Kragujevac Journal of Mathematics
Volume 50(6) (2026), Pages 855–869.

APPROXIMATION PROPERTIES OF A MODIFIED GAMMA TYPE OPERATOR

AJAY KUMAR

Abstract. This article presents a new sequence of Gamma-type operators that retains the test function $e_r(t) = t^r$, $r \in \mathbb{N}$. Initially, we derive the moment formulas for these operators. Later, we analyze the approximation properties using the standard and weighted modulus of smoothness and prove an asymptotic Voronovskaja-type theorem. Furthermore, we compare the convergence rate and error estimation of the proposed operators with existing ones that preserve test functions in various ways, using numerical examples.

1. Introduction

The classical Gamma operators were first introduced by Lupas and Müller in 1967 [17]. This is a well-known sequence of positive linear operators used for improving the approximation of a target function on the interval $[0,\infty)$. These classical Gamma operators are defined as:

$$G_n(f,x) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^\infty e^{-xu} u^n f\left(\frac{n}{u}\right) du, \quad \text{for all } x \in \mathbb{R}^+ = (0,\infty), n \in \mathbb{N}. \tag{1.1}$$

The above operators not only maintain constants but also preserve linear functions. To achieve more precise approximations compared to the original operator (1.1), numerous researchers have proposed various modified versions of the classical Gamma operators, which are extensively discussed in the literature. For additional information, refer to the relevant sources [3,5,6,11,12,14,15,19–24] and the references cited theirin.

Key words and phrases. Gamma operator, rate of convergence, modulus of continuity, Voronovskaja type theorem.
2020 *Mathematics Subject Classification.* Primary: 41A25. Secondary: 41A36.
DOI
Received: March 23, 2023.
Accepted: May 23, 2024.

A new modified form of the classical Gamma operators has been considered by Betus and Usta [4] in recent times, which is expressed as follows:

$$\widetilde{G}_n(f,x) = \frac{x^n}{\Gamma(n+1)} \int_0^\infty e^{-xu^{1/n}} f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right) du, \quad n \in \mathbb{N}. \tag{1.2}$$

The modified form of the classical Gamma operators preserves the test functions $e_0(t) = 1$ and $e_2(t) = t^2$, as noted in [4]. Furthermore, it has been observed in the same source that this modified form of the operators yields improved approximation results compared to the original operator (1.1).

In the literature, several researchers have proposed new constructions or modifications of operators for better approximation results. King [13] was the first to present a new construction of Bernstein operators that preserve the test functions $e_r = x^r$ for $r = 0, 2$. Acar et al. [2] introduced a generalized form of (1.1) that can reproduce exponential test functions and established some approximation properties for the considered sequence. Deveci et al. [7] defined a refinement of Gamma operators that preserves constants and functions of the form $e^{2\mu}$ for $\mu > 0$. Gupta and Agrawal [9] considered modified Post-Widder operators that preserve the test functions $e_r(t) = t^r$ for $r \in \mathbb{N}$ and discussed that these operators provide a better approximation for $r = 3$.

Based on the aforementioned discussion, we are motivated to modify the operator (1.2) such that it preserves the test function $e_r(t) = t^r$ for $r \in \mathbb{N}$. We shall start with the following expression:

$$G_{n,r}(f,x) = \frac{(b_{n,r}(x))^n}{\Gamma(n+1)} \int_0^\infty e^{-b_{n,r}(x)u^{1/n}} f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right) du,$$

where $b_{n,r}(x) \in \mathbb{R}^+$. Then,

$$\begin{aligned} G_{n,r}(t^r,x) = x^r &= \frac{(b_{n,r}(x))^n}{\Gamma(n+1)} \int_0^\infty e^{-b_{n,r}(x)u^{1/n}} \left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)^r du \\ &= \frac{\left(\sqrt{(n-1)(n-2)}\right)^r}{\Gamma(n+1)} (b_{n,r}(x))^n \int_0^\infty e^{-b_{n,r}(x)u^{1/n}} \frac{1}{u^{r/n}} du \\ &= \frac{\Gamma(n-r)}{\Gamma(n)} (b_{n,r}(x))^r \left(\sqrt{(n-1)(n-2)}\right)^r. \end{aligned}$$

Above implies that

$$b_{n,r}(x) = \left(\frac{\Gamma(n)}{\Gamma(n-r)}\right)^{1/r} \frac{x}{\sqrt{(n-1)(n-2)}} = \frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}, \tag{1.3}$$

where the rising factorial is given by $(-n+1)_r = (-n+1)(-n+2)\cdots(-n+r)$, $(-n+1)_1 = -n+1$ and $(-n+1)_0 = 1$. Therefore, using (1.3), the modified form of the operator $G_{n,r}(f,x)$ for $r \in \mathbb{N}$ can be written as:

$$G_{n,r}(f,x) = \frac{1}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n$$

$$\times \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}} f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)du. \tag{1.4}$$

This modified form of the operator $G_{n,r}(f,x)$, which is defined by (1.4), preserves the test function $e_r(t) = t^r$ for $r \in \mathbb{N}$, as well as the constant function. Notably, if $r = 2$, the generalized operator (1.4) reduces to the original operator (1.2). In this paper, we aim to investigate the approximation properties of the modified Gamma operator defined by (1.4).

2. Auxiliary Results

The following lemma provides a general expression for moments of the proposed operators.

Lemma 2.1. *Let $x \in \mathbb{R}^+$ and $e_m(t) = t^m$, $m = 0, 1, 2, \ldots$ Then, for $r \in \mathbb{N}$, we have*

$$G_{n,r}(e_m,x) = \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{(n-1)(n-2)(n-3)\cdots(n-m)}x^m$$

$$= \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{\Pi_{i=1}^m(n-i)}x^m, \quad m \in \mathbb{N}\cup\{0\}.$$

Proof. From (1.4), we have

$$G_{n,r}(e_m,x) = \frac{1}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n \times \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}} \left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)^m du. \tag{2.1}$$

Let $\alpha u^{1/n} = t$, where $\alpha = \frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}$. Substituting $du = \frac{n}{\alpha}\left(\frac{t}{\alpha}\right)^{n-1}dt$ and $\frac{1}{u^{m/n}} = \left(\frac{\alpha}{t}\right)^m$ in (2.1), we get

$$\begin{aligned}
G_{n,r}(e_m,x) &= \frac{(\alpha)^n\{(n-1)(n-2)\}^{m/2}}{\Gamma(n+1)}\int_0^\infty e^{-t}\left(\frac{\alpha}{t}\right)^m \frac{n}{\alpha}\left(\frac{t}{\alpha}\right)^{n-1}dt\\
&= \frac{\alpha^m\{(n-1)(n-2)\}^{m/2}}{\Gamma(n)}\int_0^\infty e^{-t}t^{(n-m)-1}dt\\
&= \frac{\alpha^m\{(n-1)(n-2)\}^{m/2}}{\Gamma(n)}\Gamma(n-m)\\
&= \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{(n-1)(n-2)(n-3)\cdots(n-m)}x^m = \frac{\{(-1)^r(-n+1)_r\}^{m/r}}{\Pi_{i=1}^m(n-i)}x^m.
\end{aligned}$$

Thus, the lemma is completed. □

Remark 2.1. Note that from Lemma 2.1, it is evident that when $r = m$, the operator (1.4) preserves the test functions $e_r(x) = x^r$ for $r \in \mathbb{N} \cup \{0\}$. If we set $r = 2$, the resulting operator (1.4) reduces to the operator (1.2) and preserves both the constant function and the test function x^2.

Lemma 2.2. *Let us define the central moment for $m \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$ as follows: $\mu_m^{G_{n,r}}(x) = G_{n,r}((t-x)^m, x)$, $n > m$, then*

$$\mu_1^{G_{n,r}}(x) = \left[\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} - 1\right]x,$$

$$\mu_2^{G_{n,r}}(x) = \left[\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - 2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\right]x^2,$$

$$\begin{aligned}\mu_4^{G_{n,r}}(x) = \Bigg[&\frac{\{(-1)^r(-n+1)_r\}^{4/r}}{(n-1)(n-2)(n-3)(n-4)} - \frac{4\{(-1)^r(-n+1)_r\}^{3/r}}{(n-1)(n-2)(n-3)} \\ &+ \frac{6\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - \frac{4\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\Bigg]x^4,\end{aligned}$$

$$\begin{aligned}\mu_6^{G_{n,r}}(x) = \Bigg[&\frac{\{(-1)^r(-n+1)_r\}^{6/r}}{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)} \\ &- \frac{6\{(-1)^r(-n+1)_r\}^{5/r}}{(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{15\{(-1)^r(-n+1)_r\}^{4/r}}{(n-1)(n-2)(n-3)(n-4)} \\ &- \frac{20\{(-1)^r(-n+1)_r\}^{3/r}}{(n-1)(n-2)(n-3)} + \frac{15\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} \\ &- \frac{6\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\Bigg]x^6.\end{aligned}$$

Proof. The proof of this lemma can be obtained through straightforward computation using (1.4) and Lemma 2.1. We omit the details of the proof. □

Lemma 2.3. *For $f \in C[0,\infty)$, we have $\|G_{n,r}(f)\| \leq \|f\|$.*

Proof. Using (1.4) and Lemma 2.1, we can obtain the following expression:

$$\begin{aligned}|G_{n,r}(f,x)| \leq &\frac{1}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n \\ &\times \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}}\left|f\left(\frac{\sqrt{(n-1)(n-2)}}{u^{1/n}}\right)\right|du, \\ \leq &\frac{\|f\|}{\Gamma(n+1)}\left(\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}\right)^n \int_0^\infty e^{-\frac{x\{(-1)^r(-n+1)_r\}^{1/r}}{\sqrt{(n-1)(n-2)}}u^{1/n}}du \\ = &\|f\|.\end{aligned}$$

Therefore, we have completed the proof. □

3. Convergence Properties of $G_{n,r}$

Let $C_B[0,\infty)$ denote the space of all real-valued uniformly continuous and bounded functions on $[0,\infty)$ equipped with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. For $f \in C_B[0,\infty)$ and $\delta > 0$, the n-th order modulus of continuity is defined as

$$\omega_n(f,\delta) = \sup_{0\le h\le\delta} \sup_{x\in[0,\infty)} |\Delta_h^n f(x)|, \quad n \in \mathbb{N},$$

where Δ denotes the forward difference operator. When $n = 1$, we obtain the usual modulus of continuity, which is denoted by $\omega(f,\delta)$.

Theorem 3.1. *If $f \in C_B[0,\infty)$, then for every $x \in [0,\infty)$, we have*

$$|G_{n,r}(f,x) - f(x)| \le 2\omega(f, er(r)),$$

where $\delta = er(r) = \sqrt{\mu_2^{G_{n,r}}(x)}$ is the error function for $r = 1, 2, 3, \ldots$

Proof. Let $x \in [0,\infty)$ and $r \in \mathbb{N}$. In view of the fact that $G_{n,r}(1;x) = 1$, we have

$$|G_{n,r}(f;x) - f(x)| = |G_{n,r}(f;x) - G_{n,r}(f(x);x)| \le G_{n,r}\left(|f(t) - f(x)|; x\right).$$

Now, using the property of modulus of continuity $|f(t) - f(x)| \le \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right)$ in the above inequality, we have

$$|G_{n,r}(f;x) - f(x)| \le \omega(f;\delta)\left(\frac{G_{n,r}((t-x)^2;x)}{\delta^2} + 1\right).$$

By choosing $\delta = \sqrt{\mu_2^{G_{n,r}}(x)}$, we get the desired result. □

Thus for different preservation of the operators $G_{n,r}$, i.e. $r = 1, 2, 3$, we have

$$|G_{n,1}(f;x) - f(x)| \le 2\omega\left(f; \frac{x}{\sqrt{(n-2)}}\right),$$

$$|G_{n,2}(f;x) - f(x)| \le 2\omega\left(f; x\sqrt{2\frac{\sqrt{(n-1)} - \sqrt{(n-2)}}{\sqrt{(n-2)}}}\right),$$

$$|G_{n,3}(f;x) - f(x)| \le 2\omega\left(f; x\sqrt{\left[\frac{(n-3)^2}{(n-1)(n-2)}\right]^{1/3} - 2\left[\frac{(n-2)(n-3)}{(n-1)^2}\right]^{1/3} + 1}\right).$$

Theorem 3.2. *Let $f \in C_B[0,\infty)$. Then,*

$$|G_{n,r}(f;x) - f(x)| \le M\omega_2(f, \sqrt{\zeta_{n,r}}) + \omega\left(f, \left|\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - x\right|\right),$$

where M is a positive constant and

$$\zeta_{n,r} = \left[\frac{(2n-3)\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)^2(n-2)} - 4\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 2\right]x^2, \quad n \ne 1, 2.$$

Proof. Let us begin with the auxiliary operators $G^*_{n,r} : C_B[0,\infty) \to C_B[0,\infty)$ defined by

$$G^*_{n,r}(f,x) = G_{n,r}(f;x) - f\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x\right) + f(x). \tag{3.1}$$

Let $h \in C^2_B[0,\infty)$, $C^2_B[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$ and $x, t \in [0,\infty)$. In view of Taylor series expansion, we have

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-\theta)h''(\theta)d\theta.$$

Applying the operator $G^*_{n,r}$ on both sides of the above equation and using Lemma 2.2, we have

$$\begin{aligned}
|G^*_{n,r}(h,x) - h(x)| \leq & \left|G^*_{n,r}\left(\int_x^t (t-\theta)h''(\theta)d\theta, x\right)\right| \\
\leq & \left|G_{n,r}\left(\int_x^t (t-\theta)h''(\theta)d\theta, x\right)\right| \\
& + \left|\int_x^{\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x} \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - \theta\right)h''(\theta)d\theta\right| \\
\leq & \mu_2^{G_{n,r}}(x)\|h''\| \\
& + \left|\int_x^{\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x} \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - \theta\right)d\theta\right|\|h''\|,
\end{aligned}$$

$$|G^*_{n,r}(h,x) - h(x)| \leq \left[\mu_2^{G_{n,r}}(x) + \left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - x\right)^2\right]\|h''\| := \zeta_{n,r}\|h''\|. \tag{3.2}$$

With the help of Lemma 2.3 and (3.1), we get

$$\|G^*_{n,r}(f,x)\| \leq \|G_{n,r}(f,x)\| + 2\|f\| \leq 3\|f\|, \quad f \in C_B[0,\infty). \tag{3.3}$$

Using (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
|G_{n,r}(f,x) - f(x)| \leq & |G^*_{n,r}(f-h,x) - (f-h)(x)| + |G^*_{n,r}(h,x) - h(x)| \\
& + \left|f\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x\right) - f(x)\right| \\
\leq & 4\|f-g\| + \zeta_{n,r}\|h''\| + \left|f\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x\right) - f(x)\right| \\
\leq & M\{\|f-h\| + \zeta_{n,r}\|h''\|\} \\
& + \omega\left(f, \left|\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x - x\right|\right).
\end{aligned}$$

Taking the infimum in the last step of the above inequality and using Peetre's K-functional, which is defined as

$$K_2(f,\beta)=\inf_{h\in C_B^2[0,\infty)}\{\|f-h\|+\beta\|h''\|:h\in C_B^2[0,\infty)\},$$

we obtain

$$|G_{n,r}(f,x)-f(x)|\leq K_2(f,\zeta_{n,r})+\omega\left(f,\left|\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}x-x\right|\right).$$

Then by using Lorentz-DeVore property [8] $K_2(f,\beta)\leq M\omega_2(f,\sqrt{\beta})$, $\beta>0$, we can conclude our desired proof. □

Remark 3.1. For $r\in\mathbb{N}$ and sufficiently large n, from Theorem 3.2 one can easily verify that the operator $G_{n,r}(f,x)\to f(x)$ as $\zeta_{n,r}\to 0$.

We observe that at $r=1$, Theorem 3.2 reduces to the result as follows.

Corollary 3.1. *Let $f\in C_B[0,\infty)$ and $x\in[0,\infty)$. Then,*

$$|G_{n,r}(f,x)-f(x)|\leq M\omega\left(f,\frac{x}{\sqrt{n-2}}\right),$$

where M is a positive constant.

4. Weighted Modulus of Continuity

Let us define the weighted space of real-valued functions $f:[0,\infty)\to\mathbb{R}$ with the property $|f(x)|\leq M_f\phi(x)$ by $B_\phi[0,\infty)=\{f:[0,\infty)\to\mathbb{R}:|f(x)|\leq M_f\phi(x),x\in[0,\infty)\}$, where M_f is a positive constant depending on f but independent of x and a weight function $\phi(x)=1+x^2$, which is continuous on $\mathbb{R}$.

Let $C_\phi[0,\infty)=C[0,\infty)\cap B_\phi[0,\infty)$ and by $C_\phi^J[0,\infty)$, we denote the subspace of all continuous functions $f\in C_\phi[0,\infty)$ for which $\lim_{x\to\infty}\frac{|f(x)|}{1+x^2}=J_f$, exists and finite, where J_f is a constant depending on f. Then, for each $f\in C_\phi[0,\infty)$, the weighted modulus of continuity is defined as (see [1])

$$\Omega(f,\delta)=\sup_{|h|<\delta,x\in(0,\infty)}\frac{|f(x+h)-f(x)|}{1+x^2+h^2+h^2x^2}.$$

The next result is a quantitative Voronovskaja type asymptotic formula.

Theorem 4.1. *Let $f''\in C_\phi^J[0,\infty)$ and $x>0$. Then, we have*

$$\begin{aligned}&\Bigg|G_{n,r}(f,x)-f(x)-\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}-1\right)xf'(x)\\&-\left(\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)}-2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}+1\right)x^2f''(x)\Bigg|\\&\leq 16(1+x^2)\Omega\left(f'',\left(\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right)^{1/4}\right)\mu_2^{G_{n,r}}(x),\end{aligned}$$

where $\mu_2^{G_{n,r}}(x)$ *and* $\mu_6^{G_{n,r}}(x)$ *are defined in Lemma 2.2.*

Proof. Let $f'' \in C_\phi^J[0,\infty)$ and $x \in (0,\infty)$. Then, by Taylor's expansion, we have

$$f(t)-f(x)=(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+h(x,t)(t-x)^2,$$

where $h(x,t) := \frac{f''(\xi)-f''(x)}{2}$, is a continuous function and $\xi \in (x,t)$. Now, applying $G_{n,r}$ on both sides of the above equation, we have

$$\begin{aligned}G_{n,r}(f(t)-f(x),x) =& G_{n,r}\Big((t-x)f'(x),x\Big)+G_{n,r}\left(\frac{(t-x)^2}{2!}f''(x),x\right)\\ &+G_{n,r}\Big(h(x,t)(t-x)^2,x\Big).\end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned}&\Bigg|G_{n,r}(f,x)-f(x)-\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}-1\right)xf'(x)\\ &-\left(\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)}-2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}+1\right)x^2f''(x)\Bigg|\\ \leq& G_{n,r}(|h(x,t)|(t-x)^2,x).\end{aligned}$$

In view of the inequality $|\xi-x|\leq|x-t|$ and by simple computation, we can write

$$|h(t,x)|\leq 8(1+x^2)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(f'',\delta).$$

In view of Lemma 2.2, we obtain

$$\begin{aligned}G_{n,r}(|h(x,t)|(t-x)^2,x)&\leq 8(1+x^2)\Omega(f'',\delta)\left\{\mu_2^{G_{n,r}}(x)+\frac{\mu_6^{G_{n,r}}(x)}{\delta^4}\right\}\\ &=8(1+x^2)\Omega(f'',\delta)\left\{1+\frac{1}{\delta^4}\cdot\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right\}\mu_2^{G_{n,r}}(x).\end{aligned}$$

Choosing $\delta=\left(\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right)^{1/4}$, we have

$$\begin{aligned}&\Bigg|G_{n,r}(f,x)-f(x)-\left(\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}-1\right)xf'(x)\\ &-\left(\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)}-2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1}+1\right)x^2f''(x)\Bigg|\\ \leq&16(1+x^2)\Omega\left(f'',\left(\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)}\right)^{1/4}\right)\mu_2^{G_{n,r}}(x),\end{aligned}$$

as desired. □

Remark 4.1. For $r \in \mathbb{N}$ and fixed $x \in [0, \infty)$, we observe that

$$\frac{\mu_6^{G_{n,r}}(x)}{\mu_2^{G_{n,r}}(x)} \to 0, \quad \text{as } n \to \infty,$$

which guarantees the convergence of Theorem 4.1.

Let $[0, \alpha]$, $\alpha \geq 0$, be the closed interval. The standard modulus of continuity is denoted by $\omega_\alpha(f, \delta)$ and defined as

$$\omega_\alpha(f, \delta) = \sup_{|t-x| \leq \delta, t, x \in [0,\alpha]} |f(t) - f(x)|.$$

It is also clear that, for any $f \in C_B[0, \infty)$, the modulus of continuity $\omega_\alpha(f, \delta) \to 0$ as $\delta \to 0$.

Theorem 4.2. *Let $f \in C_B[0, \alpha]$ and $\alpha > 0$. Then, the following inequality satisfies*

$$|G^*_{n,r}(f, x) - f(x)| \leq 4M_f(1 + x^2)\delta_n^2(x) + 2\omega_{\alpha+1}(f; \delta_n(x)),$$

where

$$\delta_n^2(x) = \left[\frac{\{(-1)^r(-n+1)_r\}^{2/r}}{(n-1)(n-2)} - 2\frac{\{(-1)^r(-n+1)_r\}^{1/r}}{n-1} + 1\right] x^2$$

and M_f is a constant depending on f.

Proof. From [10], for all $0 \leq x \leq \alpha$ and $t > \alpha + 1$, we have

$$|f(t) - f(x)| \leq 4M_f(1 + x^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{\alpha+1}(f; \delta_n(x)), \quad \delta > 0.$$

Applying the operator $G^*_{n,r}$ and Cauchy-Schwartz inequality on both sides of the above equation, we have

$$\begin{aligned}|G^*_{n,r}(f, x) - f(x)| \leq & 4M_f(1 + x^2)G^*_{n,r}((t - x)^2, x) \\ & + \left(1 + \frac{\sqrt{G^*_{n,r}((t - x)^2, x)}}{\delta}\right)\omega_{\alpha+1}(f; \delta_n(x)), \\ = & 4M_f(1 + x^2)\delta_n^2(x) + \left(1 + \frac{\delta_n(x)}{\delta}\right)\omega_{\alpha+1}(f; \delta_n(x)).\end{aligned}$$

By choosing $\delta = \delta_n(x)$, we get the desired result. □

Remark 4.2. For $r = 1$ and $x \in [0, \alpha]$, from Theorem 4.2, we have

$$|G^*_{n,1}(f, x) - f(x)| \leq 4M_f(1 + x^2)\frac{x^2}{n - 2} + 2\omega_{\alpha+1}\left(f; \frac{x}{\sqrt{n - 2}}\right), \quad n > 2.$$

Also, for $x \in [0, \alpha]$ and $r = 2$, Theorem 4.2 reduced to Theorem 4 of [4] as

$$|G^*_{n,2}(f, x) - f(x)| \leq 4M_f(1 + x^2)x^2\left[2 - 2\sqrt{\frac{n - 2}{n - 1}}\right]$$

$$+ 2\omega_{\alpha+1}\left(f; \sqrt{\left[2 - 2\sqrt{\frac{n-2}{n-1}}\right]x^2}\right), \quad n > 1.$$

It is also observed that $G^*_{n,1}(f,x)$ and $G^*_{n,2}(f,x)$ converges to $f(x)$, as $n \to \infty$.

5. Point-wise Estimates

This section is dedicated to some point-wise estimates of the rate of convergence for the modified Gamma operator $G^*_{n,r}$ defined in (1.4). First, we define the relationship between the local smoothness of f and local approximation.

Let $\eta \in (0,1]$ and $S \subset [0,\infty)$. A function $f \in C_B[0,\infty)$ is in $L_M(\eta)$ on S, if it satisfies the following condition

$$|f(t) - f(x)| \leq M|t-x|^\eta, \quad t \in [0,\infty), x \in S,$$

where M is a constant depending on f and η.

Theorem 5.1. *Let $f \in C_B[0,\infty) \cap L_M(\eta)$. Then, we have*

$$|G^*_{n,r}(f,x) - f(x)| \leq M\left(\left\{\mu_2^{G_{n,r}}(x)\right\}^{\frac{\eta}{2}} + 2d^\eta(x,S)\right),$$

where $\mu_2^{G_{n,r}}(x)$ is defined in Lemma 2.2 and $d(x,S)$ is a distance function from x to S and defined as

$$d(x,S) = \inf\{|t-x| : t \in S\}.$$

Proof. Let $\bar{S}$ be the closure of S in $[0,\infty)$. Then there exists at least one point s_0 in $\bar{S}$ such that $d(x,S) = |x - s_0|$. By the monotonicity property of $G^*_{n,r}$, we get

$$\begin{aligned}|G^*_{n,r}(f,x) - f(x)| &\leq G^*_{n,r}(|f(t) - f(s_0)|, x) + G^*_{n,r}(|f(x) - f(s_0)|, x)\\ &\leq M\left(G^*_{n,r}\left(|t-s_0|^\eta, x\right) + |x-s_0|^\eta\right)\\ &\leq M\left(G^*_{n,r}\left(|t-x|^\eta, x\right) + 2|x-s_0|^\eta\right).\end{aligned}$$

Thus, applying Hölder's inequality with $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$, we have

$$|G^*_{n,r}(f,x) - f(x)| \leq M\left(\left\{G^*_{n,r}((t-x)^2, x)\right\}^{\frac{\eta}{2}} + 2|x-s_0|^\eta\right).$$

Finally, using Lemma 2.2, we obtain the desired result. □

Next, we discuss the local direct estimation for the operator (1.4), with the help of Lipschitz-type maximal function of order η defined by B. Lenze [16] as

$$\widetilde{\omega}_\eta(f,x) = \sup_{t\neq x, t\in[0,\infty)} \frac{|f(t) - f(x)|}{|t-x|^\eta}, \quad x \in [0,\infty) \text{ and } \eta \in (0,1]. \tag{5.1}$$

Theorem 5.2. *Let $f \in C_B[0,\infty)$ and $0 < \eta \leq 1$. Then, for all $x \in [0,\infty)$, we have*

$$|G^*_{n,r}(f,x) - f(x)| \leq \widetilde{\omega}_\eta(f,x)\left\{\mu_2^{G_{n,r}}(x)\right\}^{\frac{\eta}{2}},$$

where $\mu_2^{G_{n,r}}(x)$ is defined in Lemma 2.2.

Proof. In view of (5.1), we have

$$|G^*_{n,r}(f,x) - f(x)| \leq \tilde{\omega}_\eta(f,x) G^*_{n,r}\Big(|t-x|^\eta, x\Big).$$

Using Hölder's inequality in the above equation, we obtain

$$|G^*_{n,r}(f,x) - f(x)| \leq \tilde{\omega}_\eta(f,x) G^*_{n,r}\left(|t-x|^2, x\right)^{\frac{\eta}{2}} \leq \tilde{\omega}_\eta(f,x)\left\{\mu_2^{G_{n,r}}(x)\right\}^{\frac{\eta}{2}}.$$

Thus, the theorem is completed. □

Let us consider the Lipschitz-type space with two parameters defined in [18], for any $u, v > 0$, such that

$$L_M^{u,v}(\eta) = \left\{ f \in C[0,\infty) : |f(t) - f(x)| \leq \frac{M|t-x|^\eta}{(ux^2+vx+t)^{\frac{\eta}{2}}},\ x, t \in (0,\infty) \right\},$$

where M is a positive constant and $0 < \eta \leq 1$.

Theorem 5.3. *For $f \in L_M^{u,v}(\eta)$. Then, for all $x > 0$, we have*

$$|G^*_{n,r}(f,x) - f(x)| \leq M\left(\frac{\mu_2^{G_{n,r}}(x)}{ux^2+vx}\right)^{\frac{\eta}{2}}.$$

Proof. The proof of this theorem is divided into two parts. In the first part, we prove our theorem for $\eta = 1$. Then, for $f \in L_M^{u,v}(1)$ and $x \in (0,\infty)$, we have

$$\begin{aligned}|G^*_{n,r}(f,x) - f(x)| &\leq G^*_{n,r}(|f(t)-f(x)|, x) \leq M G^*_{n,r}\left(\frac{|t-x|}{(ux^2+vx+t)^{\frac{1}{2}}}, x\right)\\ &\leq \frac{M}{(ux^2+vx)^{\frac{1}{2}}} G^*_{n,r}\left(|t-x|, x\right).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we obtain

$$|G^*_{n,r}(f,x) - f(x)| \leq \frac{M}{(ux^2+vx)^{\frac{1}{2}}}\left(G^*_{n,r}(t-x)^2, x\right)^{\frac{1}{2}} \leq M\left(\frac{\mu_2^{G_{n,r}}(x)}{ux^2+vx}\right)^{\frac{1}{2}}.$$

Thus the result holds for $\eta = 1$.

Now, we prove the result for $0 < \eta < 1$. Then, for $x \in (0,\infty)$ and $f \in L_M^{u,v}(\eta)$, we obtain

$$\begin{aligned}|G^*_{n,r}(f,x) - f(x)| &\leq G^*_{n,r}(|f(t)-f(x)|a, x) \leq M G^*_{n,r}\left(\frac{|t-x|^\eta}{(ux^2+vx+t)^{\frac{\eta}{2}}}, x\right)\\ &\leq \frac{M}{(ux^2+vx)^{\frac{\eta}{2}}} G^*_{n,r}\left(|t-x|^\eta, x\right).\end{aligned}$$

Applying Hölder's inequality by taking $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$, we obtain

$$|G^*_{n,r}(f,x) - f(x)| \leq \frac{M}{(ux^2+vx)^{\frac{\eta}{2}}} G^*_{n,r}\Big((t-x)^2, x\Big)^{\eta/2}.$$

Finally, using Lemma 2.2, we get the desired result. □

Remark 5.1. At particular $r = 2$, Theorem 5.2 and Theorem 5.3 are reduced to the Theorem 6 and Theorem 7 respectively of [4].

6. Numerical Experiments

In this section, we provide numerical examples to verify the approximation properties of the modified Gamma operator (1.4) with different preservation of test functions. The implementation is carried out in Mathematica.

Example 1. Consider the test function $f(x) = x^8 + 8x + 2$ on the interval $[1, 2]$. Figure 1 indicates that the operator $G_{n,r}(f, x)$ approaches the function $f(x)$ faster as the value of r increases. Moreover, Figure 2 suggests that the error function $E_{n,r}(f, x) = |G_{n,r}(f, x) - f(x)|$ tends to the x-axis, which means that the error decreases as the value of r increases.

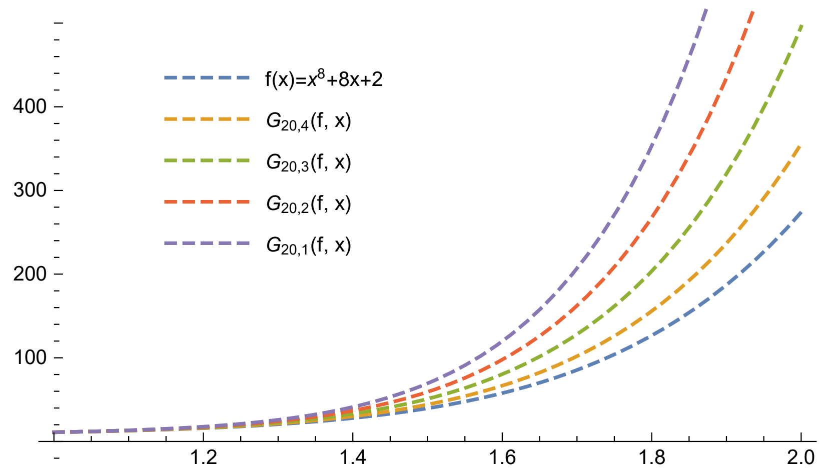

Figure 1. Convergence of $G_{n,r}(f; x)$ to $f(x) = x^8 + 8x + 2$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[1, 2]$.

Example 2. Consider the test function $f(x) = xe^{-x}$ on the interval $[0, 4]$. Figure 3 indicates that the operator $G_{n,r}(f, x)$ approaches the function $f(x)$ faster as the value of r increases. Moreover, Figure 4 suggests that the error function $E_{n,r}(f, x) = |G_{n,r}(f, x) - f(x)|$ tends to the x-axis, which means that the error decreases as the value of r increases.

Therefore, based on the aforementioned observations and figures, we can conclude that the approximation gets better as the value of r increases. Further, we can also notice that the operator (1.4) produces better convergence over the operator (1.2) for $r \geq 2$.

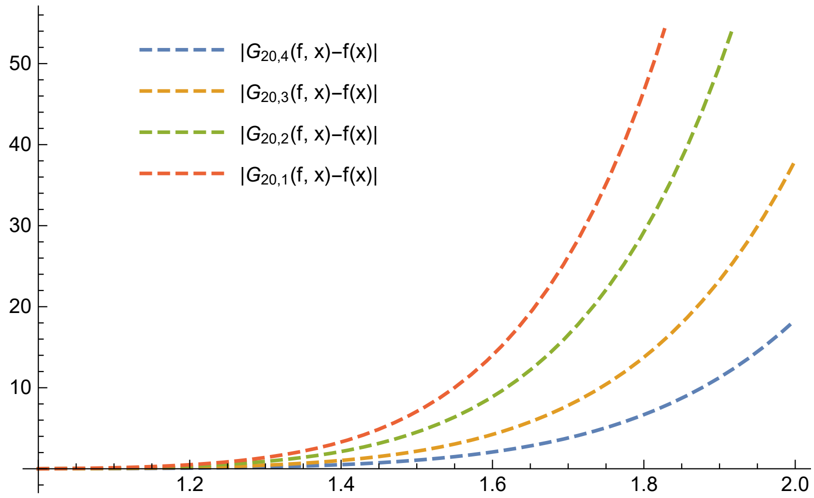

FIGURE 2. Error estimation of the operator $G_{n,r}(f;x)$ to the function $f(x) = x^8 + 8x + 2$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[1, 2]$.

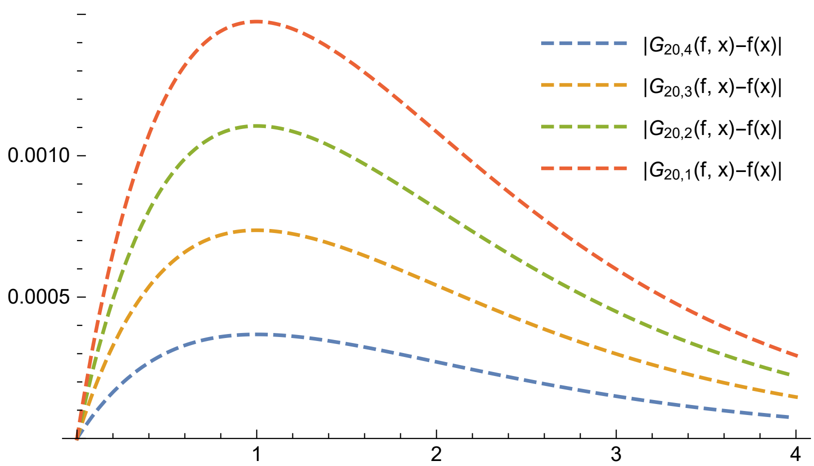

FIGURE 3. Convergence of $G_{n,r}(f;x)$ to $f(x) = xe^{-x}$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[0, 4]$.

7. CONCLUSION

The aim of the present study was to construct a modified sequence of Gamma-type operators, which preserves the test function $e_r(t) = t^r$, $r \in \mathbb{N}$. These newly defined Gamma-type operators play a crucial role in encompassing existing Gamma-type operators and facilitating the definition of new ones that can yield improved approximation results under suitable conditions. To establish that these newly defined operators constitute an approximation process, we also present some of their fundamental properties. Finally, we offer numerical experiments to validate our theoretical findings.

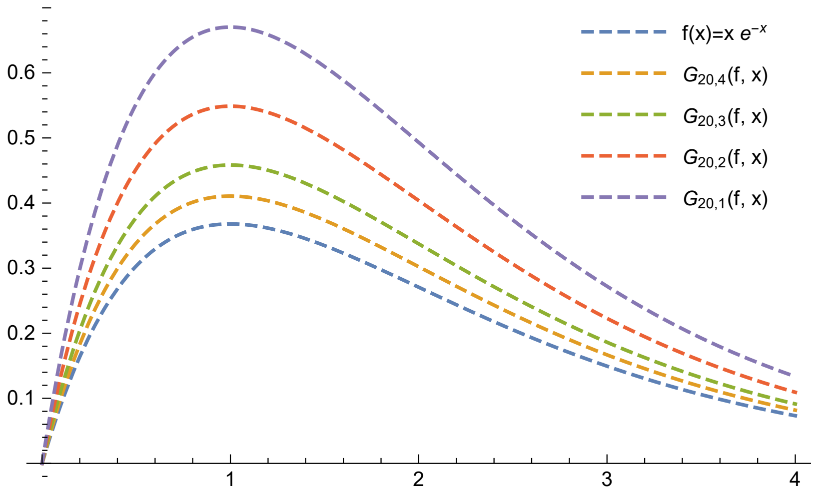

FIGURE 4. Error estimation of the operator $G_{n,r}(f;x)$ to the function $f(x) = xe^{-x}$ for $n = 20$ and $r = 1, 2, 3, 4$ on the interval $[0, 4]$.

Acknowledgements. I sincerely thank the reviewers and editors for their invaluable feedback and guidance, which greatly enhanced the quality and clarity of this research paper. I am also extremely grateful to the Ministry of Human Resource Development of India (Grant code: MHR-01-23-200-428) for their support in carrying out this research work.

REFERENCES

[1] T. Acar, A. Aral and I. Rasa, *The new forms of Voronovskaya's theorem in weighted spaces*, Positivity **2** (2016), 25–40. https://doi.org/10.1007/s11117-015-0338-4
[2] T. Acar, M. Mursaleen and S. N. Deveci, *Gamma operators reproducing exponential functions*, Adv. Difference Equ. **2020** (2020), 1–13. https://doi.org/10.1186/s13662-020-02880-x
[3] V. Artee, *Approximation by modified Gamma type operators*, Int. J. Adv. Appl. Math. Mech. **5** (2018), 12–19.
[4] Ö. Betus and F. Usta, *Approximation of functions by a new type of Gamma operators*, Numer. Methods Partial Differ. Equ. **39** (2023), 3520–3531. https://doi.org/10.1002/num.22660
[5] Q. B. Cai and X. M. Zeng, *On the convergence of a kind of q-Gamma operators*, J. Inequalities Appl. **2013** (2013), 1–9. https://doi.org/10.1186/1029-242X-2013-105
[6] W. T. Cheng, W. H. Zhang and J. Zhang, *Approximation properties of modified q-Gamma operators preserving linear functions*, Filomat **34** (2020), 1601–1609. https://doi.org/10.2298/FIL2005601C
[7] S. N. Deveci, T. Acar and O. Alagoz, *Approximation by Gamma type operators*, Math. Methods Appl. Sci. **43** (2020), 2772–2782. https://doi.org/10.1002/mma.6083
[8] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer Science & Business Media, 1993.
[9] V. Gupta and D. Agrawal, *Convergence by modified Post-Widder operators*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **113** (2019), 1475–1486. https://doi.org/10.1007/s13398-018-0562-4

[10] E. Ibikli and E. A. Gadjieva, *The order of approximation of some unbounded function by the sequences of positive linear operators*, Turkish J. Math. **19** (1995), 331–337.
[11] A. İzgi, *Voronovskaya type asymptotic approximation by modified Gamma operators*, Appl. Math. Comput. **217** (2011), 8061–8067. https://doi.org/10.1016/j.amc.2011.03.005
[12] H. Karsli, P. N. Agrawal and M. Goyal, *General Gamma type operators based on q-integers*, Math. Methods Appl. Sci. **251** (2015), 564–575. https://doi.org/10.1016/j.amc.2014.11.085
[13] J. P. King, *Positive linear operators which preserve* x^2, Acta Math. Hungar. **99** (2003), 203–208. https://doi.org/10.1023/a:1024571126455
[14] A. Kumar, *General Gamma type operators in* L_p *spaces*, Palest. J. Math. **7** (2018), 73–79.
[15] A. Kumar, A. Verma, L. Rathour, L. N. Mishra and V. N. Mishra, *Convergence analysis of modified Szász operators associated with Hermite polynomials*, Rend. Circ. Mat. Palermo (2) (2023), 1–15. https://doi.org/10.1007/s12215-023-00931-2
[16] B. Lenze, *On Lipschitz-type maximal functions and their smoothness spaces*, Indag. Math. (Proceedings), Vol. 91, Elsevier, Elsevier, 1988, 53–63. https://doi.org/10.1016/1385-7258(88)90007-8
[17] A. Lupaş and M. Müller, *Approximationseigenschaften der gammaoperatoren*, Math. Z. **98** (1967), 208–226. https://doi.org/10.1007/BF01112415
[18] M. A. Özarslan and H. Aktuğlu, *Local approximation properties for certain King type operators*, Filomat **27** (2013), 173–181. https://doi.org/10.2298/FIL1301173O
[19] R. Özçelik, E. E. Kara, F. Usta and K. J. Ansari, *Approximation properties of a new family of gamma operators and their applications*, Adv. Differ. Equ. **2021** (2021), 1–13. https://doi.org/10.1186/s13662-021-03666-5
[20] L. Rempulska and M. Skorupka, *Approximation properties of modified Gamma operators*, Adv. Difference Equ. **18** (2007), 653–662. https://doi.org/10.1080/10652460701510527
[21] J. K. Singh, P. N. Agrawal and A. Kajla, *Approximation by modified q-Gamma type operators via a-statistical convergence and power series method*, Linear and Multilinear Algebra **70** (2022), 6548–6567. https://doi.org/10.1080/03081087.2021.1960260
[22] F. Usta and Ö. Betus, *A new modification of gamma operators with a better error estimation*, Linear and Multilinear Algebra **70** (2022), 2198–2209. https://doi.org/10.1080/03081087.2020.1791033
[23] X. M. Zeng, *Approximation properties of Gamma operators*, J. Math. Anal. Appl. **311** (2005), 389–401. https://doi.org/10.1016/j.jmaa.2005.02.051
[24] C. Zhao, W. T. Cheng and X. M. Zeng, *Some approximation properties of a kind of q-Gamma-Stancu operators*, J. Inequalities Appl. **2014** (2014), 1–13. https://doi.org/10.1186/1029-242X-2014-94

Department of Mathematics,
Indian Institute of Technology Roorkee,
Roorkee-247667, Uttarakhand, India
Email address: akumar1@ma.iitr.ac.in
ORCID iD: https://orcid.org/0000-0002-4487-9200