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BRYANT-SCHNEIDER GROUP OF BASARAB LOOP

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Abstract. A loop $(Q, \circ)$ is called Basarab loop if it is both a left and a right Basarab loop; $(x \circ yx^{\rho}) \circ xz = x \circ yz$ and $yx \circ (x^{\lambda}z \circ x) = yz \circ x$ hold for all $x, y, z \in Q$ respectively. In this paper, the characterizations of the Bryant-Schneider group of a Basarab loop are studied using the left and right Basarab loop identities. It is shown that the element, x^{λ} (x^{ρ}) is in the left (right) nucleus if and only if the middle inner map T_x (inverse T_x^{-1}) is an automorphism. It is revealed that every crypto-automorphism of a Basarab loop is an element of the Bryant-Schneider group. Some related algebraic properties were also characterized. Furthermore, elements of the Bryant-Schneider group of a Basarab loop in terms of pseudo-automorphism and automorphism are also characterized. A subgroup of the Bryant-Schneider group, characterized by the Basarab loop, is established. Finally, a right pseudo-automorphic characterization of the isotopy-isomorphy of a Basarab loop is carried out.

1. Introduction

Let Q be a non-empty set. We can define a binary operation, denoted by '$\circ$', on this set. If the result of $s \circ t$ is always an element of Q for any s and t in Q, then the pair $(Q, \circ)$ is called a groupoid. Moreover, if the equations $k \circ s = l$ and $t \circ k = l$ have unique solutions s and t for all k and l in Q, then $(Q, \circ)$ is called a quasigroup.

Furthermore, if $(Q, \circ)$ is a quasigroup and there exists a unique element, denoted as the identity $e \in Q$, with the property that for any $s \in Q$ we have $s \circ e = e \circ s = s$,

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then $(Q,\circ)$ is known as a loop. We use st to represent $s\circ t$, with the understanding that '$\circ$' has lower priority than juxtaposition among factors to be multiplied.

Additionally, if $(Q,\circ)$ is a groupoid and k is a fixed element in Q, then the left and right translations of k, denoted by L_k and R_k respectively, are defined as $sL_k = k\circ s$ and $sR_k = s\circ k$ for all s in Q. It is evident that $(Q,\circ)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, the inverse mappings L_k^{-1} and R_k^{-1} exist.

Let

$$k\backslash l = lL_k^{-1} = kM_l \quad \text{and} \quad k/l = kR_l^{-1} = lM_k^{-1},$$

and note that

$$k\backslash l = m \Leftrightarrow k\circ m = l \quad \text{and} \quad k/l = m \Leftrightarrow m\circ l = k.$$

Thus, for any quasigroup $(Q,\circ)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$. M_k is the middle translation for any fixed $k\in Q$. Consequently, $(Q,\backslash)$ and $(Q,/)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1.1. A *loop* $(Q,\circ,/,\backslash,e)$ is a set Q together with three binary operations $(\circ)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(i) $k\circ(k\backslash l) = l$, $(l/k)\circ k = l$ for all $k,l\in Q$;
(ii) $k\backslash k = l/l$ or $e\circ k = k\circ e = k$ for all $k,l\in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\circ)$ among factors to be multiplied. For instance, $k\circ l/m$ and $k\circ l\backslash m$ stand for $k(l/m)$ and $a(b\backslash c)$, respectively.

In a loop $(Q,\circ)$ with identity element e, the *left inverse element* of $k\in Q$ is the element $kJ_\lambda = k^\lambda\in Q$ such that

$$k^\lambda\circ k = e,$$

while the *right inverse element* of $k\in Q$ is the element $kJ_\rho = k^\rho\in Q$ such that

$$k\circ k^\rho = e.$$

If $x^\rho = x^\lambda$, then we shall write $J_\lambda = J_\rho = J$ where $xJ = x^{-1}$.

For more study on quasigroup and loop theories, readers can check [10, 11, 18, 20–23, 25, 26, 29, 30].

The study of a Bryant-Schneider group on an arbitrary loop was first introduced in the work of Robinson [28] in 1980. The concept has been extended to different loop structures, e.g., the study of the isotopy-isomorphy of Bol loops, Moufang loops, and Osborn loops. In 2003 Adeniran [1] studied some properties of Bryant-Schneider groups of certain Bol loops. Jaiyéọlá [13], and Jaiyéọlá et al. [14, 15] extended the concept of the Bryant-Schneider group to the study of Smarandache loop, Osborn loop, and their universality. A characterization of isostrophy Bryant-Schneider group-invariant of Bol loops was studied in 2023 by Jaiyéọlá et al. [12]. The isotopic and pseudo-automorphic characterizations of a loop were presented by (Capodaglio [3], 1993), where the crypto-automorphism of a loop was defined as a generalization of pseudo-automorphism. Further study of crypto-automorphism was carried out by (Oyebo et al. [24], 2024). The work revealed that crypto-automorphisms of a quasigroup with left and right identity elements forms a group.

In this study, we shall study the properties of the crypto-automorphism group and Bryant-Schneider group of left (right) Basarab loop.

2. Preliminaries

Definition 2.1. The set of all permutations on a non-empty set Q forms a group called the symmetric group of Q, denoted as $SYM(Q)$. Let $(Q, \circ)$ be a loop, and let A, B, and C be elements of $SYM(Q)$. If $xA \circ yB = (x \circ y)C$ for all $x, y \in Q$, then the triple (A, B, C) is termed an autotopism, and these triples form a group known as the autotopism group of $(Q, \circ)$, denoted as $AUT(Q, \circ)$. If $A = B = C$, then A is called an automorphism of $(Q, \circ)$, and the set of all such automorphisms forms a group called the automorphism group of $(Q, \circ)$, denoted as $AUM(Q, \circ)$. See [26].

Definition 2.2. Let $(Q, \circ)$ be a loop.

(a) $\phi \in SYM(Q)$ is called a left pseudo-automorphism with companion $a \in Q$ if $(\phi L_a, \phi, \phi L_a) \in AUT(Q, \circ)$. The set of such maps forms a group (see [26]).

(b) $\phi \in SYM(Q)$ is called a right pseudo-automorphism with companion $a \in Q$ if $(\phi, \phi R_a, \phi R_a) \in AUT(Q, \circ)$. The set of such maps forms a group (see [26]).

(c) $\phi \in SYM(Q)$ is called a crypto-automorphism with companions $a, b \in Q$ if $(R_a\phi, L_b\phi, \phi) \in AUT(Q, \circ)$. The set of such maps forms a group (see [24]).

(d) A mapping $\phi \in SYM(Q)$ such that $(\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(Q, \circ)$ for some $f, g \in G$ is called a Bryant-Schneider map of $(Q, \circ)$. The set of such maps forms a group called the Bryant-Schneider group $BS(Q, \circ)$ of $(Q, \circ)$ (see [28]).

From Definition 2.2, it is clearly seen that

$$(\phi R_g^{-1}, \phi L_f^{-1}, \phi) = (\phi, \phi, \phi)(R_g^{-1}, L_f^{-1}, I),$$

which implies that ϕ is an isomorphism of $(G, \circ)$ onto some f, g-isotope of it.

Theorem 2.1 ([28])**.** *Let the set $BS(Q, \circ) = \{\phi \in SYM(Q) :$ exists $f, g \in Q \ni (\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(Q, \circ)\}$, then $BS(Q, \circ) \leq SYM(Q)$.*

Theorem 2.1 is associated with Theorem 2.2.

Theorem 2.2 ([26])**.** *Let $(Q, \circ)$ and $(H, \circ)$ be two isotopic loops. For some $f, g \in Q$, there exists an f, g-principal isotope $(Q, *)$ of $(Q, \circ)$ such that $(H, \circ) \cong (Q, *)$.*

Definition 2.3 ([26])**.** Let $(Q, \circ)$ be quasigroup with fixed elements $a, b \in Q$. The isotope of this form (R_a^{-1}, L_b^{-1}, I) is called LP-isotope.

Definition 2.4 ([26])**.** Let $(Q, \circ)$ be a quasigroup. Then, the following hold.

(a) Right inverse property (RIP) holds if there is a mapping $J_\rho : x \to x^\rho$ such that $(y \circ x) \circ x^\rho = y$ for all $x, y \in Q$.

(b) Left inverse property (LIP) holds if there is a mapping $J_\lambda : x \to x^\lambda$ such that $x^\lambda \circ (x \circ y) = y$ for all $x, y \in Q$.
(c) Inverse property (IP) if (a) and (b) hold.
(d) Right alternative property (RAP) if $y \circ xx = yx \circ x$ for all $x, y \in Q$.
(e) Left alternative property (LAP) if $x \circ xy = xx \circ y$ for all $x, y \in Q$.
(f) Flexible or elastic property if $(x \circ y) \circ x = x \circ (y \circ x)$ holds for all $x, y \in Q$.
(g) Cross inverse property (CIP) if there exist mapping $J_\lambda : x \to x^\lambda$ or $J_\rho : x \to x^\rho$ such that $xy \circ x^\rho = y$ or $x \circ yx^\rho = y$ or $x^\lambda \circ yx = y$ or $x^\lambda y \circ x = y$ for all $x, y \in Q$.

Definition 2.5. A loop $(Q, \circ)$ is said to be
(a) an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$;
(b) an anti-automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$;
(c) a power associative loop if $\langle x \rangle$ is a subgroup for all $x \in Q$ and a diassociative loop if $\langle x, y \rangle$ is a subgroup for all $x, y \in Q$.

Definition 2.6. Let $(Q, \circ)$ be a loop.
(a) $\mathbf{N}_\lambda = \{v \in Q : (v \circ x) \circ y = v \circ (x \circ y) \text{ for all } x, y \in Q\}$ is called the left nucleus of Q.
(b) $\mathbf{N}_\rho = \{v \in Q : y \circ (x \circ v) = (y \circ x) \circ v \text{ for all } x, y \in Q\}$ is called the right nucleus of Q.
(c) $\mathbf{N}_\mu = \{v \in Q : (y \circ v) \circ x = y \circ (v \circ x) \text{ for all } x, y \in Q\}$ is called the middle nucleus of Q.

Definition 2.7 ([3])**.** In a loop $(Q, \circ)$, a permutation ϕ is called a crypto-automorphism if there exists $a, b \in Q$ called the companions of ϕ such that for every $x, y \in Q$

$$(x \circ a)\phi \circ (b \circ y)\phi = (x \circ y)\phi.$$

Hence, ϕ is called a crypto-automorphism with companion (a, b).

Definition 2.8. Let $(Q, \circ)$ be a quasigroup. A mapping $\phi \in SYM(Q)$ will be defined as a two-middle pseudo-automorphism if there exists elements a and b in Q such that $(\phi R_a^{-1}, \phi L_{b^\lambda}^{-1}, \phi) \in AUT(Q, \circ)$.

Remark 2.1. The collection of two-middle pseudo-automorphisms will be denoted as $PS_{\mu 2}(Q, \circ)$. It is worth noting that $PS_\mu(Q, \circ) \subseteq PS_{\mu 2}(Q, \circ)$.

Theorem 2.3 ([24])**.** *The set of crypto-automorphisms $CAUM(Q, \circ)$ of a quasigroup $(Q, \circ)$ with right and left identity elements forms a group.*

Theorem 2.4 ([24])**.** *Let $(Q, \circ)$ be a loop. $BS(Q, \circ) = CAUM(Q, \circ)$.*

Theorem 2.5 ([26])**.** *Let $(Q, \circ)$ be an inverse property loop. Then, for any $a \in Q$:*
(a) $J_\lambda R_a J_\rho = L_{a^\lambda}$;

(b) $J_\rho R_a J_\rho = L_{a^\rho}$;
(c) $J_\rho L_a J_\rho = R_{a^\rho}$;
(d) $J_\lambda L_a J_\rho = R_{a^\lambda}$.

Theorem 2.6 ([26]). *Let $P = (A, B, C) \in AUT(Q, \circ)$ of a loop $(Q, \circ)$.*

(a) *If $(Q, \circ)$ is a left inverse property loop (LIPL), then $P_\lambda = (JAJ, C, B) \in AUT(Q, \circ)$.*
(b) *If $(Q, \circ)$ is a right inverse property loop (RIPL), then $P_\rho = (C, JBJ, A) \in AUT(Q, \circ)$.*

Theorem 2.7 ([26,31]). *Let $(Q, \circ)$ be a RIP or LIP or AAIP loop. Then, $J_\rho = J_\lambda = J$, i.e., $a^\rho = a^\lambda = a^{-1}$ for all $a \in Q$.*

Definition 2.9. A loop $(Q, \circ)$ is called

(a) left Basarab loop (LBaL) if it satisfies the identity $(x \circ yx^\rho) \circ xz = x \circ yz$ for all $x, y, z \in Q$;
(b) right Basarab loop (RBaL) if it satisfies the identity $yx \circ (x^\lambda z \circ x) = yz \circ x$ for all $x, y, z \in Q$.

A Basarab loop was studied in Basarab's work before Cote et al. [2] in 2010 worked on classification of loops of generalised Bol-Moufang type. Basarab published two prominent papers [4, 5] in 1992, focusing on IK-loops in [6] and [7]. Some recent studies of Basarab loop have been carried out in various fashions. The properties of Basarab loop with invariants of inverse properties were examined in [16]. This study revealed that a Basarab loop exhibits a cross inverse property if and only if it is an abelian group, or if all left (right) translations of the loop are right (left) regular. The same authors also investigated the Basarab loop and the generators of its total inner mapping group [17]. In 2021, the characterization of subloops of a Basarab loop was explored [8], and a recent announcement was made by Effiong et al. [9] regarding the holomorphic characterization of a Basarab loop.

Theorem 2.8 ([16]). (a) *In a left (right) Basarab loop, the following are equivalent:*

i *flexibility;*
ii *RIP (LIP);*
iii *RAP (LAP);*
iv *AAIP.*

(b) *In a Basarab loop, the following are equivalent:*

i *flexibility;*
ii *right inverse property;*
iii *left inverse property;*
iv *inverse property;*
v *right alternative property;*
vi *left alternative property;*
vii *alternative property.*

3. Main Results

Lemma 3.1. *Let $(Q,\circ)$ be a loop. Then, $(Q,\circ)$ is a*

(a) *left Basarab loop if and only if any of the following hold for all $a,b\in Q$:*
   i $(R_{a^\rho}L_a, L_a, L_a)\in AUT(Q,\circ)$;
   ii $R_{a^\rho}L_aR_{ab}=R_bL_a$;
   iii $R_{a^\rho}L_a=T_a^{-1}=R_bL_aR_{ab}^{-1}$;
   iv $L_aL_{a\circ ba^\rho}=L_bL_a$;
   v $L_{a\circ ba^\rho}=L_a^{-1}L_bL_a$;

(b) *right Basarab loop if and only if any of the following hold for all $a,b\in Q$:*
   i $(R_a, L_{a^\lambda}R_a, R_a)\in AUT(Q,\circ)$;
   ii $R_aL_{a^\lambda b\circ a}=R_bR_a$;
   iii $L_{a^\lambda b\circ a}=R_a^{-1}R_bR_a$;
   iv $L_{a^\lambda}R_aL_{ba}=L_bR_a$;
   v $L_{a^\lambda}R_a=T_a=L_bR_aL_{ba}^{-1}$.

Proof. Use Definition 2.9 with the aid of autotopisms and translations. □

Theorem 3.1. *Let $(Q,\circ)$ ba a Basarab loop.*

(a) *$a^\rho\in\mathbf{N}_\rho$ if and only if $T_a^{-1}\in AUM(Q,\circ)$.*
(b) *$a^\lambda\in\mathbf{N}_\lambda$ if and only if $T_a\in AUM(Q,\circ)$.*

Proof. (a) If $(Q,\circ)$ is a left Basarab loop, then from Lemma 3.1, $(R_{a^\rho}L_a, L_a, L_a)\in AUT(Q,\circ)$ for all $a\in Q$. If $a^\rho\in\mathbf{N}_\rho$, then $(I, R_{a^\rho}, R_{a^\rho})$ is an autotopism of $(Q,\circ)$ for all $a\in Q$. The product

$$(I,R_{a^\rho},R_{a^\rho})(R_{a^\rho}L_a,L_a,L_a)=(R_{a^\rho}L_a,R_{a^\rho}L_a,R_{a^\rho}L_a)$$

is equivalent to $(T_a^{-1},T_a^{-1},T_a^{-1})$ is an autotopism of $(Q,\circ)$ for all $a\in Q$. Thus, T_a^{-1} is automorphism of $(Q,\circ)$.

For the converse, we reverse the procedures to obtain $a^\rho\in\mathbf{N}_\rho$ if and only if $(R_{a^\rho}L_a,R_{a^\rho}L_a,R_{a^\rho}L_a)\in AUT(Q,\circ)$ is equivalent to $T_a^{-1}\in AUM(Q,\circ)$.

(b) If $(Q,\circ)$ is a right Basarab loop, in Lemma 3.1, $(R_a, L_{a^\rho}R_a, R_a)\in AUT(Q,\circ)$ for all $a\in Q$. If $a^\lambda\in\mathbf{N}_\lambda$, then $(L_{a^\lambda}, I, L_{a^\lambda})$ is an autotopism of $(Q,\circ)$ for all $a\in Q$. The product

$$(L_{a^\lambda},I,L_{a^\lambda})(R_a,L_{a^\rho}R_a,R_a)=(L_{a^\lambda}R_a,L_{a^\rho}R_a,L_{a^\lambda}R_a)$$

is equivalent to $(T_a,T_a,T_a)\in AUT(Q,\circ)$. Hence, T_a is an automorphism of $(Q,\circ)$. For the converse, we simply do the reverse procedure. □

Theorem 3.2. *Let U be a crypto-automorphism of LBaL $(Q,\circ)$ with companion (a,b). Then, $L_xU\in BS(Q,\circ)$ for all $x\in Q$.*

Proof. If $(Q,\circ)$ is a LBaL then $P=(R_{x^\rho}L_x,L_x,L_x)$ is an autotopism of $(Q,\circ)$ for all $x\in Q$. If U be a crypto-automorphism of $(Q,\circ)$, then $T=(R_aU,L_bU,U)$ is an autotopism of $(Q, \circ)$ with a twin (a, b). The product

(3.1)

$$PT = (R_{x^\rho} L_x, L_x, L_x)(R_a U, L_b U, U) = (R_{x^\rho} L_x R_a U, L_x L_b U, L_x U) \in AUT(Q, \circ).$$

Set $L_x U = \omega$ in (3.1) to obtain

$$(R_{x^\rho} L_x R_a U, L_x L_b U, \omega) \in AUT(Q, \circ). \tag{3.2}$$

Writing this in an equivalent form, for all $y, z \in Q$, we have

$$y R_{x^\rho} L_x R_a U \circ z L_x L_b U = (y \circ z)\omega. \tag{3.3}$$

Put $y = e$ in (3.3) to obtain

$$\begin{aligned} e R_{x^\rho} L_x R_a U \circ z L_x L_b U = (e \circ z)\omega \Rightarrow & (x \circ e x^\rho) R_a U \circ z L_x L_b U = z\omega \\ \Rightarrow & z L_x L_b U L_{aU} = z\omega \\ \Rightarrow & z L_x L_b U = z\omega L_{aU}^{-1} \Rightarrow L_x L_b U = \omega L_{aU}^{-1}. \qquad (3.4) \end{aligned}$$

Also, put $z = e$ in (3.3) to get

$$\begin{aligned} y R_{x^\rho} L_x R_a U \circ e L_x L_b U = y\omega \Rightarrow & y R_{x^\rho} L_x R_a U \circ (ex) L_b U = y\omega \\ \Rightarrow & y R_{x^\rho} L_x R_a U R_{(bx)U} = y\omega \qquad (3.5) \\ \Rightarrow & y R_{x^\rho} L_x R_a U = y\omega R_{(bx)U}^{-1} \\ \Rightarrow & R_{x^\rho} L_x R_a U = \omega R_{(bx)U}^{-1}. \qquad (3.6) \end{aligned}$$

Using (3.4) and (3.6) in (3.2), we have $(\omega R_{(bx)U}^{-1}, \omega L_{aU}^{-1}, \omega) \in AUT(Q, \circ)$. Thus, $\omega \in BS(Q, \circ)$ for all $a, b \in Q$. □

Corollary 3.1. *Let U be a crypto-automorphism of LBaL $(Q, \circ)$ with companion (a, b). Then, $L_a U \in PS_{\mu 2}(Q, \circ)$ with companion $((bx)U, (aU)^\rho)$.*

Proof. This follows as a consequence of Theorem 3.2. □

Theorem 3.3. *Let U be a crypto-automorphism of RBaL $(Q, \circ)$ with companion (a, b). Then,*

(a) *$R_x U \in BS(Q, \circ)$ for all $x \in Q$;*
(b) *$R_x U \in PS_{\mu 2}(Q, \circ)$ with companion $(bU, ((xa)U)^\rho)$ for all $x \in Q$*

Proof. Follow similar steps as in Theorem 3.2. □

Theorem 3.4. *Let $(Q, \circ)$ be a LBaL with RIP. If the map $\theta \in BS(Q, \circ)$ such that $\theta = \omega L_x^{-1}$ where $\omega : e \to e$, then ω is a unique right pseudo-automorphism with companion $(xg)^{-1}$ for all $x \in Q$ and for some $g \in Q$.*

Proof. Let $\theta \in BS(Q, \circ)$. Then, $M = (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q, \circ)$ for some $f, g \in Q$. Since $(Q, \circ)$ is a LBaL, $N = (R_{x^\rho} L_x, L_x, L_x) \in AUT(Q, \circ)$ for all $x \in Q$. The product

(3.7)

$$MN = (\theta R_g^{-1}, \theta L_f^{-1}, \theta)(R_{x^\rho} L_x, L_x, L_x) = (\theta R_g^{-1} R_{x^\rho} L_x, \theta L_f^{-1} L_x, \theta L_x) \in AUT(Q, \circ),$$

for some $f, g \in Q$ and for all $x \in Q$.

Let $(Q, \circ)$ be a RIPL, applying Theorem 2.6 to the autotopism in (3.7), we get

$$(\theta L_x, J\theta L_f^{-1} L_x J, \theta R_g^{-1} R_{x^\rho} L_x) \in AUT(Q, \circ), \tag{3.8}$$

for some $f, g \in Q$ and for all $x \in Q$.

Let $\alpha = J\theta L_f^{-1} L_x J$ in (3.8). Then,

$$(\theta L_x, \alpha, \theta R_g^{-1} R_{x^\rho} L_x) \in AUT(Q, \circ), \tag{3.9}$$

for some $g \in Q$ and for all $x \in Q$. For all $y, z \in Q$, we have

$$y\theta L_x \circ z\alpha = (y \circ z)\theta R_g^{-1} R_{x^\rho} L_x.$$

If $\theta = \omega L_x^{-1}$, then $\theta L_x = \omega$, so, $\alpha = J\omega L_x^{-1} L_f^{-1} L_x J$. Next, we have

$$y\omega \circ z\alpha = (y \circ z)\omega L_x^{-1} R_g^{-1} R_{x^\rho} L_x. \tag{3.10}$$

Let $y = e$ in (3.10). Then, we have

$$e\omega \circ z\alpha = z\omega L_x^{-1} R_g^{-1} R_{x^\rho} L_x \Rightarrow z\alpha = z\omega L_x^{-1} R_g^{-1} R_{x^\rho} L_x \Rightarrow \alpha = \omega L_x^{-1} R_g^{-1} R_{x^\rho} L_x. \tag{3.11}$$

Using (3.10) and (3.11), (3.9) becomes

$$(\omega, \omega L_x^{-1} R_g^{-1} R_{x^\rho} L_x, \omega L_x^{-1} R_g^{-1} R_{x^\rho} L_x) \in AUT(Q, \circ),$$

for some $g \in Q$ and for all $x \in Q$.

Since, $(Q, \circ)$ is a left Basarab loop, $R_{x^\rho} L_x = R_g L_x R_{xg}^{-1}$. Hence,

$$\begin{aligned} &(\omega, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}) \\ =&(\omega, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}, \omega L_x^{-1} R_g^{-1} R_g L_x R_{xg}^{-1}) = (\omega, \omega R_{xg}^{-1}, \omega R_{xg}^{-1}) \end{aligned}$$

is an autotopism of $(Q, \circ)$ for some $g \in Q$ and for all $x \in Q$. Thus, ω is a right pseudo-automorphism with companion $(xg)^{-1}$.

Let $\omega_1 L_{x_1}^{-1} = \omega_2 L_{x_2}^{-1}$, where $\omega_1, \omega_2 : e \to e$, for all $x_1, x_2 \in Q$. Then, $L_{x_1}^{-1} L_{x_2} = \omega_1^{-1}\omega_2$. So, $eL_{x_1}^{-1} L_{x_2} = e\omega_1^{-1}\omega_2$ implies $L_{x_1}^{-1} L_{x_2} = I$, implies $L_{x_1} = L_{x_2}$. Hence, $x_1 = x_2$ and so $\omega_1 = \omega_2$. This implies that for all $x \in Q$, there exists a unique ω such that $\theta = \omega L_x^{-1}$. Hence, $\theta = \omega L_x^{-1}$ where ω is a unique right pseudo-automorphism with companion $(xg)^{-1}$ for all $x \in Q$ and for some $g \in Q$. □

Corollary 3.2. *Let $(Q, \circ)$ be a LBaL with RIP. For some $\theta \in BS(Q, \circ)$ such that $\omega L_x^{-1} = \theta$, where $\omega : e \to e$, $\omega \in AUM(Q, \circ)$ is unique for all $x \in Q$.*

Proof. Set $x = g^{-1}$ in Theorem 3.4. □

Theorem 3.5. *Let $(Q, \circ)$ be a RBaL with LIP. If the map $\theta \in BS(Q, \circ)$ such that $\theta = \omega R_x^{-1}$, where $\omega : e \to e$, then ω is a unique left pseudo-automorphism with companion $(fx)^{-1}$ for all $x \in Q$ and for some $f \in Q$.*

Proof. Let $\theta \in BS(Q,\circ)$ then $M = (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q,\circ)$ for some $f,g \in Q$. Since $(Q,\circ)$ is a RBaL, $N = (R_x, L_{x^\lambda} R_x, R_x) \in AUT(Q,\circ)$ for all $x \in Q$. The product

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta)(R_x, L_{x^\lambda} R_x, R_x) = (\theta R_g^{-1} R_x, \theta L_f^{-1} L_{x^\lambda} R_x, \theta R_x) \in AUT(Q,\circ),$$

for some $f,g \in Q$ and all $x \in Q$. If $(Q,\circ)$ is LIPL, by Theorem 2.6,

$$(J\theta R_g^{-1} R_x J, \theta R_x, \theta L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q,\circ), \tag{3.12}$$

for some $f,g \in Q$ and all $x \in Q$. Let $\alpha = J\theta R_g^{-1} R_x J$ in (3.12), to get

$$(\alpha, \theta R_x, \theta L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q,\circ),$$

for all $x \in Q$ and some $f \in Q$.

If $\theta = \omega R_x^{-1}$, then $\omega = \theta R_x$. Thus, $\alpha = J\omega R_x^{-1} R_g^{-1} R_x J$. Here, (3.12) becomes

$$(\alpha, \omega, \omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q,\circ), \tag{3.13}$$

for all $x \in Q$ and some $f \in Q$. For all $y,z \in Q$, we have $y\alpha \circ z\omega = (y\circ z)\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x$. Put $z = e$. Then,

(3.14)
$$y\alpha \circ e\omega = y\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x \Rightarrow y\alpha = y\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x \Rightarrow \alpha = \omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x.$$

Now, using (3.14) in (3.13), we get

$$(\omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x, \omega, \omega R_x^{-1} L_f^{-1} L_{x^\lambda} R_x) \in AUT(Q,\circ), \tag{3.15}$$

for all $x \in Q$ and some $f \in Q$. Since $(Q,\circ)$ is a RBaL, by Lemma 3.1, $L_{x^\lambda} R_x = L_f R_x L_{fx}^{-1}$ for all $x,f \in Q$. Thus, we have the last autotopism

$$(\omega R_x^{-1} L_f^{-1} L_f R_x L_{fx}^{-1}, \omega, \omega R_x^{-1} L_f^{-1} L_f R_x L_{fx}^{-1}) = (\omega L_{fx}^{-1}, \omega, \omega L_{fx}^{-1}) \in AUT(Q,\circ),$$

for all $x \in Q$ and some $f \in Q$. Thus, ω is a left pseudo-automorphism with companion $(fx)^{-1}$ for all $x \in Q$ and for some $f \in Q$. The proof of the uniqueness of ω is similar to the one in Theorem 3.4. □

Corollary 3.3. *Let $(Q,\circ)$ be a RBaL with LIP. For some $\theta \in BS(Q,\circ)$ such that $\theta = \omega R_x^{-1}$, where $\omega : e \to e$, $\omega \in AUM(Q,\circ)$ is unique for all $x \in Q$ and for some $f \in Q$.*

Proof. Set $x = f^{-1}$ in Theorem 3.5. □

Theorem 3.6. *Let $(Q,\circ)$ be a LBaL and let $\omega = \omega(f,g) \in BS(Q,\circ)$ such that $\omega : e \to e$. Then, $\omega = \omega(f,g) \equiv \omega(f,f^\rho)$ and $\omega \equiv \omega(f,g) = \omega(g^\lambda, g)$.*

Proof. Let $\omega \in BS(Q,\circ)$, then $(\omega R_g^{-1}, \omega L_f^{-1}, \omega) \in AUT(Q,\circ)$ for some $f,g \in Q$. Since $(Q,\circ)$ is a LBaL, $(R_{x^\rho} L_x, L_x, L_x) \in AUT(Q,\circ)$ for all $x \in Q$. The product

$$(\omega R_g^{-1}, \omega L_f^{-1}, \omega)(R_{x^\rho} L_x, L_x, L_x) = (\omega R_g^{-1} R_{x^\rho} L_x, \omega L_f^{-1} L_x, \omega L_x) \tag{3.16}$$

is an autotopism of $(Q,\circ)$ for some $f,g \in Q$ and for all $x \in Q$. For all $y,z \in Q$ we have

$$y\omega R_g^{-1} R_{x^\rho} L_x \circ z\omega L_f^{-1} L_x = (y \circ z)\omega L_x. \tag{3.17}$$

Put $z = e$ in (3.17) to obtain

$$\begin{aligned} y\omega R_g^{-1}R_{x^\rho}L_x \circ e\omega L_f^{-1}L_x = y\omega L_x \Rightarrow & y\omega R_g^{-1}R_{x^\rho}L_x \circ eL_f^{-1}L_x = y\omega L_x \\ \Rightarrow & y\omega R_g^{-1}R_{x^\rho}L_xR_{xf^\rho} = y\omega L_x \\ \Rightarrow & R_g^{-1}R_{x^\rho}L_xR_{xf^\rho} = L_x \\ \Rightarrow & R_g^{-1} = L_xR_{xf^\rho}^{-1}L_x^{-1}R_{x^\rho}^{-1}, \end{aligned}$$

for some $f \in Q$ and for all $x \in Q$. Set $x = f$ in the last equality to get

$$R_g^{-1} = L_fL_f^{-1}R_{f^\rho}^{-1} = R_{f^\rho}^{-1}.$$

Hence, $g = f^\rho$. On the other hand, set $y = e$ in (3.17) to get

$$\begin{aligned} e\omega R_g^{-1}R_{x^\rho}L_x \circ z\omega L_f^{-1}L_x = z\omega L_x \Rightarrow & eR_g^{-1}R_{x^\rho}L_x \circ z\omega L_f^{-1}L_x = z\omega L_x \\ \Rightarrow & \left(x \circ g^{-1}x^\rho\right)\left(z\omega L_f^{-1}L_x\right) = z\omega L_x \\ \Rightarrow & L_f^{-1} = L_xL_{(x\circ g^{-1}x^\rho)}^{-1}L_x^{-1}. \end{aligned}$$

Set $x = g^\lambda$, to get

$$L_f^{-1} = L_{g^\lambda}L_{(g^\lambda\circ g^{-1}(g)^{\lambda\rho})}^{-1}L_{g^\lambda}^{-1} = L_{g^\lambda}^{-1}.$$

Thus, $L_f^{-1} = L_{g^\lambda}^{-1}$ implies $f = g^\lambda$. □

Corollary 3.4. *Let $(Q,\circ)$ be a LBaL. If $\omega \equiv \omega(f,g) \in BS(Q,\circ)$, such that $\omega : e \to e$, for some $f, g \in Q$, the following hold:*

(a) *ω is a right pseudo-automorphism with companion f;*
(b) *$|f| = 2$ or $|g| = 2$.*

Proof. Putting $R_g^{-1} = R_{f\rho}^{-1}$ and $L_f^{-1} = L_{g^\lambda}^{-1}$ in (3.16), we get the autotopism

$$(\omega R_{f\rho}^{-1}R_{x^\rho}L_x, \omega L_{g^\lambda}^{-1}L_x, \omega L_x)$$

of $(Q,\circ)$ for some $f, g \in Q$ and for all $x \in Q$. Set $x = f$, to get $(\omega L_f, \omega L_{g^\lambda}^{-1}L_f, \omega L_f) \in AUT(Q,\circ)$ for some $f, g \in Q$. Let $g = f$. Then, $(\omega L_f, \omega L_{f^\lambda}^{-1}L_f, \omega L_f) \in AUT(Q,\circ)$ for some $f \in Q$. For all $y, z \in Q$, we have $y\omega L_f \circ z\omega L_{f^\lambda}^{-1}L_f = (y\circ z)\omega L_f$. Setting $z = e$, we get

$$y\omega L_f \circ e\omega L_{f^\lambda}^{-1}L_f = y\omega L_f \Rightarrow y\omega L_f \circ f^2 = y\omega L_f \Rightarrow y\omega L_fR_{f^2} = y\omega L_f \Rightarrow |f| = 2.$$

Thus, $(\omega L_f, \omega, \omega L_f) \in AUT(Q,\circ)$ for some $f \in Q$. Therefore, ω is a pseudo-automorphism with companion f for some $f \in Q$. □

Theorem 3.7. *Let $(Q,\circ)$ be a LBaL with AAIP. Then,*

$$\begin{aligned}BS'(Q,\circ) =&\Big\{\omega \in BS(Q,\circ) \mid \omega : e \to e \text{ and } (x\omega)^{-1} = \Big(x^{-1}\Big)\omega\Big\}\\ =&\Big\{\omega \in SYM(Q) \mid \text{exists } f \in Q \ni \Big(\omega R_{f^{-1}}^{-1}, \omega L_f^{-1}, \omega\Big) \in AUT(Q),\ e\omega = e\\ &\text{and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q\Big\}\\ =&\Big\{\omega \in SYM(Q) \mid \text{exists } g \in Q \ni \Big(\omega R_g^{-1}, \omega L_{g^{-1}}^{-1}, \omega\Big) \in AUT(Q),\\ &e\omega = e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q\Big\}\\ \le& BS(Q,\circ).\end{aligned}$$

Proof. Let

$$BS'(Q,\circ) = \Big\{\omega \in BS(Q,\circ) \mid \omega : e \to e \text{ and } (x\omega)^{-1} = \Big(x^{-1}\Big)\omega\Big\} \subseteq BS(Q,\circ).$$

Using the connection from Theorem 3.6,

$$\begin{aligned}BS'(Q,\circ) =&\Big\{\omega \in BS(Q,\circ) \mid \omega : e \to e \text{ and } (x\omega)^{-1} = \Big(x^{-1}\Big)\omega\Big\}\\ =&\Big\{\omega \in SYM(Q) \mid \text{exists } f \in Q \ni \Big(\omega R_{f^{-1}}^{-1}, \omega L_f^{-1}, \omega\Big) \in AUT(Q),\\ &e\omega = e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q\Big\}\\ =&\Big\{\omega \in SYM(Q) \mid \text{exists } g \in Q \ni \Big(\omega R_g^{-1}, \omega L_{g^{-1}}^{-1}, \omega\Big) \in AUT(Q),\\ &e\omega = e \text{ and } (x\omega)^{-1} = (x^{-1})\omega \text{ for all } x \in Q\Big\}.\end{aligned}$$

Note that $eI = e$ and $(gI)^{-1} = (g^{-1})I$ for all $g \in Q$ and $(IR_e^{-1}, IL_e^{-1}, I) = (I, I, I) \in AUT(Q,\circ)$. Then, $I \in BS'(Q,\circ)$. Hence, $BS'(Q,\circ)$ contains the identity map, thus $BS'(Q,\circ)$ is not empty.

Let $\alpha, \sigma \in BS'(Q,\circ)$. Then, $\alpha, \sigma \in BS(Q,\circ)$ and $e\alpha = e$ and $(x\alpha)^{-1} = (x^{-1})\alpha$, $e\sigma = e$ and $(x\sigma)^{-1} = (x^{-1})\sigma$ for all $x \in Q$.

In addition, there exist $f_1, g_1, f_{11}, g_{11} \in Q$ with $g_1 = f_1^{-1}$, $f_{11} = g_{11}^{-1}$ such that

$$\begin{aligned}P =&(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), \quad T = (\sigma R_{g_{11}}^{-1}, \sigma L_{f_{11}}^{-1}, \sigma),\\ T^{-1} =&(R_{g_{11}}\sigma^{-1}, L_{f_{11}}\sigma^{-1}, \sigma^{-1}) \in AUT(Q,\circ),\\ PT^{-1} =&(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_{11}}\sigma^{-1}, L_{f_{11}}\sigma^{-1}, \sigma^{-1})\\ =&(\alpha R_{g_1}^{-1} R_{g_{11}}\sigma^{-1}, \alpha L_{f_1}^{-1} L_{f_{11}}\sigma^{-1}, \alpha\sigma^{-1}) \in AUT(Q,\circ).\end{aligned}$$

Let $\varrho = \sigma R_{g_1}^{-1} R_{g_{11}}\sigma^{-1}$ and $\gamma = \sigma L_{f_1}^{-1} L_{f_{11}}\sigma^{-1}$, such that $(\alpha\sigma^{-1}\varrho, \alpha\sigma^{-1}\gamma, \alpha\sigma^{-1}) \in AUT(Q,\circ)$ if and only if for all $a, b \in Q$

$$a\alpha\sigma^{-1}\varrho \circ b\alpha\sigma^{-1}\gamma = (a \circ b)\alpha\sigma^{-1}. \qquad (3.18)$$

Setting $a = e$ in Q and replacing b by $b\sigma\alpha^{-1}$ in (3.18), we have

$$(e\alpha\sigma^{-1}\varrho) \circ (b\gamma) = b \Rightarrow b\gamma L_{(e\alpha\sigma^{-1}\varrho)} = b \Rightarrow \gamma = L^{-1}_{(e\alpha\sigma^{-1}\varrho)}.$$

Analogously, setting $b = e$ in Q, the identity element and replacing a by $a\sigma\alpha^{-1}$ in (3.18), we have

$$(a\varrho) \circ (e\alpha\sigma^{-1}\gamma) = a \Rightarrow a\varrho R_{(e\alpha\sigma^{-1}\gamma)} = a \Rightarrow \varrho = R^{-1}_{(e\alpha\sigma^{-1}\gamma)}.$$

Thus, $g = e\alpha\sigma^{-1}\gamma = e\gamma = e\sigma L^{-1}_{f_1} L_{f_{11}}\sigma^{-1} = [f_{11} \circ (f_1\backslash e)]\sigma^{-1} = [f_{11} \circ f_1^{-1}]\sigma^{-1}$ and $f = e\alpha\sigma^{-1}\varrho = e\varrho = e\sigma R^{-1}_{f_1^{-1}} R_{f_{11}^{-1}}\sigma^{-1} = eR^{-1}_{f_1^{-1}} R_{f_{11}^{-1}}\sigma^{-1} = [(e/f_1^{-1}) \circ f_{11}^{-1}]\sigma^{-1} = (f_1 \circ f_{11}^{-1})\sigma^{-1}$. Then, using Theorem 2.7, $f^{-1} = [(f_1 \circ f_{11}^{-1})\sigma^{-1}]^{-1} = (f_1 \circ f_{11}^{-1})^{-1}\sigma^{-1} = (f_{11} \circ f_1^{-1})\sigma^{-1} = g$. Hence, $PT^{-1} = (\alpha\sigma^{-1}\varrho, \alpha\sigma^{-1}\gamma, \alpha\sigma^{-1}) = (\alpha\sigma^{-1}R^{-1}_{f^{-1}}, \alpha\sigma^{-1}L^{-1}_{f}, \alpha\sigma^{-1})$ is an autotospism of $(Q, \circ)$, $e\alpha\sigma^{-1} = e$ and $(x^{-1})\alpha\sigma^{-1} = (x\alpha\sigma^{-1})^{-1}$ for all $x \in Q$. So, $\alpha\sigma^{-1} \in BS'(Q, \circ)$. Also,

$$PT^{-1} = (\alpha\sigma^{-1}R^{-1}_{g}, \alpha\sigma^{-1}L^{-1}_{g^{-1}}, \alpha\sigma^{-1})$$

is an autotopism of $(Q, \circ)$ and $e\alpha\sigma^{-1} = e$ and $(x^{-1})\alpha\sigma^{-1} = (x\alpha\sigma^{-1})^{-1}$ for all $x \in Q$. So, $\alpha\sigma^{-1} \in BS'(Q, \circ)$. Hence, $BS'(Q, \circ) \leq BS(Q, \circ)$. □

Corollary 3.5. *Let $(Q, \circ)$ be a LBaL with AAIP. Then, $AUM(Q, \circ) \leq BS'(Q, \circ) \leq BS(Q, \circ)$.*

Proof. This follows as a consequence of Theorem 3.7. □

Theorem 3.8. *Let $(Q, \circ)$ be a LBaL with a RIP and let $H = (A, B, C) \in AUT(Q, \circ)$. Then, there exists a right pseudo-automorphism ω with companion yx, where $eA = x$ and $eB = y$ such that*

$$(A, B, C) = (\omega, \omega R_{yx}, \omega R_{yx})(L_{x^{-1}}, JL_{x^{-1}}J, R^{-1}_{x^\rho}L_{x^{-1}})^{-1}.$$

Proof. Let $(Q, \circ)$ be a LBaL, then $(R^{-1}_{x^\rho}L_{x^{-1}}, L_{x^{-1}}, L_{x^{-1}}) \in AUT(Q, \circ)$ for all $x \in Q$. Applying Theorem 2.6, $(L_{x^{-1}}, JL_{x^{-1}}J, R^{-1}_{x^\rho}L_{x^{-1}}) \in AUT(Q, \circ)$ for all $x \in Q$. Suppose that $K = (A, B, C)$ is an autotopism of $(Q, \circ)$, then the product

$$\begin{aligned}&(A, B, C)(L_{x^{-1}}, JL_{x^{-1}}J, R^{-1}_{x^\rho}L_{x^{-1}})\\ =&(AL_{x^{-1}}, BJL_{x^{-1}}J, CR^{-1}_{x^\rho}L_{x^{-1}}) \in AUT(Q, \circ), \quad \text{for all } x \in Q.\end{aligned}$$

Replace $AL_{x^{-1}}$ with ω and note that $e\omega = eAL_{x^{-1}} = x^{-1} \circ x = e$,

$$t\omega \circ zBJL_{x^{-1}}J = (t \circ z)CR^{-1}_{x^\rho}L_{x^{-1}} \tag{3.19}$$

for all $t, z \in Q$. Set $t = e$ in (3.19), we obtain

$$\begin{aligned}e\omega \circ zBJL_{x^{-1}}J = zCR^{-1}_{x^\rho}L_{x^{-1}} &\Rightarrow zBJL_{x^{-1}}J = zCR^{-1}_{x^\rho}L_{x^{-1}}\\ &\Rightarrow BJL_{x^{-1}}J = CR^{-1}_{x^\rho}L_{x^{-1}},\end{aligned}$$

for all $x \in Q$. So, (3.19) becomes

$$t\omega \circ zBJL_{x^{-1}}J = (t \circ z)BJL_{x^{-1}}J, \tag{3.20}$$

for all $t, z \in Q$. Let $z = e$, then it follows from (3.20) that $y\omega \circ eBJL_{x^{-1}}J = yBJL_{x^{-1}}J$ implies $\omega yJL_{x^{-1}}J = BJL_{x^{-1}}J$, implies $\omega R_{yx} = BJL_{x^{-1}}J$. Using this information in (3.20), we get $(\omega, \omega R_{yx}, \omega R_{yx}) \in AUT(Q, \circ)$. Hence, ω is a right pseudo-automorphism with companion (yx). So,

$$\begin{aligned}(A, B, C)(L_{x^{-1}}, JL_{x^{-1}}J, R_{x^\rho}^{-1}L_{x^{-1}}) &= (\omega, \omega R_{yx}, \omega R_{yx})\\ \Rightarrow (A, B, C) &= (\omega, \omega R_{yx}, \omega R_{yx})(L_{x^{-1}}, JL_{x^{-1}}J, R_{x^\rho}^{-1}L_{x^{-1}})^{-1}. \qquad \square\end{aligned}$$

Theorem 3.9. *Let $(Q, \circ)$ be a LBaL with a RIP such that $|xz| = 2$ for all $z \in Q$, and let $a \oplus b = aR_x^{-1} \circ bL_y^{-1}$, for any arbitrarily fixed $a, b \in Q$ and for all $x, y \in Q$. Then, $(Q, \oplus) \cong (Q, \circ)$ if and only if there exists right pseudo-automorphism ω of $(Q, \circ)$ with companion yx.*

Proof. Let C be isomorphism between $(Q, \oplus)$ and $(Q, \circ)$. Then,

$$(a \circ b)C = aC \oplus bC = aCR_x^{-1} \circ bCL_y^{-1} = aB \circ bA,$$

for all $a, b \in Q$, where $B = CR_x^{-1}$ and $A = CL_y^{-1}$. So, $(B, A, C) \in AUT(Q, \circ)$.

Let e and e^* denote the identity elements in $(Q, \circ)$ and $(Q, \oplus)$, respectively. So, since C is an isomorphism, we have $eC = e^*$, where $e^* = y \circ x$, and

$$eA = eCR_x^{-1} = e^*R_x^{-1} = (y \circ x)R_x^{-1} = y, \tag{3.21}$$

$$eB = eCL_y^{-1} = e^*L_y^{-1} = (y \circ x)L_y^{-1} = x. \tag{3.22}$$

By Theorem 3.8, and going by (3.21), and (3.22), we have

$$(B, A, C) = (\omega, \omega R_{yx}, \omega R_{yx})(L_{x^{-1}}, JL_{x^{-1}}J, R_{x^\rho}^{-1}L_{x^{-1}})^{-1},$$

where ω is a right pseudo-automorphism of $(Q, \circ)$ with companion yx.

Conversely, if ω is right pseudo-automorphism with companion yx, then $(\omega, \omega R_{yx}, \omega R_{yx})$ is a autotopism of $(Q, \circ)$. On the other hand, since $(Q, \circ)$ is LBaL, we have that $(L_{y^{-1}}, JL_{y^{-1}}J, R_{y^\rho}^{-1}L_{y^{-1}})^{-1}$ is an autotopism of $(Q, \circ)$. Now,

$$\begin{aligned}(B, A, C) &= (\omega, \omega R_{yx}, \omega R_{yx})(L_{y^{-1}}, JL_{y^{-1}}J, R_{y^\rho}^{-1}L_{y^{-1}})^{-1}\\ &= (\omega, \omega R_{yx}, \omega R_{yx})(L_{y^{-1}}^{-1}, JL_{y^{-1}}^{-1}J, L_{y^{-1}}^{-1}R_{y^\rho})\\ &= \left(\omega L_{y^{-1}}^{-1}, \omega R_{yx}JL_{y^{-1}}^{-1}J, \omega R_{yx}L_{y^{-1}}^{-1}R_{y^\rho}\right) \in AUT(Q, \circ).\end{aligned}$$

Writing it in an equivalent relation form, for all $a, b \in Q$, we have

$$a\omega L_{y^{-1}}^{-1} \circ b\omega R_{yx}JL_{y^{-1}}^{-1}J = (a \circ b)\omega R_{yx}L_{y^{-1}}^{-1}R_{y^\rho} \Rightarrow (a \circ b)C = aB \circ bA, \tag{3.23}$$

where $C = \omega R_{yx}L_{y^{-1}}^{-1}R_{y^\rho}$, $B = \omega L_{y^{-1}}^{-1}$ and $A = \omega R_{yx}JL_{y^{-1}}^{-1}J$. So,

$$eB = e\omega L_{y^{-1}}^{-1} = eL_{y^{-1}}^{-1} = y^{-1}\backslash e = y,$$

$$eA = e\omega R_{yx}JL_{y^{-1}}^{-1}J = eR_{yx}JL_{y^{-1}}^{-1}J = \left[y^{-1}\backslash(yx)^{-1}\right]^{-1}$$

$$\Rightarrow eA = [y\backslash(yx)]^{-1} = x^{-1} = x. \tag{3.24}$$

Note that we used the assumptions that $|z| = 2$ for all $z \in Q$ and Q has RIP, which by Theorem 2.8 is equivalent to flexibility in a left Basarab loop.

Now, setting $b = e$ in (3.23), to get

$$aC = aB \circ eA \Rightarrow aC = aB \circ x \Rightarrow aC = aBR_x \Rightarrow B = CR_x^{-1}. \tag{3.25}$$

Put $a = e$ in (3.23) to obtain

$$bC = eB \circ bA \Rightarrow bC = y \circ bA \Rightarrow bC = bAL_y \Rightarrow A = CL_y^{-1}. \tag{3.26}$$

So, using (3.23), (3.25) and (3.26), we obtain

$$(a \circ b)C = aB \circ bA = aCR_x^{-1} \circ bCL_y^{-1} = aC \oplus bC,$$

for all $a, b \in Q$. Hence, $(Q, \oplus) \cong (Q, \circ)$. □

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