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STUDY OF NONLINEAR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING ψ-HILFER GENERALIZED PROPORTIONAL DERIVATIVE VIA TOPOLOGICAL DEGREE THEORY

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Abstract. In this paper, we investigate the existence and uniqueness of a solution for a hybrid fractional differential equation involving the generalized proportional fractional derivative of the ψ-Hilfer type. We first establish the equivalence between the original problem and an integral equation. Using topological degree theory for condensing maps, we investigate the existence of the solution. Then we apply Banach's fixed point theorem to study the uniqueness of the solution. Finally, we present an illustrative example to demonstrate our main results.

1. Introduction

The origins of fractional calculus go back to the late 17th century, when Newton and Leibniz laid the foundations of differential and integral calculus. But it is only in the last three decades that fractional calculus has gained significant interest and seen a proliferation of applications, with the concept of fractional derivatives having evolved considerably. Fractional calculus extends the concept of conventional derivatives to non-integer orders, allowing for greater flexibility in modeling. For further insights into fractional derivatives, see references [2, 3, 11, 18].

Dynamical models of fractional order, which use differentiation or integration of non-integer orders are often more accurate in describing a variety of complex systems. Compared to integer-order models, fractional-order systems tend to be more sensitive

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and exhibit richer dynamics. They are sometimes referred to as “memory systems” because they can incorporate past behavior into their responses, especially when considering initial conditions.

The potential of fractional calculus to reshape our understanding of the natural world is considerable. Several theoretical and experimental studies suggest that non-integer derivatives can better describe certain physical systems, including those in electrochemical, thermal and viscoelastic contexts (see [1, 5, 8, 18]). The use of traditional models based on integer-order derivatives may not be suitable in these cases.

As a result, new models based on differential equations with non-integer derivatives have been developed (see [1,4,7,17,20]). These fractional-order models provide a powerful tool to accurately represent complex systems, which can lead to more effective analysis and control in various scientific and engineering applications. Recently, much attention has been paid to the study of hybrid fractional differential equations. These systems have attracted interest in both the automation and computing communities. The main goal of studying hybrid dynamical systems is to provide solutions in terms of models, methods, performance and overall quality for problems that can be inadequately solved by homogeneous approaches. More details about the theory of hybrid systems can be found in [7, 19–21].

The generalized proportional fractional derivative, or more specifically the ψ-Hilfer generalized proportional fractional derivative, is the new type of derivative that interests us in this work and that has been proposed and developed in several papers (see [10, 13, 15, 16]).

The authors of [14], have introduced and studied the following nonlocal mixed boundary value problem involving ψ-Hilfer generalized proportional fractional derivative of order $\alpha \in (1, 2]$

$$
\begin{cases} D_{c+}^{\alpha,\beta,\sigma,\psi} y(t) = f(t, y(t)), \quad t \in [c, d], \\ y(c) = 0, \\ y(d) = \sum_{j=1}^{m} \eta_j y(\gamma_j) + \sum_{i=1}^{n} \gamma_i I_{c^+}^{\phi_i,\sigma,\psi} y(\lambda_i) + \sum_{k=1}^{r} \rho_i D_{c^+}^{\delta_k,\beta,\sigma,\psi} y(\mu_k), \end{cases} \tag{1.1}
$$

where $D_{c+}^{\alpha,\beta,\sigma,\psi}$ is the ψ-Hilfer generalized proportional fractional derivative of order $\alpha \in (1, 2]$ and type $\beta \in [0, 1]$, such that $f \in C([c, d] \times \mathbb{R}, \mathbb{R})$, $\eta_j, \gamma_i, \rho_i \in \mathbb{R}$, $I_{c^+}^{\phi_i,\sigma,\psi}$ is the generalized proportional fractional integral operator of order $\phi_i > 0$, and $\gamma_j, \lambda_i, \mu_k \in [c, d]$.

In [20] we have investigated the existence of solutions for the following p-Laplacian hybrid fractional differential equation involving the generalized Caputo proportional fractional derivative:

$$
\begin{cases} {}^{\mathfrak{C}}_{\delta}D_{0^+}^{\alpha,g} \Phi_p \left({}^{\mathfrak{C}}_{\delta}D_{0^+}^{\vartheta,g} \left(\frac{x(t)}{\mathcal{G}(t,x(t))} \right) \right) = \mathcal{H}(t, x(t)), \quad t \in \Theta := [0, b], \\ \left(\frac{x(t)}{\mathcal{G}(t,x(t))} \right)_{t=0} = w_0, \quad w_0 \in \mathbb{R}, \\ \left(\frac{x(t)}{\mathcal{G}(t,x(t))} \right)'_{t=0} = 0, \end{cases}
$$

where $0 < \alpha < 1$, $1 < \vartheta < 2$, ${}^{\mathfrak{C}}_{\delta}D^{\alpha,g}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order α, $\Phi_p(x) = |x|^{p-2}x$, $p > 1$ is the p-Laplacian operator, $g : \Theta \to \mathbb{R}$, $\mathcal{G} \in C(\Theta \times \mathbb{R}, \mathbb{R}^*)$ and $\mathcal{H} \in C(\Theta \times \mathbb{R}, \mathbb{R})$.

Motivated by the above mentioned works, in the present paper we investigate the existence and uniqueness results of the following nonlinear hybrid differential equation with ψ-Hilfer generalized proportional fractional derivative with order $0 < \beta < 1$

$$(1.2)\qquad \begin{cases} {}^{H}_{\delta}D^{\beta,\sigma,\psi}_{a^+}\left(\frac{x(t)-\mathcal{Q}(t,{}_{\delta}I^{\beta,\psi}_{a^+}x(t))}{\mathcal{L}(t,{}_{\delta}I^{\beta,\psi}_{a^+}x(t))}\right) = \mathcal{M}(t,{}_{\delta}I^{\beta,\psi}_{a^+}x(t)), \quad t \in \mathcal{T} := [a,b], \\ {}_{\delta}I^{1-\zeta,\psi}_{a^+}\left(\frac{x(t)-\mathcal{Q}(t,{}_{\delta}I^{\beta,\psi}_{a^+}x(t))}{\mathcal{L}(t,{}_{\delta}I^{\beta,\psi}_{a^+}x(t))}\right)_{t=a} = \sum_{k=1}^{n}\lambda_k x(\epsilon_k), \quad \epsilon_k \in [a,b], \lambda_k \in \mathbb{R}, \end{cases}$$

where $\mathcal{T} := [a,b]$ is a finite interval of $\mathbb{R}$ with $0 \leq a < b < +\infty$, ${}^{H}_{\delta}D^{\beta,\sigma,\psi}_{a^+}(\cdot)$ is the ψ-Hilfer generalized proportional fractional derivative of order β and type σ, such that $0 < \beta < 1$, $0 \leq \sigma \leq 1$, $0 \leq \delta \leq 1$ $\zeta = \beta + \sigma(1-\beta)$, $\zeta > \beta$, $\zeta > \sigma$, $\zeta \in (0,1]$, ${}_{\delta}I^{\beta,\psi}_{a^+}(\cdot)$ is the ψ-Hilfer generalized proportional fractional integral, $\mathcal{L} \in C([a,b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $\mathcal{Q}, \mathcal{M} \in C([a,b] \times \mathbb{R}, \mathbb{R})$.

This paper is structured as follows. Section 2 introduces some definitions and lemmas. In Section 3, we present our main results concerning the existence and uniqueness of solution to the problem described above. In Section 4, we present an application to illustrate our main results. Finally, we present our conclusion in Section 5.

2. Preliminaries

In this section, we present definitions and lemmas related to the ψ-Hilfer generalized proportional fractional derivative and the Kuratowski measure of noncompactness, which will be consistently utilized throughout the following sections of this study.

- We denote by $C(\mathcal{T}, \mathbb{R})$ the space of all continuous functions with the norm $\|f\| = \sup\{|f(t)| : t \in \mathcal{T}\}$.
- We consider the Banach space $X = (C(\mathcal{T}, \mathbb{R}), \|\cdot\|)$.
- We denote by $B_\eta(0) = \{u \in X : \|u\| \leq \eta\}$ the closed ball centered at 0 with radius η.

Throughout this paper we consider the function $\psi : [a,b] \to \mathbb{R}$, that is an increasing differentiable function.

Definition 2.1 ([11, 12])**.** Let $\delta \in (0,1]$, $\beta > 0$, $\Phi \in L^1([a,b], \mathbb{R})$, The left-sided generalized proportional fractional integral with respect to ψ of order β of the function Φ is defined by

$${}_{\delta}I^{\beta;\psi}_{a^+}\Phi(t) = \frac{1}{\delta^\beta \Gamma(\beta)} \int_{a^+}^{t} e^{\frac{\delta-1}{\delta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\Phi(s)ds,$$

where $\Gamma(\beta) = \int_0^{+\infty} e^{-\tau}\tau^{\beta-1}d\tau$, $\beta > 0$, is the Euler gamma function.

Definition 2.2 ([11,12])**.** Let $\delta \in [0,1]$, $\gamma, \varrho : [0,1] \times \mathbb{R} \to [0, +\infty)$ be continuous such that $\lim_{\delta\to 0^+}\gamma(\delta,t) = 0$, $\lim_{\delta\to 1^-}\gamma(\delta,t) = 1$, $\lim_{\delta\to 0^+}\varrho(\delta,t) = 1$, $\lim_{\delta\to 1^-}\varrho(\delta,t) = 0$, and $\gamma(\delta,t)\neq 0$, $\delta\in(0,1]$, $\varrho(\delta,t)\neq 0$, $\delta\in[0,1)$. Then, the proportional derivative of order δ with respect to ψ of the function Φ is given by

$$ {}_{\delta}D^{\psi}\Phi(t)=\varrho(\delta,t)\Phi(t)+\gamma(\delta,t)\frac{\Phi'(t)}{\psi'(t)}. $$

In particular, if $\gamma(\delta,t)=\delta$ and $\varrho(\delta,t)=1-\delta$, then we have

$$ {}_{\delta}D^{\psi}\Phi(t)=(1-\delta)\Phi(t)+\delta\frac{\Phi'(t)}{\psi'(t)}. $$

Definition 2.3 ([11, 12])**.** Let $\delta\in(0,1]$. The left-sided ψ-Riemann-Liouville generalized proportional fractional derivative of order $n-1<\beta<n$ of the function $\Phi\in C^{n}([a,b])$ is given by

$$ \begin{aligned} {}_{\delta}D^{\beta;\psi}_{a^+}\Phi(t) &= {}_{\delta}D^{n,\psi}_{\delta}I^{n-\beta;\psi}_{a^+}\Phi(t)\\ &=\frac{{}_{\delta}D^{n,\psi}}{\delta^{n-\beta}\Gamma(n-\beta)}\int_a^t e^{\frac{\delta-1}{\delta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{n-\beta-1}\Phi(s)ds, \end{aligned} $$

where $n=[\beta]+1$ and ${}_{\delta}D^{n,\psi}=\underbrace{{}_{\delta}D^{\psi}\cdot{}_{\delta}D^{\psi}\cdots{}_{\delta}D^{\psi}}_{n-\text{times}}$.

Definition 2.4 ([16])**.** Let $\delta\in(0,1]$. The left-sided ψ-Hilfer generalized proportional fractional derivative of order $n-1<\beta<n$ and type $\sigma\in[0,1]$ of the function $\Phi\in C^{n}([a,b],\mathbb{R})$ is given by

$$ {}^{H}_{\delta}D^{\beta,\sigma,\psi}_{a^+}\Phi(t)={}_{\delta}I^{\sigma(n-\beta);\psi}_{a^+}({}_{\delta}D^{n,\psi})_{\delta}I^{(1-\sigma)(n-\beta);\psi}_{a^+}\Phi(t). $$

In other way

$$ {}^{H}_{\delta}D^{\beta,\sigma,\psi}_{a^+}\Phi(t)={}_{\delta}I^{\sigma(n-\beta);\psi}_{a^+}({}_{\delta}D^{\zeta,\psi}_{a^+}\Phi(t)), $$

where $\zeta=\beta+\sigma(n-\beta)$ and $n=[\beta]+1$.

Lemma 2.1 ([11, 12])**.** *Let $\delta\in(0,1]$, $\beta,\rho>0$ and $\Phi\in L^{1}([a,b],\mathbb{R})$. Then, we have*

$$ {}_{\delta}I^{\beta;\psi}_{a^+}({}_{\delta}I^{\rho;\psi}_{a^+}\Phi(t))={}_{\delta}I^{\rho;\psi}_{a^+}({}_{\delta}I^{\beta;\psi}_{a^+}\Phi(t))={}_{\delta}I^{\beta+\rho;\psi}_{a^+}\Phi(t). $$

Throughout this paper, as a simplification, we set

$$ \Omega^{\zeta-1}_{\psi}(t,a)=e^{\frac{\delta-1}{\delta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\zeta-1}. $$

Lemma 2.2 ([16])**.** *Let $\delta\in(0,1]$, $n-1<\beta<n$, $\sigma\in[0,1]$, $\Phi\in C([a,b],\mathbb{R})$ and ${}_{\delta}I^{n-\zeta;\psi}_{a^+}\Phi(t)\in C^{n}([a,b],\mathbb{R})$. Then, we have*

$$ {}_{\delta}I^{\beta;\psi}_{a^+}({}^{H}_{\delta}D^{\beta,\sigma,\psi}_{a^+}\Phi(t))=\Phi(t)-\sum_{k=1}^{n}\frac{\Omega^{\zeta-k}_{\psi}(t,a)}{\delta^{\zeta-k}\Gamma(\zeta-k+1)}({}_{\delta}I^{k-\zeta;\psi}_{a^+}\Phi(a)), $$

where $\zeta=\beta+\sigma(n-\beta)$ and $n=[\beta]+1$.

Lemma 2.3 ([11, 12])**.** *Let $\delta\in(0,1]$, $n-1<\rho<\beta<n$, $n\in\mathbb{N}$, $\sigma\in[0,1]$ and $\Phi\in C^{n}([a,b],\mathbb{R})$. Then, we have*

$$ {}^{H}_{\delta}D^{\rho,\sigma,\psi}_{a^+}({}_{\delta}I^{\beta;\psi}_{a^+}\Phi(t))={}_{\delta}I^{\beta-\rho;\psi}_{a^+}\Phi(t). $$

Lemma 2.4 ([11, 12]). *Let $\beta > 0$, $\rho > 0$ and $\delta \in (0,1]$. Then, we have*

(i) $\left({}_{\delta}I^{\beta;\psi}_{a^+} e^{\frac{\delta-1}{\delta}(\psi(\tau)-\psi(a))}(\psi(\tau)-\psi(a))^{\rho-1}\right)(t) = \frac{\Gamma(\rho)}{\delta^{\beta}\Gamma(\beta+\rho)}\Omega^{\beta+\rho-1}_{\psi}(t,a)$;

(ii) $\left({}_{\delta}D^{\beta;\psi}_{a^+} e^{\frac{\delta-1}{\delta}(\psi(\tau)-\psi(a))}(\psi(\tau)-\psi(a))^{\rho-1}\right)(t) = \frac{\delta^{\beta}\Gamma(\rho)}{\Gamma(\rho-\beta)}\Omega^{\rho-\beta-1}_{\psi}(t,a)$.

Lemma 2.5 ([11, 12]). *Let $\beta > 0$, $\delta > 0$ and $\Phi \in L^1([a,b],\mathbb{R})$. Then, we have* $\lim_{t\to a}({}_{\delta}I^{\beta;\psi}_{a^+}\Phi(t)) = 0$.

We will now introduce definitions and properties related to the Kuratowski measure of noncompactness, which serve as the basis for proving the existence of a solution to the problem (1.2).

Definition 2.5 ([6]). Let X be a Banach space and E be a bounded subset of X. We define the Kuratowski measure of noncompactness by the mapping $\omega : E \to [0,+\infty)$ defined as follows

$$\omega(F) = \inf\left\{\varrho > 0/F \subseteq \cup_{i=1}^{n} F_i \text{ and } \operatorname{diam}(F_i) \leq \varrho\right\}.$$

Lemma 2.6 ([11, 12]). *Let F and G be two bounded subset of the Banach space X. The Kuratowski measure of noncompactness ω satisfies the following proprieties:*

(1) $F \subseteq G \Rightarrow \omega(F) \leq \omega(G)$;
(2) $\omega(F+G) \leq \omega(F) + \omega(G)$;
(3) $\omega(\overline{F}) = \omega(F) = \omega(\operatorname{conv}(F))$, *where $\overline{F}$ and $\operatorname{conv}(F)$ denote the closure and the convex hull of F, respectively;*
(4) $\omega(\alpha F) = |\alpha|\omega(F)$, $\alpha \in \mathbb{R}$;
(5) $\omega(F) = 0$ *if and only if F is relatively compact in X.*

Definition 2.6 ([11, 12]). Let $\Lambda : E \to X$ be a continuous bounded operator. Then, Λ is ω-Lipschitz if there exists a constant $\theta \geq 0$, such that for all $F \subset E$ we have

$$\omega(\Lambda(F)) \leq \theta\omega(F).$$

Remark 2.1. If the constant $\theta < 1$, then the operator Λ becomes strict ω-contraction.

Definition 2.7 ([11, 12]). Let F be a bounded subset of E. We say that the operator $\Lambda : E \to X$ is ω-condensing if

$$\omega(\Lambda(F)) \leq \omega(F).$$

Remark 2.2. If $\omega(\Lambda(F)) \geq \omega(F)$, then this implies that $\omega(F) = 0$.

Lemma 2.7 ([9]). *Let $\Lambda, \Upsilon : E \to X$ are two ω-Lipschitz operators with constants θ and ϑ, respectively. Then, the operator $\Lambda + \Upsilon : E \to X$ is ω-Lipschitz with the constant $\varsigma = \theta + \vartheta$.*

Lemma 2.8 ([9]). *Let $\Lambda : E \to X$. Then, we have the following.*

(*i*) *If the map Λ is compact, then Λ is ω-Lipschitz with constant $\theta = 0$.*

(*ii*) *If the map Λ is Lipschitz with constant ϑ, then Λ is ω-Lipschitz with the same constant.*

Theorem 2.1 ([9]). *Let $\Pi : E \to X$ be ω-condensing and*

$$\Delta = \{x \in X : x = \lambda \Pi x, \lambda \in [0,1]\}.$$

If Δ is a bounded set in X, then there exists $\eta > 0$ such that $\Delta \subset B_\eta(0)$ and

$$\deg(I - \lambda\Pi, B_\eta(0), 0) = 1, \quad \text{for all } \lambda \in [0,1].$$

Therefore, Π has at least one fixed point and the set of the fixed points of the operator Π belongs to $B_\eta(0)$.

3. Auxiliary Results

In this section, we derive the solution formula for the problem (1.2) and identify the conditions under which it has a solution. We also study the existence and uniqueness of solutions to the problem (1.2).

Lemma 3.1. *Let $\mathcal{L} \in C([a,b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\mathcal{Q}, \mathcal{M} \in C([a,b] \times \mathbb{R}, \mathbb{R})$ and*

$$\chi = \delta^{\zeta-1}\Gamma(\zeta) - \sum_{k=1}^{n} \lambda_k L(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k))\Omega_\psi^{\zeta-1}(\epsilon_k, a) \neq 0. \tag{3.1}$$

The solution of the hybrid fractional differential equation (1.2) is then given by

$$\begin{aligned} x(t) =& \mathcal{Q}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)) + \mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)) \frac{\Omega_\psi^{\zeta-1}(t,a)}{\chi} \sum_{k=1}^{n} \lambda_k \Big(\mathcal{Q}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)) \\ &+ \mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)) I_{a^+}^{\beta,\psi} \mathcal{M}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)) \Big) + \mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)) {}_\delta I_{a^+}^{\beta,\psi} \mathcal{M}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)). \end{aligned} \tag{3.2}$$

Proof. Let $t \in \mathcal{T}$, then we consider that $x(t)$ is a solution of the problem (1.2), then by applying the operator ${}_\delta I_{a^+}^{\beta;\psi}(\cdot)$ on both sides of the problem (1.2) and using Lemma 2.2, we obtain

$$\begin{aligned} x(t) =& \mathcal{Q}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)) + \mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)) \frac{\Omega_\psi^{\zeta-1}(t,a)}{\delta^{\zeta-1}\Gamma(\zeta)} {}_\delta I_{a^+}^{1-\zeta,\psi} \left(\frac{x(t) - Q(t, {}_\delta I_{a^+}^{\beta,\psi} x(t))}{\mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t))} \right)_{t=a} \\ &+ \mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)) I_{a^+}^{\beta,\psi} \mathcal{M}(t, {}_\delta I_{a^+}^{\beta,\psi} x(t)). \end{aligned} \tag{3.3}$$

In (3.3) we take $t = \epsilon_k$ and multiply its two members by λ_k, and we get

$$\begin{aligned} \lambda_k x(\epsilon_k) =& \lambda_k \mathcal{Q}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)) \\ &+ \lambda_k \mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)) \frac{\Omega_\psi^{\zeta-1}(\epsilon_k,a)}{\delta^{\zeta-1}\Gamma(\zeta)} {}_\delta I_{a^+}^{1-\zeta,\psi} \left(\frac{x(a) - \mathcal{Q}(a, {}_\delta I_{a^+}^{\beta,\psi} x(a))}{\mathcal{L}(a, {}_\delta I_{a^+}^{\beta,\psi} x(a))} \right) \\ &+ \lambda_k \mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)) {}_\delta I_{a^+}^{\beta,\psi} \mathcal{M}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi} x(\epsilon_k)). \end{aligned}$$

Then, we have

(3.4)

$$\begin{aligned}\sum_{k=1}^{n}\lambda_k x(\epsilon_k) =& \sum_{k=1}^{n}\lambda_k Q(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\\ &+\sum_{k=1}^{n}\lambda_k \mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\frac{\Omega_\psi^{\zeta-1}(\epsilon_k,a)}{\delta^{\zeta-1}\Gamma(\zeta)}\, {}_\delta I_{a^+}^{1-\zeta,\psi}\left(\frac{x(a)-Q(a, {}_\delta I_{a^+}^{\beta,\psi}x(a)}{\mathcal{L}(a, {}_\delta I_{a^+}^{\beta,\psi}x(a))}\right)\\ &+\sum_{k=1}^{n}\lambda_k \mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k)) I_{a^+}^{\beta,\psi}\mathcal{M}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k)).\end{aligned}$$

From the initial condition ${}_\delta I_{a^+}^{1-\zeta,\psi}\left(\frac{x(a)-\mathcal{Q}(a, {}_\delta I_{a^+}^{\beta,\psi}x(a)}{\mathcal{L}(a, {}_\delta I_{a^+}^{\beta,\psi}x(a))}\right) = \sum_{k=1}^{n}\lambda_k x(\epsilon_k)$ and (3.4), we obtain

(3.5)

$$\begin{aligned}{}_\delta I_{a^+}^{1-\zeta,\psi}\left(\frac{x(a)-\mathcal{Q}(a, {}_\delta I_{a^+}^{\beta,\psi}x(a)}{\mathcal{L}(a, {}_\delta I_{a^+}^{\beta,\psi}x(a))}\right) =& \frac{\delta^{\zeta-1}\Gamma(\zeta)}{\chi}\left(\sum_{k=1}^{n}\lambda_k \mathcal{Q}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\right.\\ &\left.+\sum_{k=1}^{n}\lambda_k \mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k)) I_{a^+}^{\beta,\psi}\mathcal{M}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\right).\end{aligned}$$

Substituting (3.5) into (3.3) we get

$$\begin{aligned}x(t) =& \mathcal{Q}(t, {}_\delta I_{a^+}^{\beta,\psi}x(t)) + \mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi}x(t))\frac{\Omega_\psi^{\zeta-1}(t,a)}{\chi}\sum_{k=1}^{n}\lambda_k\Big(\mathcal{Q}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\\ &+\mathcal{L}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k)) I_{a^+}^{\beta,\psi}\mathcal{M}(\epsilon_k, {}_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\Big)\\ &+\mathcal{L}(t, {}_\delta I_{a^+}^{\beta,\psi}x(t)) {}_\delta I_{a^+}^{\beta,\psi}\mathcal{M}(t, {}_\delta I_{a^+}^{\beta,\psi}x(t)).\end{aligned}$$

The opposite follows by direct computation. □

As we prepare to provide the main results of this investigation, we first present the following hypotheses.

(H_1) There exists a constant $C_1 > 0$ such that for all $p, q \in \mathbb{R}$, and $t \in \mathcal{T}$ we have

$$|\mathcal{M}(t,p) - \mathcal{M}(t,q)| \leq C_1|p-q|.$$

(H_2) There exists a constant $\xi > 0$ such that for all $p \in \mathbb{R}$, $r \in (0,1)$, and $t \in \mathcal{T}$ we have:

$$|\mathcal{M}(t,p)| \leq \xi|p|^r.$$

(H_3) There exists a constant $C_2 > 0$ such that for all $p, q \in \mathbb{R}$, and $t \in \mathcal{T}$ we have

$$|\mathcal{Q}(t,p) - \mathcal{Q}(t,q)| \leq C_2|p-q|.$$

(H_4) There exist constants $\varphi > 0$ and $N > 0$ such that for all $p \in \mathbb{R}$, and $t \in \mathcal{T}$ we have

$$|\mathcal{L}(t,p)| \leq \varphi, \quad |\mathcal{Q}(t,p)| \leq N.$$

Define the two operators $\mathcal{A}, \mathcal{B} : X \to X$ as follows:

$$\mathcal{A}x(t) = \mathcal{Q}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)) + \mathcal{L}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)) \frac{\Omega_{\psi}^{\zeta-1}(t,a)}{\chi} \sum_{k=1}^{n} \lambda_k \Big(\mathcal{Q}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) + \mathcal{L}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) I_{a^+}^{\beta,\psi} \mathcal{M}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) \Big), \qquad t \in \mathcal{T},$$

and

$$\mathcal{B}x(t) = \mathcal{L}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)) {}_{\delta}I_{a^+}^{\beta,\psi} \mathcal{M}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)), \quad t \in \mathcal{T}.$$

We consider the operator $\Pi : X \to X$ defined by

$$\Pi x(t) = \mathcal{A}x(t) + \mathcal{B}x(t), \quad t \in \mathcal{T}.$$

Next, we have the following lemmas.

Lemma 3.2. *The operator $\mathcal{A}$ is Lipschitz and satisfies the following condition*

$$\|\mathcal{A}x\| \le N + \frac{N\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|} \sum_{k=1}^{n} |\lambda_k| + \frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)+\zeta-1}}{|\chi|\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}} \|x\|^r \sum_{k=1}^{n} |\lambda_k|.$$

Proof. Let $x, y \in X$, then by using hypotheses H_1, H_3, H_4, Lemma 2.4 (i), and using the fact that $e^{\frac{\delta-1}{\delta}(\psi(t)-\psi(a))} \le 1$ we get

$$\begin{aligned}
&|\mathcal{A}x(t) - \mathcal{A}y(t)| \\
\le& \left| \mathcal{Q}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)) - \mathcal{Q}(t, {}_{\delta}I_{a^+}^{\beta,\psi} y(t)) \right| \\
&+ \frac{\Omega_{\psi}^{\zeta-1}(t,a)}{|\chi|} \left| \mathcal{L}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)) \sum_{k=1}^{n} \lambda_k \mathcal{Q}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) - \mathcal{L}(t, {}_{\delta}I_{a^+}^{\beta,\psi} y(t)) \sum_{k=1}^{n} \lambda_k \mathcal{Q}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} y(\epsilon_k)) \right| \\
&+ \frac{\Omega_{\psi}^{\zeta-1}(t,a)}{|\chi|} \left| \mathcal{L}(t, {}_{\delta}I_{a^+}^{\beta,\psi} x(t)) \sum_{k=1}^{n} \lambda_k L(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) I_{a^+}^{\beta,\psi} \mathcal{M}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) \right. \\
&\left. - \mathcal{L}(t, {}_{\delta}I_{a^+}^{\beta,\psi} y(t)) \sum_{k=1}^{n} \lambda_k L(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} y(\epsilon_k)) I_{a^+}^{\beta,\psi} \mathcal{M}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} y(\epsilon_k)) \right| \\
\le& \frac{C_2}{\delta^{\beta}\Gamma(\beta)} \int_a^t \Omega_{\psi}^{\beta-1}(t,s)\psi'(s)|x(s)-y(s)|ds + \frac{\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|} \\
&\times \sum_{k=1}^{n} |\lambda_k| \cdot \left| \mathcal{Q}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) - \mathcal{Q}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} y(\epsilon_k)) \right| \\
&+ \frac{\varphi^2(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|} \sum_{k=1}^{n} |\lambda_k| {}_{\delta}I_{a^+}^{\beta,\psi} \left| \mathcal{M}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} x(\epsilon_k)) - \mathcal{M}(\epsilon_k, {}_{\delta}I_{a^+}^{\beta,\psi} y(\epsilon_k)) \right|
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{C_2}{\delta^\beta\Gamma(\beta)}\int_a^t(\psi(t)-\psi(s))^{\beta-1}\psi'(s)|x(s)-y(s)|ds+\frac{\varphi C_2(\psi(b)-\psi(a))^{\zeta-1}}{\delta^\beta|\chi|\Gamma(\beta)}\\
&\times\sum_{k=1}^n|\lambda_k|\int_a^{\epsilon_k}\Omega_\psi^{\beta-1}(\epsilon_k,s)\psi'(s)|x(s)-y(s)|ds+\frac{C_1\varphi^2(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|\delta^\beta\Gamma(\beta)}\\
&\times\sum_{k=1}^n|\lambda_k|\int_a^{\epsilon_k}(\psi(\epsilon_k)-\psi(s))^{\beta-1}\psi'(s)\left(\frac{1}{\delta^\beta\Gamma(\beta)}\int_a^s\Omega_\psi^{\beta-1}(s,\tau)\psi'(\tau)|x(\tau)-y(\tau)|d\tau\right)ds\\
\leq & \frac{(\psi(b)-\psi(a))^\beta C_2}{\delta^\beta\Gamma(\beta+1)}\|x-y\|+\frac{\varphi C_2(\psi(b)-\psi(a))^{\beta+\zeta-1}}{\delta^\beta|\chi|\Gamma(\beta+1)}\sum_{k=1}^n|\lambda_k|\cdot\|x-y\|\\
&+\frac{C_1\varphi^2(\psi(b)-\psi(a))^{\beta+\zeta-1}}{|\chi|\delta^{2\beta}\Gamma(\beta)\Gamma(\beta+1)}\sum_{k=1}^n|\lambda_k|\int_a^{\epsilon_k}(\psi(\epsilon_k)-\psi(s))^{\beta-1}\psi'(s)ds\|x-y\|\\
\leq & \frac{(\psi(b)-\psi(a))^\beta C_2}{\delta^\beta\Gamma(\beta+1)}\|x-y\|+\frac{\varphi C_2(\psi(b)-\psi(a))^{\beta+\zeta-1}}{\delta^\beta|\chi|\Gamma(\beta+1)}\sum_{k=1}^n|\lambda_k|\cdot\|x-y\|\\
&+\frac{C_1\varphi^2(\psi(b)-\psi(a))^{2\beta+\zeta-1}}{|\chi|\delta^{2\beta}(\Gamma(\beta+1))^2}\sum_{k=1}^n|\lambda_k|\cdot\|x-y\|\\
\leq & R\|x-y\|.
\end{aligned}
$$

Therefore, the operator $\mathcal{A}$ is Lipschitz with a constant

$$
\begin{aligned}
R=&\frac{(\psi(b)-\psi(a))^\beta}{\delta^\beta\Gamma(\beta+1)}\left(C_2+\frac{\varphi C_2(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^n|\lambda_k|\right.\\
&\left.+\frac{C_1\varphi^2(\psi(b)-\psi(a))^{\beta+\zeta-1}}{|\chi|\delta^\beta(\Gamma(\beta+1))}\sum_{k=1}^n|\lambda_k|\right).
\end{aligned}
\tag{3.6}
$$

Moreover, using Hypotheses (H_2) and (H_4), we get

$$
\begin{aligned}
|\mathcal{A}x(t)|\leq & N+\frac{N\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^n|\lambda_k|+\frac{\xi\varphi^2(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|\delta^\beta\Gamma(\beta)}\|x\|^r\\
&\times\sum_{k=1}^n|\lambda_k|\int_a^{\epsilon_k}\Omega_\psi^{\beta-1}(\epsilon_k,s)\psi'(s)\left|\frac{1}{\delta^\beta\Gamma(\beta)}\int_a^s\Omega_\psi^{\beta-1}(s,\tau)\psi'(\tau)d\tau\right|^r ds\\
\leq & N+\frac{N\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^n|\lambda_k|+\frac{\xi\varphi^2(\psi(b)-\psi(a))^{r\beta+\zeta-1}}{|\chi|\delta^{\beta(r+1)}\Gamma(\beta)(\Gamma(\beta+1))^r}\|x\|^r\\
&\times\sum_{k=1}^n|\lambda_k|\int_a^{\epsilon_k}(\psi(\epsilon_k)-\psi(s))^{\beta-1}\psi'(s)\\
\leq & N+\frac{N\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^n|\lambda_k|\\
&+\frac{\xi\varphi^2(\psi(b)-\psi(a))^{\beta(r+1)+\zeta-1}}{|\chi|\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\|x\|^r\sum_{k=1}^n|\lambda_k|.
\end{aligned}
\tag{3.7}
$$

Therefore,

$$\|\mathcal{A}x\| \leq N + \frac{N\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^{n}|\lambda_k| + \frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)+\zeta-1}}{|\chi|\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\|x\|^r\sum_{k=1}^{n}|\lambda_k|. \tag{3.8}$$

□

Lemma 3.3. *The operator $\mathcal{B}$ is continuous, and verifies the following condition*

$$\|\mathcal{B}x\| \leq \frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)}}{\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\|x\|^r.$$

Proof. Let $x_n, x \in C(\mathcal{T},\mathbb{R})$, such that $x_n \to x$ as $n \to +\infty$. Then, we have

$$\begin{aligned}&\|\mathcal{B}x_n(t)-\mathcal{B}x(t)\|\\ \leq&\frac{\varphi}{\delta^\beta\Gamma(\beta)}\int_a^t(\psi(t)-\psi(s))^{\beta-1}\psi'(s)\left|\mathcal{M}(s,{}_\delta I_{a^+}^{\beta,\psi}x_n(s))-\mathcal{M}(s,{}_\delta I_{a^+}^{\beta,\psi}x(s))\right|ds.\end{aligned}$$

Thanks to the continuity of the function $\mathcal{M}$ and Lebesgue dominated convergence theorem we get

$$\|\mathcal{B}x_n(t)-\mathcal{B}x(t)\| \to 0 \quad \text{as} \quad n \to +\infty.$$

This implies that the operator $\mathcal{B}$ is continuous.

On the other hand, by using the hypotheses (H_2) and (H_4), we get

$$\begin{aligned}|\mathcal{B}x(t)| \leq&\frac{\xi\varphi}{\delta^{\beta(1+r)}(\Gamma(\beta))^{(1+r)}}\\ &\times\int_a^t(\psi(t)-\psi(s))^{\beta-1}\psi'(s)\left|\int_a^s(\psi(s)-\psi(\tau))^{\beta-1}\psi'(\tau)x(\tau)d\tau\right|^r ds\\ \leq&\frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)}}{\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\|x\|^r.\end{aligned} \tag{3.9}$$

□

Lemma 3.4. *The operator $\mathcal{B}$ is compact.*

Proof. To prove that the operator $\mathcal{B}$ is compact, we prove that $\mathcal{B}(B_\eta)$ is relatively compact by using the Arzela-Ascoli Theorem. Let $t \in \mathcal{T}$, $x \in B_\eta$, from the inequality (3.9) we have

$$\|\mathcal{B}x\| \leq \frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)}}{\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\eta^r.$$

By extension, this suggests that the operator $\mathcal{B}(B_\eta)$ is uniformly bounded.

Now we show that operator $\mathcal{B}$ is equicontinuous. Let $t_1, t_2 \in \mathcal{T}$ and $x \in B_\eta$, then we have

$$|\mathcal{B}x(t_2)-\mathcal{B}x(t_1)|$$

$$
\begin{aligned}
\leq & \Bigg| \mathcal{L}(t_2, {}_\delta I_{a^+}^{\beta,\psi} x(t_2)) \frac{1}{\delta^\beta \Gamma(\beta)} \int_a^{t_2} \Omega_\psi^{\beta-1}(t_2,s)\psi'(s) |\mathcal{M}(s, {}_\delta I_{a^+}^{\beta,\psi} x(s))| ds \\
& - \mathcal{L}(t_1, {}_\delta I_{a^+}^{\beta,\psi} x(t_1)) \frac{1}{\delta^\beta \Gamma(\beta)} \int_a^{t_1} \Omega_\psi^{\beta-1}(t_1,s)\psi'(s) |\mathcal{M}(s, {}_\delta I_{a^+}^{\beta,\psi} x(s))| ds \Bigg| \\
\leq & \frac{\zeta\varphi \|x\|^r}{\delta^\beta \Gamma(\beta)} \Bigg| \int_a^{t_2} (\psi(t_2)-\psi(s))^{\beta-1}\psi'(s) \left| \frac{1}{\delta^\beta \Gamma(\beta)} \int_a^s (\psi(s)-\psi(\tau))^{\beta-1}\psi'(\tau) d\tau \right|^r ds \\
& - \int_a^{t_1} (\psi(t_1)-\psi(s))^{\beta-1}\psi'(s) \left| \frac{1}{\delta^\beta \Gamma(\beta)} \int_a^s (\psi(s)-\psi(\tau))^{\beta-1}\psi'(\tau) d\tau \right|^r ds \Bigg| \\
\leq & \frac{\zeta\varphi \|x\|^r (\psi(b)-\psi(a))^{r\beta}}{\delta^{\beta(r+1)} \Gamma(\beta) (\Gamma(\beta+1))^r} \\
& \times \left| \int_a^{t_2} (\psi(t_2)-\psi(s))^{\beta-1}\psi'(s) ds - \int_a^{t_1} (\psi(t_1)\psi(s))^{\beta-1}\psi'(s) ds \right| \\
\leq & \frac{\zeta\varphi \|x\|^r (\psi(b)-\psi(a))^{r\beta}}{\delta^{\beta(r+1)} (\Gamma(\beta+1))^{r+1}} \left| (\psi(t_2)-\psi(a))^\beta - (\psi(t_1)\psi(a))^\beta \right|.
\end{aligned}
$$

By using the continuity of the function ψ, we obtain $|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0$ as $t_1 \to t_2$, this implies that the operator $\mathcal{B}$ is equicontinuous.

Therefore, according to the above arguments, the operator $\mathcal{B}$ is bounded and equicontinuous. Thanks to Arzelá-Ascoli theorem, the operator $\mathcal{B}$ is compact. □

Theorem 3.1. *Let us assume that the hypotheses (H_1), (H_2), (H_3) and (H_4) are verified. Then, the problem (1.2) has at least one solution $x \in X$ provided that $R < 1$ and the set of solutions is bounded in $C(\mathcal{T}, \mathbb{R})$, where R is given by (3.6).*

Proof. Let $\mathcal{A}, \mathcal{B}$ and Π be the bounded and continuous operators defined above. Then, thanks to Lemma 3.2 and Lemma 2.8 (ii), the operator $\mathcal{A}$ is ω-Lipschitz with constant $0 < R < 1$ and according to Lemma 3.4 and Lemma 2.8, (i), the operator $\mathcal{B}$ is ω-Lipschitz with constant 0. Consequently, the operator Π is ω-Lipschitz with constant R. Since $R < 1$, Π is therefore ω-condensing.

We consider the set Δ defined as follows:

$$\Delta = \{x \in X : x = \lambda \Pi x, \lambda \in [0,1]\}.$$

The set Δ is bounded. Indeed, let $x \in \Delta$. Then, using inequalities (3.8) and (3.9), we get

$$
\begin{aligned}
\|x\| =& \|\lambda(\mathcal{A}x + \mathcal{B}x)\| \\
\leq & \lambda(\|\mathcal{A}x\| + \|\mathcal{B}x\|) \\
\leq & N + \frac{N\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|} \sum_{k=1}^n |\lambda_k| + \frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)+\zeta-1}}{|\chi|\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}} \|x\|^r \sum_{k=1}^n |\lambda_k| \\
& + \frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)}}{\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}} \|x\|^r
\end{aligned}
$$

$$\begin{aligned}\leq & N\left(1+\frac{\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^{n}|\lambda_k|\right)+\frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)}\|x\|^r}{\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\\ & \times\left(\frac{(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^{n}|\lambda_k|+1\right).\end{aligned}$$

This implies that the set Δ is bounded in X. If this is not the case, we assume that $\varepsilon:=\|x\|\to+\infty$ and divide both sides of the above inequality by ε, we get:

$$\begin{aligned}1\leq & \lim_{\varepsilon\to+\infty}\frac{N}{\varepsilon}\left(1+\frac{\varphi(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^{n}|\lambda_k|\right)\\ & +\lim_{\varepsilon\to+\infty}\frac{\xi\varphi(\psi(b)-\psi(a))^{\beta(r+1)}\|x\|^r}{\varepsilon\delta^{\beta(r+1)}(\Gamma(\beta+1))^{r+1}}\left(\frac{(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^{n}|\lambda_k|+1\right)\\ = & 0.\end{aligned}$$

This is absurd; hence, the set Δ is bounded. Consequently, by Theorem 2.1, the operator Π has at least one fixed point in X, which serves as a solution to the problem (1.2).

The proof is completed. □

Theorem 3.2. *Suppose that the hypotheses* (H_1), (H_3) *and* (H_4) *are satisfied. Then, the problem* (1.2) *has a unique solution provided that*

$$R+\frac{\varphi C_1(\psi(b)-\psi(a))^{2\beta}}{\delta^{2\beta}(\Gamma(\beta+1))^2}<1, \tag{3.10}$$

where R *is given by* (3.6).

Proof. Suppose that the hypotheses (H_1), (H_3) and (H_4) are satisfied, and let $x,y\in C(\mathcal{T},\mathbb{R})$. Using the same arguments as in the proof of Lemma 3.2 we obtain the following

$$\begin{aligned}|\Pi x(t)-\Pi y(t)| = & |(\mathcal{A}x(t)+\mathcal{B}x(t))-(\mathcal{A}y(t)+\mathcal{B}y(t))|\\ \leq & |\mathcal{A}x(t)-\mathcal{A}y(t)|+|\mathcal{B}x(t)-\mathcal{B}y(t)|\\ \leq & R\|x-y\|+\Big|\mathcal{L}(t,{}_\delta I_{a^+}^{\beta,\psi}x(t))\,{}_\delta I_{a^+}^{\beta,\psi}\mathcal{M}(t,{}_\delta I_{a^+}^{\beta,\psi}x(t))\\ & -\mathcal{L}(t,{}_\delta I_{a^+}^{\beta,\psi}y(t))\,{}_\delta I_{a^+}^{\beta,\psi}\mathcal{M}(t,{}_\delta I_{a^+}^{\beta,\psi}y(t))\Big|\\ \leq & R\|x-y\|+\varphi\,{}_\delta I_{a^+}^{\beta,\psi}|\mathcal{M}(t,{}_\delta I_{a^+}^{\beta,\psi}x(t))-\mathcal{M}(t,{}_\delta I_{a^+}^{\beta,\psi}y(t))|\\ \leq & R\|x-y\|+\frac{\varphi C_1}{\delta^\beta\Gamma(\beta)}\\ & \times\int_a^t\Omega_\psi^{\beta-1}(t,s)\psi'(s)\left(\frac{1}{\delta^\beta\Gamma(\beta)}\int_a^s\Omega_\psi^{\beta-1}(s,\tau)\psi'(\tau)|x(\tau)-y(\tau)|d\tau\right)ds\\ \leq & R\parallel x-y\parallel+\frac{\varphi C_1(\psi(b)-\psi(a))^\beta}{\delta^{2\beta}\Gamma(\beta)\Gamma(\beta+1)}\end{aligned}$$

$$\times \int_a^t (\psi(t)-\psi(s))^{\beta-1}\psi'(s)ds\|x-y\|$$
$$\leq \left(R+\frac{\varphi C_1(\psi(b)-\psi(a))^{2\beta}}{\delta^{2\beta}(\Gamma(\beta+1))^2}\right)\|x-y\|.$$

From the condition (3.10) it follows that the operator Π is a contraction. It follows that Π has a unique fixed point according to the Banach fixed point theorem, which is the solution of the nonlinear hybrid fractional differential equation (1.2).

The proof is completed. □

4. Example

In this section, we illustrate our main results with an example. Consider the following problem:

$$(4.1)\quad \begin{cases} {}^{H}_{\frac{1}{2}}D_{0_+}^{\frac{1}{2},\frac{1}{4},t}\left(x(t)-\frac{1}{2t+54}\left(\frac{\left|{}_{\frac{2}{3}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}{1+\left|{}_{\frac{1}{2}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}\right)\right)=\frac{t\cos\left({}_{\frac{1}{2}}I_{0^+}^{\frac{1}{2},t}x(t)\right)}{17}, & t\in[0,1],\\ {}_{\frac{1}{2}}I_{0^+}^{1-\frac{5}{8},t}\left(x(t)-\frac{\left|{}_{\frac{1}{2}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}{1+\left|{}_{\frac{1}{2}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}\right)_{t=0}=\sum_{k=1}^{3}\lambda_k x(\epsilon_k), & \epsilon_k\in[0,1],\lambda_k\in\mathbb{R}. \end{cases}$$

In this example, we take $\mathcal{T}=[0,1]$, $\psi(t)=t$, $\beta=\frac{1}{2}$, $\sigma=\frac{1}{4}$, $\delta=\frac{1}{2}$, $\zeta=\frac{5}{8}$, $\lambda_1=\frac{1}{8}$, $\lambda_2=\frac{1}{11}$, $\lambda_3=\frac{2}{23}$, $\varepsilon_1=\frac{2}{15}$, $\varepsilon_2=\frac{3}{8}$, and $\varepsilon_3=\frac{1}{16}$.

Comparing Problem (4.1) with Problem (1.2), we find

$$\mathcal{Q}(t,{}_{\delta}I_{a^+}^{\beta,\psi}x(t))=\frac{1}{2t+54}\left(\frac{\left|{}_{\frac{2}{3}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}{1+\left|{}_{\frac{2}{3}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}\right),\quad \mathcal{M}(t,{}_{\delta}I_{a^+}^{\beta,\psi}x(t))=\frac{t\cos\left({}_{\frac{2}{3}}I_{0^+}^{\frac{1}{2},t}x(t)\right)}{16},$$

and

$$\mathcal{L}(t,{}_{\delta}I_{a^+}^{\beta,\psi}x(t))=1.$$

Then for all $p,q\in\mathbb{R}$ and $t\in[0,1]$ we have

$$|\mathcal{M}(t,p)-\mathcal{M}(t,q)|=\frac{t}{17}|\cos(p)-\cos(q)|\leq\frac{1}{17}|p-q|.$$

This implies that the function $\mathcal{T}$ is Lipschitz, with a Lipschitz constant $C_1=\frac{1}{17}$, then hypothesis (H_1) is satisfied. Also, we have

$$|\mathcal{M}(t,p)|=\left|\frac{t}{17}\cos p\right|\leq\frac{1}{17}|p|.$$

Hence, hypothesis (H_2) is satisfied with $r=1$ and $\xi=\frac{1}{17}$.

It is clear that for all $p,q\in\mathbb{R}$ and $t\in[0,1]$, we have

$$|\mathcal{Q}(t,p)-\mathcal{Q}(t,q)|=\frac{1}{2t+54}\left|\frac{|p|}{1+|p|}-\frac{|q|}{1+|q|}\right|\leq\frac{1}{54}|p-q|.$$

This implies that the function $\mathcal{Q}$ is Lipschitz, with a Lipschitz constant $C_2 = \frac{1}{54}$, then hypothesis (H_3) is satisfied.

Next, to check hypothesis H_4, we have $\mathcal{L}(t,_\delta I_{a^+}^{\beta,\psi}x(t)) = 1 = \varphi$ and

$$\mathcal{Q}(t,_\delta I_{a^+}^{\beta,\psi}x(t)) = \frac{1}{2t+54}\left(\frac{\left|_{\frac{1}{2}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}{1+\left|_{\frac{1}{2}}I_{0^+}^{\frac{1}{2},t}x(t)\right|}\right) \leq N = \frac{1}{54}.$$

Now it only remains to verify the conditions (3.1) and (3.6). We have

$$\begin{aligned}\chi &= \delta^{\zeta-1}\Gamma(\zeta) - \sum_{k=1}^{n}\lambda_k L(\epsilon_k,_\delta I_{a^+}^{\beta,\psi}x(\epsilon_k))\Omega_\psi^{\zeta-1}(\epsilon_k,a)\\ &= \left(\frac{1}{2}\right)^{\frac{-3}{8}}\Gamma\left(\frac{5}{8}\right) - \sum_{k=1}^{3}\lambda_k\epsilon_k^{\frac{-3}{8}}e^{-\epsilon_k} = 1,295 \neq 0\end{aligned}$$

and

$$\begin{aligned}R =&\frac{(\psi(b)-\psi(a))^\beta}{\delta^\beta\Gamma(\beta+1)}\left(C_2 + \frac{\varphi C_2(\psi(b)-\psi(a))^{\zeta-1}}{|\chi|}\sum_{k=1}^{n}|\lambda_k|\right.\\ &\left.+\frac{C_1\varphi^2(\psi(b)-\psi(a))^{\beta+\zeta-1}}{|\chi|\delta^\beta(\Gamma(\beta+1))}\sum_{k=1}^{n}|\lambda_k|\right)\\ =&\frac{1}{(\frac{1}{2})^{\frac{1}{2}}\Gamma(\frac{3}{2})}\left(\frac{1}{54} + \frac{1}{54\times 1,295}\sum_{k=1}^{3}|\lambda_k| + \frac{1}{17\times|\chi|\times(\frac{1}{2})^{\frac{1}{2}}\Gamma(\frac{3}{2})}\sum_{k=1}^{3}|\lambda_k|\right)\\ \simeq&0,0713 < 1.\end{aligned}$$

We remark that all the conditions and the hypotheses are satisfied, then Problem (4.1) has a solution in $C(T,\mathbb{R})$. In addition, we have:

$$\begin{aligned}R + \frac{\varphi C_1(\psi(b)-\psi(a))^{2\beta}}{\delta^{2\beta}(\Gamma(\beta+1))^2} &= 0,0713 + \frac{1}{17\times\frac{1}{2}\Gamma^2(\frac{3}{2})}\\ &\simeq 0,2211 < 1,\end{aligned}$$

this implies that the problem (4.1) has a unique solution in $C(T,\mathbb{R})$.

5. Conclusion

In this study, we have developed the theory of hybrid fractional differential equations, which includes the ψ-Hilfer generalized proportional fractional derivative. We have used the Banach fixed point theorem and topological degree theory to analyze the existence and uniqueness of solutions. An illustrative example is provided to demonstrate our main findings.

As a possible direction for future research, we intend to investigate the existence, uniqueness and Ulam-Hyers stability of a novel class of hybrid Langevin equations and their associated formulations.

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