tx) + d(y, Ty)\Big) + c\Big(d(x, Ty) + d(y, Tx)\Big), \tag{1.5}$$

where $a + 2b + 2c < 1$. Then, the following hold.

(a) *The mapping $D : E \times E \to [0, +\infty)$ defined by $D(x, x) = 0$ and*

$$D(x, y) = d(x, Tx) + d(y, Ty), \quad x \neq y,$$

is a metric on E.

(b) *If (E, d) is complete, then (E, D) is also complete.*

(c) *For all $x, y \in E$, we have*

$$D(Tx, Ty) \leq q\, D(x, y), \tag{1.6}$$

where $q = \frac{a+b+c}{1-b-c} < 1$. This means that T is a Banach contraction with respect to the metric D.

When $a + 2b + 2c = 1$, we obtain that T becomes a non-expansive mapping, and D may not be a metric on E in this case.

The authors in [8] prove the following theorems.

Theorem 1.4. *Let (E, σ) be a complete b-metric space with constant $s \geq 1$, and a mapping $T : E \to E$ satisfying*

$$\sigma(Tx, Ty) \leq \alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) + \delta\sigma(x, Ty) + L\sigma(y, Tx). \tag{1.7}$$

Moreover, we assume the following condition is fulfilled

$$\alpha + \beta + \gamma + 2s \min\{\delta, L\} < 1. \tag{R}$$

If $s\beta + s^2L < 1$ or $s\gamma + s^2\delta < 1$ holds, then T has at least a fixed point in E. If moreover $\alpha + \delta + L < 1$ is fulfilled, then T has a unique fixed point $x^* \in E$.

If (E, d) is a complete metric space, we get the following corollary.

Corollary 1.1. *Let (E, d) be a complete metric space, and a mapping $T : E \to E$ satisfying*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx), \tag{1.8}$$

where

$$\alpha + \beta + \gamma + 2 \min\{\delta, L\} < 1. \tag{M_0}$$

Then, T has at least a fixed point in E. If moreover $\alpha + \delta + L < 1$ is fulfilled, then T has a unique fixed point $x^* \in E$.

Remark 1.1. The condition (M_0) is weaker than the condition

$$\alpha+\beta+\gamma+\delta+L<1.$$

Moreover, the function D defined in Theorem 1.3 may not be a metric on E, because $D(x,y)=0$ implies that $d(x,Tx)=d(y,Ty)=0$, which does not necessarily mean that $x=y$. In other words, the fixed point is not necessarily unique.

For the case of generalized non-expansive mappings, we have the following theorem.

Theorem 1.5 ([8])**.** *Let (E,σ) be a complete b-metric space with constant $s\geq 1$, and $T:E\to E$ a self- mapping satisfying*

$$\sigma(Tx,Ty)\leq \alpha\sigma(x,y)+\beta\sigma(x,Tx)+\gamma\sigma(y,Ty)+\delta\sigma(x,Ty)+L\sigma(y,Tx), \tag{1.9}$$

where

$$\alpha+\beta+\gamma+s(\delta+L)=1,\quad \beta+\gamma\neq 0,\quad \delta+L\neq 0. \tag{R_0}$$

If T is continuous or $s\beta+s^2L<1$ or $s\gamma+s^2\delta<1$, then T has a unique fixed point $x^*\in E$.

If $s=1$, $\beta=\gamma$ and $\delta=L$, then we get Theorem 1.1.

In order to prove the above theorems, the authors use the following basic lemma.

Lemma 1.1. *Let (E,σ) be a complete b-metric space with constant $s\geq 1$, and (x_n) be a sequence in E. Assume that there exist $\lambda\in[0,1)$ and a positive number M, such that*

$$\sigma(x_{n+1},x_n)\leq M\lambda^n,\quad \text{for all } n\in\mathbb{N}. \tag{1.10}$$

Then, (x_n) converges to some element $x^*\in E$. In addition, for all $n\in\mathbb{N}$ and for every fixed integer N such that $s\,\lambda^N<1$, we have the evaluation of order of convergence is given as follows

$$\sigma(x_n,x^*)\leq s^2A\left[\frac{s^2\lambda^N}{1-s\lambda^N}+s+1\right]\lambda^{\lfloor\frac{n}{N}\rfloor N}, \tag{1.11}$$

where $\lfloor x\rfloor$ means the integer part of x and $A=M\sum_{k=1}^{N-1}s^k+s^{N-1}$.

In the next section, we need the following lemma.

Theorem 1.6 ([19])**.** *Let (E,σ) be a b-metric space with constant $s\geq 1$ and (x_n) is a convergent sequence with $\lim_{n\to+\infty}x_n=x$. Then, for all $y\in E$*

$$\frac{1}{s}\sigma(x,y)\leq \liminf_{n\to+\infty}\sigma(x_n,y)\leq \limsup_{n\to+\infty}\sigma(x_n,y)\leq s\sigma(x,y). \tag{1.12}$$

2. Main Results

First, we state the main theorem.

Theorem 2.1. *Let (E, σ) be a complete b-metric space with constant $s \geq 1$, and $T : E \to E$ a self-mapping satisfying*

$$\sigma(Tx, Ty) \leq \alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) + \delta\sigma(x, Ty) + L\sigma(y, Tx), \tag{2.1}$$

where (R_1) or (R_2) holds with

(R_1) $\quad \alpha + \beta + \gamma + 2s\delta = 1, \delta \neq 0 \quad$ *and* $\quad L < s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma,$

(R_2) $\quad \alpha + \beta + \gamma + 2sL = 1, L \neq 0 \quad$ *and* $\quad \delta < s(2s-1)L + (2s-1)\beta + (2s-1)\gamma.$

If $s\beta + sL < 1$ or $s\gamma + s\delta < 1$, then T has at least a fixed point in E. If moreover $\alpha + \delta + L < 1$, then T has a unique fixed point.

In order to prove this theorem we need some lemmas, where the key ideas are borrowed from [7], and used in [8] to prove similar lemmas in the setting of b-metric spaces.

Lemma 2.1. *Let (E, σ) be a b-metric space with constant $s \geq 1$ and a self-mapping $T : E \to E$ such that (2.1) holds for all $x, y \in E$ where $\alpha + \beta + \gamma + 2s\delta = 1$, $\delta \neq 0$ or $\alpha + \beta + \gamma + 2sL = 1$, $L \neq 0$. Then, we have for all $x \in E$*

$$\sigma(Tx, T^2x) \leq \sigma(x, Tx). \tag{2.2}$$

Proof. Assume that $\alpha + \beta + \gamma + 2s\delta = 1$ holds. By substituting $y = Tx$ in the contraction condition (2.1), we get

$$\sigma(Tx, T^2x) \leq \alpha\sigma(x, Tx) + \beta\sigma(x, Tx) + \gamma\sigma(Tx, T^2x) + \delta\sigma(x, T^2x).$$

Using (b_3), we derive

$$(1 - \gamma - s\delta)\sigma(Tx, T^2x) \leq (\alpha + \beta + s\delta)\sigma(x, Tx).$$

Since $1 - \gamma - s\delta = \alpha + \beta + s\delta \neq 0$, then $\sigma(Tx, T^2x) \leq \sigma(x, Tx)$.

If $\alpha + \beta + \gamma + 2sL = 1$ holds, we put $x = Ty$

$$(1 - \beta - sL)\sigma(Ty, T^2y) \leq (\alpha + \gamma + sL)\sigma(y, Ty),$$

which implies that $\sigma(Ty, T^2y) \leq \sigma(y, Ty)$ for all $y \in E$. ☐

Lemma 2.2. *Let (E, σ) be a b-metric space with constant $s \geq 1$ and a self-mapping $T : E \to E$ satisfying (2.1), and either (R_1) or (R_2) holds, then there exists $0 \leq k < 2s$ such that for all $x \in E$*

$$\sigma(Tx, T^3x) \leq k\sigma(x, Tx). \tag{2.3}$$

Proof. Assume that (R_1) holds. Putting $y = T^2x$ in the contraction condition (2.1), we obtain

$$\sigma(Tx, T^3x) \leq \alpha\sigma(x, T^2x) + \beta\sigma(x, Tx) + \gamma\sigma(T^2x, T^3x) + \delta\sigma(x, T^3x) + L\sigma(Tx, T^2x).$$

Using (2.2), we get the following inequalities:

$$\begin{aligned}\sigma(x,T^2x) \leq & s\sigma(x,Tx)+s\sigma(Tx,T^2x) \leq 2s\sigma(x,Tx),\\ \sigma(T^2x,T^3x) \leq & \sigma(Tx,T^2x) \leq \sigma(x,Tx),\\ \sigma(x,T^3x) \leq & s\sigma(x,Tx)+s\sigma(Tx,T^3x).\end{aligned}$$

Combining these three inequalities, we conclude that

$$\begin{aligned}\sigma(Tx,T^3x) \leq & 2\alpha s\sigma(x,Tx)+\beta\sigma(x,Tx)+\gamma\sigma(x,Tx)+s\delta\sigma(x,Tx)+s\delta\sigma(Tx,T^3x)\\ & +L\sigma(x,Tx).\end{aligned}$$

We derive $\sigma(Tx,T^3x) \leq k_1\sigma(x,Tx)$, where $k_1 = \frac{2\alpha s+\beta+\gamma+s\delta+L}{1-s\delta}$.

Now we prove that $k_1 < 2s$. We have

$$\begin{aligned}k_1-2s &= \frac{2\alpha s+\beta+\gamma+s\delta+L}{1-s\delta}-2s\\ &= \frac{2\alpha s+\beta+\gamma+s\delta+L}{\alpha+\beta+\gamma+s\delta}-2s\\ &= \frac{2\alpha s+\beta+\gamma+s\delta+L-2s\left(\alpha+\beta+\gamma+s\delta\right)}{\alpha+\beta+\gamma+s\delta}\\ &= \frac{-2\delta s^2+\left(\delta-2\beta-2\gamma\right)s+\beta+\gamma+L}{\alpha+\beta+\gamma+s\delta}\\ &= \frac{P(s)}{\alpha+\beta+\gamma+s\delta},\end{aligned}$$

where $P(t) = -2\delta t^2+\left(\delta-2\beta-2\gamma\right)t+\beta+\gamma+L$. Since $\delta \neq 0$, then the polynomial $P(t)$ is of degree 2, and we have

$$\begin{aligned}\Delta &= \left(\delta-2\beta-2\gamma\right)^2+8\delta\left(\beta+\gamma+L\right)\\ &= \delta^2+4\beta^2+4\gamma^2-4\beta\delta-4\gamma\delta+8\beta\gamma+8\delta\beta+8\delta\gamma+8\delta L\\ &= \delta^2+4\beta^2+4\gamma^2+4\beta\delta+4\gamma\delta+8\beta\gamma+8\delta L.\end{aligned}$$

Since $\Delta > 0$ then the polynomial $P(t)$ has two roots:

$$t_1 = \frac{\delta-2\beta-2\gamma-\sqrt{\Delta}}{4\delta}, \quad t_2 = \frac{\delta-2\beta-2\gamma+\sqrt{\Delta}}{4\delta}.$$

Obviously we have $t_1 < t_2$. One can observe that if $t_2 < s$, then $P(s) < 0$, therefore $k_1 - 2s < 0$. We have:

$$\begin{aligned}t_2 < s \Leftrightarrow & \delta-2\beta-2\gamma+\sqrt{\Delta} < 4s\delta\\ \Leftrightarrow & \Delta < \left((4s-1)\delta+2\beta+2\gamma\right)^2\\ \Leftrightarrow & \delta^2+4\beta^2+4\gamma^2+4\beta\delta+4\gamma\delta+8\beta\gamma+8\delta L < (4s-1)^2\delta^2+4\beta^2+4\gamma^2\\ & +4(4s-1)\beta\delta+4(4s-1)\gamma\delta+8\beta\gamma\\ \Leftrightarrow & 0 < (16s^2-8s)\delta^2+(16s-8)\beta\delta+(16s-8)\gamma\delta-8\delta L\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow 0 < 8\delta\left[s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma - L\right]\\&\Leftrightarrow 0 < s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma - L\\&\Leftrightarrow L < s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma.\end{aligned}$$

We conclude that $k_1 - 2s < 0$, i.e., $k_1 < 2s$.

If (R_2) holds, then we put $x = T^2y$ and we get

$$\sigma(Ty, T^3y) \leq k_2\sigma(y, Ty),$$

where $k_2 = \frac{2\alpha s+\beta+\gamma+\delta+sL}{1-sL}$. With the same arguments, we prove that $k_2 < 2s$. For the assertion of the lemma, one can take $k = \min\{k_1, k_2\}$. □

Lemma 2.3. *Let* (E, σ) *be a b-metric space with constant* $s \geq 1$ *and* $T : E \to E$ *a self-mapping satisfying* (2.1)*, where* (R_1) *or* (R_2) *holds. Then, there exists* $0 < \varepsilon < 1$ *such that for all* $x \in E$

$$\sigma(T^2x, T^3x) \leq \varepsilon\sigma(x, Tx). \tag{2.4}$$

Proof. Assume (R_1) holds, we have

$$\begin{aligned}\sigma(T^2x, T^3x) \leq&\alpha\sigma(Tx, T^2x) + \beta\sigma(Tx, T^2x) + \gamma\sigma(T^2x, T^3x)\\&+ \delta\sigma(Tx, T^3x) + L\sigma(T^2x, T^2x)\\\leq&\alpha\sigma(x, Tx) + \beta\sigma(x, Tx) + \gamma\sigma(x, Tx) + k_1\delta\sigma(x, Tx)\\=&(\alpha + \beta + \gamma + k_1\delta)\sigma(x, Tx)\\=&\varepsilon_1\sigma(x, Tx),\end{aligned}$$

where $\varepsilon_1 = \alpha + \beta + \gamma + k_1\delta < \alpha + \beta + \gamma + 2s\delta = 1$, since $\delta \neq 0$.

If (R_2) holds, then we get also

$$\sigma(T^2x, T^3x) \leq \varepsilon_2\sigma(x, Tx),$$

where $\varepsilon_2 = \alpha + \beta + \gamma + k_2L < \alpha + \beta + \gamma + 2sL = 1$, since $L \neq 0$. □

Proof of Theorem 2.1. According to (2.4), for all $x \in E$, we have $\sigma(T^2x, T^3x) \leq \varepsilon\sigma(x, Tx)$, where $0 < \varepsilon = \alpha + \beta + \gamma + k\min\{\delta, L\} < 1$. For all $n \in \mathbb{N}$ and $x \in E$, let us denote $x_n = T^nx$. Hence, from (2.4), we obtain for all $n \geq 0$

$$\sigma(T^{n+3}x, T^{n+2}x) \leq \varepsilon\sigma(T^{n+1}x, T^nx).$$

If n is even, then we obtain by induction $\sigma(T^{n+1}x, T^nx) \leq \varepsilon^{\frac{n}{2}}\sigma(x, Tx)$. If n is odd, then $n-1$ is even and we get $\sigma(T^{n+1}x, T^nx) \leq \varepsilon^{\frac{n-1}{2}}\sigma(x, Tx)$.

Therefore, for all $n \geq 0$, the following inequality holds

$$\sigma(x_{n+1}, x_n) \leq \left(\sqrt{\varepsilon}\right)^n\left(\sqrt{\varepsilon}\right)^{-1}\sigma(x, Tx). \tag{2.5}$$

Thus, the condition (1.10) of Lemma 1.1 is fulfilled for

$$\lambda = \sqrt{\varepsilon} = \sqrt{\alpha + \beta + \gamma + k\min\{\delta, L\}} < 1$$

and $M = (\sqrt{\varepsilon})^{-1}\sigma(x, Tx)$. Hence, the sequence $\{T^nx\}$ converges to $x^* \in E$.

If T is continuous, then it is clear that x^* is a fixed point of T.

Suppose that $s\beta + sL < 1$. Let us take $x = x^*$ and $y = x_n$ in (2.1), then we obtain

$$\begin{aligned}\sigma(Tx^*, Tx_n) \leq& \alpha\sigma(x^*, x_n) + \beta\sigma(x^*, Tx^*) + \gamma\sigma(x_n, Tx_n)\\ &+ \delta\sigma(x^*, Tx_n) + L\sigma(x_n, Tx^*)\\ \leq& \alpha\sigma(x^*, x_n) + s\beta\sigma(x^*, Tx_n) + s\beta\sigma(Tx_n, Tx^*) + \gamma\sigma(x_n, x_{n+1})\\ &+ \delta\sigma(x^*, x_{n+1}) + sL\sigma(x_n, Tx_n) + sL\sigma(Tx_n, Tx^*).\end{aligned}$$

This yields

$$\begin{aligned}(1 - s\beta - sL)\,\sigma(Tx^*, Tx_n) \leq& \alpha\sigma(x^*, x_n) + s\beta\sigma(x^*, x_{n+1}) + \gamma\sigma(x_n, x_{n+1})\\ &+ \delta\sigma(x^*, x_{n+1}) + sL\sigma(x_n, x_{n+1}).\end{aligned}$$

By taking the limit as $n \to +\infty$ on both sides of this last inequality, we get $\sigma(Tx_n, Tx^*) = 0$. On the other hand we obtain

$$\sigma(x^*, Tx^*) \leq s\left(\sigma(x^*, Tx_n) + \sigma(Tx_n, Tx^*)\right) = s\left(\sigma(x^*, x_{n+1}) + \sigma(Tx_n, Tx^*)\right).$$

Taking the limit as $n \to +\infty$ we obtain $\sigma(x^*, Tx^*) = 0$, which implies that $Tx^* = x^*$.

By assuming the condition $s\gamma + s\delta < 1$ and substituting $x = x_n$ and $y = x^*$ in (2.1), we get $Tx^* = x^*$ by using the same sketch of proof.

Assume that $\alpha + \delta + L < 1$ holds. Let x^* and y^* two fixed points such that $y^* \neq x^*$. We have

$$\sigma(Tx^*, Ty^*) \leq \alpha\sigma(x^*, y^*) + \beta\sigma(x^*, Tx^*) + \gamma\sigma(y^*, Ty^*) + \delta\sigma(x^*, Ty^*) + L\sigma(y^*, Tx^*).$$

This latter inequality yields $\sigma(x^*, y^*) \leq \alpha\sigma(x^*, y^*) + \delta\sigma(x^*, y^*) + L\sigma(y^*, x^*)$ or equivalently $(1 - \alpha - \delta - L)\,\sigma(x^*, y^*) \leq 0$. Since $y^* \neq x^*$, then $1 - \alpha - \delta - L \leq 0$ and this is in contradiction with $\alpha + \delta + L < 1$. Then, T has a unique fixed point in E. □

The following example shows that all the conditions of Theorem 2.1 are satisfied, while some classical contraction fail to hold.

Example 2.1. Let $E = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and

$$\sigma(x, y) = \begin{cases} 0, & \text{if } x = y,\\ 1, & \text{if } x, y \in \{0, 1\} \text{ and } x \neq y,\\ |x - y|, & \text{if } x, y \in \{0\} \cup \left\{\frac{1}{2n} : n = 1, 2, \ldots\right\} \text{ and } x \neq y,\\ 4, & \text{otherwise.}\end{cases}$$

Then, we get σ is a b-metric with $s = 4$ (see [21]).

We define the self-mapping $T : E \to E$ by

$$\begin{aligned} &T(0) = 0, \quad T(1) = 0,\\ &T\left(\tfrac{1}{2n}\right) = \frac{1}{2(n+1)}, \quad T\left(\tfrac{1}{2n+1}\right) = 0 \quad n \geq 1.\end{aligned}$$

The inequality (2.1) is satisfied for the constants

$$\alpha = 0.4, \quad \beta = 0.1, \quad \gamma = 0.1, \quad \delta = 0.05, \quad L = 2.0.$$

We distinguish the following cases.

(a) If $x, y \in \{0\} \cup \{\frac{1}{2n} : n \geq 1\}$, then $\sigma(x,y) = |x-y|$, $\sigma(Tx,Ty) = |Tx - Ty|$. For $x = \frac{1}{2n}$ and $y = \frac{1}{2m}$ with $n < m$, we have $Tx = \frac{1}{2(n+1)}$, $Ty = \frac{1}{2(m+1)}$, and therefore

$$|Tx - Ty| = \left|\frac{1}{2(n+1)} - \frac{1}{2(m+1)}\right| = \frac{m-n}{2(n+1)(m+1)}.$$

Moreover,

$$|x-y| = \frac{m-n}{2nm}, \qquad |x - Tx| = \frac{1}{2n(n+1)}, \qquad |y - Ty| = \frac{1}{2m(m+1)},$$

$$|x - Ty| = \frac{m-n+1}{2n(m+1)}, \qquad |y - Tx| = \frac{m-n+1}{2m(n+1)}.$$

Thus, $|Tx - Ty| \leq \frac{m-n}{2m(n+1)} \leq L\sigma(y, Tx)$.

If $x = \frac{1}{2n}$ and $y = 0$, then $|Tx - Ty| = \frac{1}{2n}$, and

$$|x-y| = \frac{1}{n}, \qquad |x - Tx| = \frac{1}{2n(n+1)}, \quad |y - Ty| = 0,$$

$$|x - Ty| = \frac{1}{n}, \quad |y - Tx| = \frac{1}{2n},$$

so the inequality holds as well.

(b) If $x = 0$ and $y = 1$, then $\sigma(Tx, Ty) = 0$.

(c) If $x = \frac{1}{2m+1}$ and $y = \frac{1}{2n}$, then

$$Tx = 0, \quad Ty = \tfrac{1}{2(n+1)}.$$

So, $\sigma(Tx, Ty) = \frac{1}{2(n+1)}$.

On the other hand,

$$\begin{aligned} &\alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \delta\sigma(x,Ty) + L\sigma(y,Tx) \\ =&0.4 \cdot 4 + 0.1 \cdot 4 + \frac{0.1}{2n(n+1)} + 0.05 \cdot 4 + \frac{2}{2n} \\ =&0.4 + \frac{2.1 + 2n}{2n(n+1)}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sigma(Tx,Ty) - [\alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \delta\sigma(x,Ty) + L\sigma(y,Tx)] \\ =&\frac{1}{2(n+1)} - 0.4 - \frac{2.1 + 2n}{2n(n+1)} \\ =&\frac{-0.8n^2 - 1.8n - 2.1}{2n(n+1)} \leq 0, \end{aligned}$$

which shows that (2.1) holds in this case too.

(d) If $x \in \{0,1\}$ and $y = \frac{1}{2n+1}$, then $Tx = 0$ and $Ty = 0$.

Therefore, in all cases

$$\sigma(Tx,Ty) \leq \alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \delta\sigma(x,Ty) + L\sigma(y,Tx),$$

with the constants

$$\alpha = 0.4, \quad \beta = 0.1, \quad \gamma = 0.1, \quad \delta = 0.05, \quad L = 2.0.$$

Next, we check the remaining assumptions of Theorem 2.1

$$\alpha + \beta + \gamma + 2s\delta = 0.4 + 0.1 + 0.1 + 2 \cdot 4 \cdot 0.05 = 1,$$
$$s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma = 2.8 > 2 = L,$$
$$s\gamma + s\delta = 0.8 < 1.$$

Thus, all conditions of Theorem 2.1 are satisfied.

Next, we show that the Banach, Kannan, Reich and Hardy-Rogers contraction conditions are not satisfied by T.

For $x = \frac{1}{2n}$ and $y = 0$, we have

$$\sigma(Tx,Ty) = \tfrac{1}{2(n+1)}, \qquad \sigma(x,y) = \tfrac{1}{2n}, \qquad \sigma(x,Tx) = \tfrac{1}{2n(n+1)}, \qquad \sigma(y,Ty) = 0,$$
$$\sigma(x,Ty) = \tfrac{1}{2n}, \qquad \sigma(y,Tx) = \tfrac{1}{2(n+1)}.$$

(a) Banach. Assuming $\sigma(Tx,Ty) \leq k\,\sigma(x,y)$ with $k < 1$ gives $\frac{1}{2(n+1)} \leq k\frac{1}{2n}$, i.e., $\frac{n}{n+1} \leq k$, which is impossible as $n \to +\infty$. Hence, T is not a Banach contraction.

(b) Kannan. If

$$\sigma(Tx,Ty) \leq b\sigma(x,Tx) + c\sigma(y,Ty),$$

then

$$\frac{1}{2(n+1)} \leq \frac{b}{2n(n+1)},$$

which yields $n \leq b < 1$, a contradiction.

(c) Reich. If

$$\sigma(Tx,Ty) \leq a\sigma(x,y) + b\sigma(x,Tx) + c\sigma(y,Ty),$$

then

$$\frac{1}{2(n+1)} \leq \frac{a}{2n} + \frac{b}{2n(n+1)},$$

which implies $(1-a)n \leq a + b$, which is impossible.

(d) Hardy-Rogers. If

$$\sigma(Tx,Ty) \leq a\sigma(x,y) + b\sigma(x,Tx) + c\sigma(y,Ty) + d\sigma(x,Ty) + e\sigma(y,Tx),$$

we obtain

$$\frac{1}{2(n+1)} \leq \frac{a}{2n} + \frac{b}{2n(n+1)} + \frac{d}{2n} + \frac{e}{2(n+1)},$$

which leads to $(1-a-d-e)n \leq a + b + d$, again impossible since $1 - a - d - e > 0$.

(e) Theorem 1.4. We have

$$s\beta + s^2 L = 32.4 > 1 \quad \text{and} \quad s\gamma + s^2\delta = 1.2 > 1.$$

Therefore, Theorem 1.4 cannot be applied, which further highlights the significance and validity of our results.

The completeness of the space: Assume that (x_p) is a Cauchy sequence in (E, σ). Then, for any $\varepsilon \in (0,1)$, there exists $p_0 \in \mathbb{N}$ such that

$$\sigma(x_p, x_r) \le \varepsilon, \quad \text{for all } p, r \ge p_0.$$

In particular, this implies that $x_p \in \{0\} \cup \{\frac{1}{2n} : n \ge 2\}$ for every $p \ge p_0$. The set $\{0\} \cup \{\frac{1}{2n} : n \ge 2\}$ is closed in $\mathbb{R}$ (since $\sigma(x,y) = |x-y|$), hence it is complete. Therefore, (x_p) converges in E.

3. Discussions

3.1. **Comparison with Theorem** 1.5**.** Suppose that the assumptions in Theorem 1.5 hold.

If $\delta = L \neq 0$, then all assumptions (R_1), (R_2) and (R_0) are equivalent, because

$$\alpha + \beta + \gamma + s(\delta + L) = \alpha + \beta + \gamma + 2s\delta = \alpha + \beta + \gamma + 2sL.$$

Moreover, the supplementary conditions in both (R_1) and (R_2) become

$$\delta < s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma,$$
$$L < s(2s-1)L + (2s-1)\beta + (2s-1)\gamma.$$

Since $\beta + \gamma \neq 0$, these conditions also hold. Hence, we conclude that the assumptions of Theorem 1.5, 2.1 are equivalent in the case $\delta = L$.

Let us consider the second case: $\delta \neq L$, without loss of generality we suppose that $\delta < L$. Then,

$$\alpha + \beta + \gamma + 2s\delta = \alpha + \beta + \gamma + 2s\min\{\delta, L\} < \alpha + \beta + \gamma + s(\delta + L) = 1,$$

so, we use only Theorem 1.4. On the other hand, we have

$$\delta < L < s(2s-1)\delta + (2s-1)\beta + (2s-1)\gamma,$$

which is true, since $\beta + \gamma \neq 0$.

As a conclusion, the condition $[(R_1)$ or (R_2) or $(R)]$ is weaker than (R_0).

On the other hand, we only assume that $s\beta + sL < 1$ or $s\gamma + s\delta < 1$, whereas in Theorem 1.4, the assumption is $s\beta + s^2L < 1$ or $s\gamma + s^2\delta < 1$. This represents an additional advantage.

The next theorem is a weak form of both Theorems 1.4 and 2.1.

Theorem 3.1. *Let (E, σ) be a complete b-metric space with constant $s \ge 1$ and $T : E \to E$ a self- mapping satisfying* (2.1)*, where (R) or (R_1) or (R_2) holds.*

If T is continuous or $s\beta + sL < 1$ or $s\gamma + s\delta < 1$, then T has at least a fixed point in E. If moreover $\alpha + \delta + L < 1$, then T has a unique fixed point.

3.2. **In the setting of metric spaces.** In Theorem 2.1, if we put $s = 1$, we get a general form for [7, Theorem 1] established in the particular class of complete metric spaces. Moreover, if (R_1) holds, then the condition $L < s(2s-1)\delta+(2s-1)\beta+(2s-1)\gamma$ become $L < \delta + \beta + \gamma$ which is equivalent to the condition

$$\alpha + \delta + L < 1. \tag{3.1}$$

If (R_2) holds, then the condition $\delta < s(2s-1)L + (2s-1)\beta + (2s-1)\gamma$ is equivalent to (3.1). On the other hand, under the condition $\alpha + \beta + \gamma + 2\delta = 1, \delta \neq 0$ we get $\gamma + \delta < 1$, and from the condition $\alpha + \beta + \gamma + 2L = 1, L \neq 0$ we derive $\beta + L < 1$.

From these remarks we get the following theorems.

Theorem 3.2. *Let (E, d) be a complete metric space, and $T : E \to E$ a self-mapping satisfying*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx), \tag{3.2}$$

such that either (M_1) or (M_2) holds, where

$$(M_1) \qquad \alpha + \beta + \gamma + 2\delta = 1, \quad \delta \neq 0,$$

$$(M_2) \qquad \alpha + \beta + \gamma + 2L = 1, \quad L \neq 0,$$

with $\alpha + \delta + L < 1$. Then, T has a unique fixed point.

Theorem 3.3. *Let (E, d) be a complete metric space, and $T : E \to E$ a self-mapping satisfying (3.2) for all $x, y \in E$, such that the following condition holds:*

$$(M_0) \qquad \alpha + \beta + \gamma + 2\min\{\delta, L\} < 1.$$

Then, T has a fixed point in E. If moreover $\alpha + \delta + L < 1$, then T has a unique fixed point.

Remark 3.1. (a) Comparing the conditions (M_1) and (M_2) with the condition (B), we see that in the case $\beta = \gamma = b$ and $\delta = L = c$ the two conditions (M_1) and (M_2) are equivalent to the following:

$$(M) \qquad \alpha + 2b + 2c = 1, \quad c \neq 0.$$

One can deduce that the condition (M) is weaker than the condition (B), since the constant b can vanish in (M). Hence, Theorem 3.2 is more general than Theorem 1.1.

(b) Theorem 3.2 in the case $\beta = \gamma = 0$ and $\delta = L = c$ is more general than the Theorem 1.2, since we assume only that the space is complete, while in Theorem 1.2 the author assumes that the space is compact.

Example 3.1. Let $E = \{0, 1, 2\}$ be endowed with the metric $d(x, y) = |x - y|$. We define a mapping $T : E \to E$ by

$$T(0) = 0, \quad T(1) = 0, \quad T(2) = 1.$$

A simple calculation gives: $d(0, 1) = 1$, $d(0, 2) = 2$, $d(1, 2) = 1$, and $d(T0, T1) = 0$, $d(T0, T2) = 1$, $d(T1, T2) = 1$.

We consider the inequality

$$d(Tx,Ty) \leq 0.1\, d(x,y) + 0.1\, d(x,Tx) + 0.2\, d(y,Ty) + 0.3\, d(x,Ty) + 0.4\, d(y,Tx),$$

which shows that T is a non-expansive mapping with the coefficients

$$\alpha = 0.1, \quad \beta = 0.1, \quad \gamma = 0.2, \quad \delta = 0.3, \quad L = 0.4.$$

It is easy to verify that

$$\alpha + \beta + \gamma + 2\delta = 1 \quad \text{and} \quad \alpha + \delta + L = 0.8 < 1,$$

so all the conditions of Theorem 3.2 are satisfied.

On the other hand, T does not satisfy the classical contraction conditions.

(a) Banach. For the pair $x = 1$, $y = 2$, we have $d(T1,T2) = 1 \leq k\, d(1,2) = k$, which implies $k \geq 1$, contradicting the requirement $k < 1$. Hence, T is not a Banach contraction.

(b) Kannan. For any coefficient $\alpha' < \frac{1}{2}$, consider $x = 1, y = 2$. Then, $d(T1,T2) = 1 \leq 2\alpha'$, which requires $\alpha' \geq \frac{1}{2}$. Therefore, T is not a Kannan contraction.

(c) Reich. For the pair $x = 1$, $y = 2$, we obtain $1 \leq a + b + c$, which contradicts the condition $a + b + c < 1$. Thus, T is not a Reich contraction.

(d) Hardy-Rogers. Suppose there exist non-negative coefficients p, q, r, u, v with $p+q+r+u+v < 1$ satisfying the Hardy-Rogers inequality for all pairs. Considering the pairs $(1,2)$ and $(2,1)$, we obtain

$$1 \leq p + q + r + 2v, \quad 1 \leq p + q + r + 2u.$$

Adding these inequalities yields $2 \leq 2(p + q + r + u + v)$, or equivalently $1 \leq p + q + r + u + v$, contradicting the assumption $p+q+r+u+v < 1$. Therefore, no choice of Hardy-Rogers coefficients with sum less than 1 can satisfy the inequality for all pairs.

3.3. **Conclusion.** The example above fully satisfies the conditions of theorem under (M_1), ensuring the existence (and uniqueness) of a fixed point, while it does not fall under the classical Banach, Kannan, Reich, or Hardy-Rogers contraction results.

Remark 3.2. If we use (M_1) or (M_2) together with the condition $\alpha + \delta + L < 1$, then the function D defined in Theorem 1.3 is a metric on E. However, the mapping T is not a contraction with respect to D; it is only a non-expansive mapping. Therefore, we cannot deduce the existence of a fixed point of T.

Remark 3.3. We obtain similar results when working in b-metric-like spaces instead of b-metric spaces.

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