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ON THE EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A STRONGLY DAMPED NONLINEAR COUPLED PETROVSKY-WAVE SYSTEM

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Abstract. In this paper, we consider the initial-boundary value problem for a class of nonlinear coupled wave equation and Petrovesky system in a bounded domain. The strong damping is nonlinear. First, we prove the existence of global weak solutions by using the energy method combined with Faedo-Galarkin method and the multiplier method.

In addition, under suitable conditions on functions $g_i(\cdot)$, $i = 1, 2$ and $a(\cdot)$, we obtain both exponential and polynomial decay estimates. The method of proofs is direct and based on the energy method combined with the multipliers technique, on some integral inequalities due to Haraux and Komornik.

1. Introduction

The study of nonlinear wave phenomena was performed by certain eminent scientists. The theory of nonlinear waves, on the other hand, emerged as a coherent science in the late 1960s and early 1970s, which were the years of its rapid growth. While study in this area was undertaken only recently, the theory of nonlinear damped waves is still an emerging theme. In this paper, we study the existence and decay properties of solutions for the initial boundary value problem of the Petrovsky-wave system of

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the type

$$
\begin{cases}
y_1'' + \Delta^2 y_1 - a(x)\Delta y_2 - g_1(\Delta y_1') = 0, & x \in \Omega, t \geq 0,\\
y_2'' - \Delta y_2 - a(x)\Delta y_1 - g_2(\Delta y_2') = 0, & x \in \Omega, t \geq 0,\\
\Delta y_1 = y_1 = y_2 = 0, & x \in \Gamma, t \geq 0,\\
y_i(x,0) = y_i^0(x), \quad y_i'(x,0) = y_i^1(x), & x \in \Omega, i = 1,2,
\end{cases} \tag{1.1}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary Γ and $g_i : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $g_i(0) = 0$, $i = 1, 2$.

When $a(x) = 0$, the Petrovsky equation has been investigated in [7] by Komornik. The author has used the semigroup approach to present the existence and uniqueness of a global solution y_1 for (1.1). Then, using a multiplier technique, he directly proved exponential and polynomial decay estimates for the associated energy.

Bahlil et al. [4], studied the system:

$$
\begin{cases}
y_1'' + a(x)y_2 + \Delta^2 y_1 - g_1(y_1'(x,t)) = f_1(y_1, y_2), & \text{in } \Omega \times \mathbb{R}^+,\\
y_2'' + a(x)y_1 - \Delta y_2 - g_2(y_2'(x,t)) = f_2(y_1, y_2), & \text{in } \Omega \times \mathbb{R}^+,\\
\partial_\nu y_1 = y_1 = v = y_2 = 0, & \text{on } \Gamma \times \mathbb{R}^+,
\end{cases} \tag{1.2}
$$

under suitable assumptions on the weight of the damping, they proved the global existence of solutions by use of the potential well method due to Payne and Sattinger [13] and Sattinger [14] combined with the Faedo-Galerkin method.

Also they proved general stability estimates using some properties of convex functions and the multiplier method.

In [5] Guesmia studied problem (1.2) with $f_i(y_1, y_2) = 0$. He proved the existence of a global weak solution and uniform decay of solutions.

Motivated by previous works, it is interesting to investigate the global existence and decay of solutions to problem (1.1). Firstly, we show that, under suitable conditions on the functions g_i and a, the solutions are global in time. After that, we establish the rate of decay of solutions by the multiplier method. Precisely, we show that the decay rate of energy function is exponential or polynomial.

This article is organized as follows: in the next section, we give some preliminaries. In Section 3, we study the existence of global solutions of the problem (1.1). Then in Section 4, we are devoted to the proof of decay estimate.

2. Preliminaries and Main Results

In this section, we present some material for the proof of our result.

We first introduce the following spaces: $H = L^2(\Omega) \times L^2(\Omega)$, $W = H_0^1(\Omega) \times H_0^1(\Omega)$, $H_\Delta^3(\Omega) = \{v \in H^3(\Omega) : \ v = \Delta v = 0 \text{ on } \Gamma\}$ and $\|u\|^2_{H_\Delta^3(\Omega)} = \int_\Omega |\nabla \Delta v|^2 \, dx$, and

$$
V = \left(H_\Delta^3(\Omega) \cap H^2(\Omega)\right) \times H^2(\Omega), \quad \tilde{V} = \left(H_\Delta^3(\Omega) \cap H^4(\Omega)\right) \times \left(H_\Delta^3(\Omega) \cap H^2(\Omega)\right).
$$

Let H', V', $\tilde{V}'$, W' the dual spaces of H, $V, \tilde{V}, W$, respectively. We have

$$
\tilde{V} \subset V \subset W \subset H = H' \subset W' \subset \tilde{V}' \subset V.
$$

For the relaxation function g and a we assume the following.

(**H0**) Let $a : \Omega \to \mathbb{R}$ be non-increasing differentiable function bounded such that

$$a(x) \in W^{1,\infty}(\Omega), \quad \|a\|_{L^\infty(\Omega)} = \min\left\{\frac{1}{c'}, 1\right\}, \tag{2.1}$$

where $c' > 0$ is the constant $\|\nabla\Delta v\| \le c'\|\Delta v\|$.

(**H1**) $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are non-increasing differentiable functions such that g_i is a C^1 and globally lipschitz with $g_i(0) = 0$ and there exists $p \ge 1$, c_j, $j = 1, \ldots, 4$, τ_0, τ_1 are strictly positive constants for all $s \in \mathbb{R}$ satisfying

$$c_1|s|^p \le g_i(s) \le c_2|s|^{\frac{1}{p}}, \quad \text{if } |s| \le 1, \tag{2.2}$$

$$c_3|s| \le g_i(s) \le c_4|s|, \quad \text{if } = |s| > 1, \tag{2.3}$$

$$\text{exists } \tau_0, \quad \tau_1 > 0, \quad \tau_0 \le g_i'(s) \le \tau_1, \quad \text{for all } s \in \mathbb{R}. \tag{2.4}$$

Now inspired by Komornik [7], we define the energy associated with the solution of system (1.1).

Lemma 2.1. *The energy associated with the solution of the problem* (1.1) *by the following formula*

$$E(t) = \frac{1}{2}\int_\Omega \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 + |\nabla\Delta y_1|^2 + |\Delta y_2|^2\right) dx + \int_\Omega a(x)\Delta y_1 \Delta y_2 dx \tag{2.5}$$

is a nonnegative function and satisfies $E'(t) \le 0$.

Proof. Multiplying the first equation in (1.1) by $-\Delta y_1'$ and the second equation by $-\Delta y_2'$, integrating over Ω using integration by part and Green's formula, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 + |\nabla\Delta y_1|^2 + |\Delta y_2|^2\right) dx + 2\int_\Omega a(x)\Delta y_1\Delta y_2\, dx\right] \\
&= -\int_\Omega \Delta y_1' g_1(\Delta y_1') + \Delta y_2' g_2(\Delta y_2')\, dx.
\end{aligned}$$

Using Hölder's inequality, Sobolev embedding and condition (2.1), we get

$$\begin{aligned}
\int_\Omega a(x)\Delta y_1\Delta y_2 dx &\ge -\frac{1}{2}\|a\|_{L^\infty(\Omega)}\frac{\sqrt{c'}}{\sqrt{c'}}\int_\Omega |\Delta y_1 \Delta y_2|\, dx \\
&\ge -\frac{1}{2}\|a\|_{L^\infty(\Omega)}\int_\Omega \left(\frac{1}{c'}|\Delta y_1|^2 + c'|\Delta y_2|^2\right) dx \\
&\ge -\frac{1}{2}\|a\|_{L^\infty(\Omega)}\int_\Omega \left(\frac{c'^2}{c'}|\nabla\Delta y_1|^2 + c'|\Delta y_2|^2\right) dx \\
&\ge -\frac{c'}{2}\|a\|_{L^\infty(\Omega)}\int_\Omega \left(|\nabla\Delta y_1|^2 + |\Delta y_2|^2\right) dx.
\end{aligned}$$

Then

$$E(t) \ge \frac{1}{2}\int_\Omega \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 + (1 - c'\|a\|_{L^\infty(\Omega)})(|\nabla\Delta y_1|^2 + |\Delta y_2|^2)\right) dx \ge 0.$$

Now, E is a nonnegative function

$$E'(t) = -\int_{\Omega} (\Delta y_1' g_1(\Delta y_1') + \Delta y_2' g_2(\Delta y_2'))\, dx. \tag{2.6}$$
□

3. Global Existence

In this section, we use the Faedo-Galerkin approximation to construct an approximate solutions of (1.1). We are now in the position to state our results.

Theorem 3.1. *Let* $(y_1^0, y_2^0) \in \widetilde{V}$ *and* $(y_1^1, y_2^1) \in V$*, arbitrarily. Assume that* (2.1) *and* (2.2)–(2.4) *hold. Then system* (1.1) *has a unique weak solution satisfying*

$$(y_1, y_2) \in L^\infty(\mathbb{R}_+, \widetilde{V}), \quad (y_1', y_2') \in L^\infty(\mathbb{R}_+, V)$$

and

$$(y_1'', y_2'') \in L^\infty(\mathbb{R}_+, W).$$

Proof. We use the Faedo-Galerkin method to prove the existence of global solutions. Let $T > 0$ be fixed and denoted by V^k the space generated by $\{w_i^1, w_i^2, \dots, w_i^k\}$, where the set $\{w_i^k,\ k \in \mathbb{N}\}$ is a basis of $\widetilde{V}$.

We construct approximate solution y_i^k, $k = 1, 2, 3, \dots$, in the form

$$y_i^k(x,t) = \sum_{j=1}^{k} c^{jk}(t) w_i^j(x),$$

where c^{jk}, $j = 1, 2, \dots, k$, are determined by the following ordinary differential equations

$$\begin{cases} (\ddot{y}_1^k + \Delta^2 y_1^k - a(x)\Delta y_2^k - g_1(\Delta \dot{y}_1^k), w_1^j) = 0, & \text{for all } w_1^j \in V^k, \\ (\ddot{y}_2^k - \Delta u_2^k - a(x)\Delta y_1^k - g_2(\Delta \dot{y}_2^k), w_2^j) = 0, & \text{for all } w_2^j \in V^k, \\ y_i^k(0) = y_i^{0k}, \quad \dot{y}_i^k(0) = y_i^{1k}, & x \in \Omega, i = 1, 2, \end{cases} \tag{3.1}$$

with initial conditions

$$y_1^k(0) = y_1^{0k} = \sum_{j=1}^{k} \langle y_1^0, w_1^j \rangle w_1^j \to y_1^0, \text{ in } H^4(\Omega) \cap H^3_\Delta(\Omega) \text{ as } k \to +\infty, \tag{3.2}$$

$$y_2^k(0) = y_2^{0k} = \sum_{j=1}^{k} \langle y_2^0, w_2^j \rangle w_2^j \to y_2^0, \quad \text{in } H^3_\Delta(\Omega) \cap H^2(\Omega) \text{ as } k \to +\infty, \tag{3.3}$$

$$\dot{y}_1^k(0) = y_1^{1k} = \sum_{j=1}^{k} \langle y_1^1, w_1^j \rangle w_1^j \to y_1^1, \quad \text{in } H^3_\Delta(\Omega) \cap H^2(\Omega) \text{ as } k \to +\infty, \tag{3.4}$$

$$\dot{y}_2^k(0) = y_2^{1k} = \sum_{j=1}^{k} \langle y_2^1, w_2^j \rangle w_2^j \to y_2^1, \quad \text{in } H^2(\Omega) \text{ as } k \to +\infty, \tag{3.5}$$

and

(3.6)
$$-\Delta^2 y_1^{0k} + a(x)\Delta y_2^{0k} + g_1(\Delta y_1^{1k}) \to -\Delta^2 y_1^0 + a(x)\Delta y_2^0 + g_1(\Delta y_1^1), \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty,$$
(3.7)
$$\Delta y_2^{0k} + a(x)\Delta y_1^{0k} + g_2(\Delta y_2^{1k}) \to \Delta y_2^0 + a(x)\Delta y_1^0 + g_2(\Delta y_2^1), \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty.$$

By using some a priori estimates to show that $t_k = \infty$. Then, we show that the sequence of solutions to (3.1) converges to a solution of (1.1) with the claimed smoothness.

The first estimate. Taking $w_i^j = -2\Delta \dot{y}_i^k$ in (3.1), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(|\nabla \dot{y}_1^k|^2 + |\nabla \dot{y}_2^k|^2 + |\nabla\Delta y_1^k|^2 + |\Delta y_2^k|^2\right) dx + 2a(x)\Delta y_1^k \Delta y_2^k\, dx\\&+ 2\int_\Omega \Delta\dot{y}_1^k g_1(\Delta\dot{y}_1^k)\, dx + 2\int_\Omega \Delta\dot{y}_2^k g_2(\Delta\dot{y}_2^k)\, dx = 0.\end{aligned} \tag{3.8}$$

Integrating it over $(0,t)$, we obtain

$$\begin{aligned}&\int_\Omega \left(|\nabla \dot{y}_1^k(t)|^2 + |\nabla \dot{y}_2^k(t)|^2\right) dx + \left(1 - c'\|a\|_{L^\infty(\Omega)}\right)\int_\Omega \left(|\nabla\Delta y_1^k(t)|^2 + |\Delta y_2^k(t)|^2\right) dx\\&+ 2\int_0^t\int_\Omega \Delta\dot{y}_1^k(s) g_1(\Delta\dot{y}_1^k(s))\, dx\, ds + 2\int_0^t\int_\Omega \Delta\dot{y}_2^k(s) g_2(\Delta\dot{y}_2^k(s))\, dx\, ds\\&\le A^k(0) \le C_1,\end{aligned} \tag{3.9}$$

where

$$A^k(0) = \int_\Omega \left(|\nabla \dot{y}_1^k(t)|^2 + |\nabla \dot{y}_2^k(t)|^2\right) dx + (1 + c'\|a\|_{L^\infty(\Omega)})\int_\Omega \left(|\nabla\Delta y_1^k(t)|^2 + |\Delta y_2^k(t)|^2\right) dx,$$

for some C_1 independent of k. These estimates imply that the solutions y_i^k exist globally in $]0,+\infty[$. Estimate (3.9) yields

$$y_1^k \text{ is bounded in } L^\infty(0,T; H^3_\Delta(\Omega)), \tag{3.10}$$
$$y_2^k \text{ is bounded in } L^\infty(0,T; H^2(\Omega)), \tag{3.11}$$
$$\dot{y}_1^k \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)), \tag{3.12}$$
$$\dot{y}_2^k \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)), \tag{3.13}$$
$$\Delta\dot{y}_i^k g_i(\Delta\dot{y}_i^k) \text{ is bounded in } L^1(\mathcal{A}), \tag{3.14}$$

where $\mathcal{A} = \Omega \times (0,T)$.

The second estimate. Taking $w_i^j = \Delta^2 \dot{y}_i^k$ in (3.1), implies

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(|\Delta\dot{y}_1^k|^2 + |\Delta\dot{y}_2^k|^2 + |\Delta^2 y_1^k|^2 + |\nabla\Delta y_2^k|^2 + 2a(x)\nabla\Delta y_1^k \nabla\Delta y_2^k\right) dx\\&+ 2\int_\Omega \nabla a(x)\Delta y_2^k \nabla\Delta\dot{y}_1^k\, dx + 2\int_\Omega \nabla a(x)\Delta y_1^k \nabla\Delta\dot{y}_2^k\, dx\\&+ 2\int_\Omega |\nabla\Delta\dot{y}_1^k|^2 g_1'(\Delta\dot{y}_1^k)\, dx + 2\int_\Omega |\nabla\Delta\dot{y}_2^k|^2 g_2'(\Delta\dot{y}_2^k)\, dx = 0\end{aligned} \quad . \tag{3.15}$$

By Using Hölder's inequality and Sobolev embedding, (3.10) and condition (2.2), we have

$$\begin{aligned}
\left|2\int_\Omega a(x)\nabla\Delta y_1^k\nabla\Delta y_2^k\,dx\right| &\le 2\|a\|\int_\Omega |\nabla\Delta y_1^k||\nabla\Delta y_2^k|\,dx\\
&\le 2\|a\|^2\int_\Omega |\nabla\Delta y_1^k|^2\,dx+\frac12\int_\Omega|\nabla\Delta y_2^k|^2\,dx\\
&\le 2\|a\|^2C'+\frac12\int_\Omega|\nabla\Delta y_2^k|^2\,dx
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
2\left|\int_\Omega \nabla a(x)\Delta y_2^k\nabla\Delta\dot y_1^k\,dx\right| &\le 2\int_\Omega |\nabla a(x)||\Delta y_2^k||\nabla\Delta\dot y_1^k|\,dx\\
&\le \frac{2}{\sqrt{\tau_0}}\int_\Omega |\nabla a(x)||\Delta y_2^k||\nabla\Delta\dot y_1^k|\sqrt{g_1'(\Delta\dot y_1^k)}\,dx\\
&\le \int_\Omega |\nabla\Delta\dot y_1^k|^2 g_1'(\Delta\dot y_1^k)\,dx+\frac{1}{\tau_0}\|\nabla a\|^2\int_\Omega|\Delta y_2^k|^2\,dx\\
&\le \int_\Omega |\nabla\Delta\dot y_1^k|^2 g_1'(\Delta\dot y_1^k)\,dx+\frac{1}{\tau_0}\|\nabla a\|^2C'.
\end{aligned}\tag{3.17}$$

Similarly, we have

$$\begin{aligned}
2\left|\int_\Omega \nabla a(x)\Delta y_1^k\nabla\Delta\dot y_2^k\,dx\right| &\le \int_\Omega |\nabla\Delta\dot y_2^k|^2 g_2'(\Delta\dot y_2^k)\,dx+\frac{1}{\tau_0}\|\nabla a\|^2\int_\Omega|\Delta y_1^k|^2\,dx\\
&\le \int_\Omega |\nabla\Delta\dot y_2^k|^2 g_2'(\Delta\dot y_2^k)\,dx+\frac{c'}{\tau_0}\|\nabla a\|^2\int_\Omega|\nabla\Delta y_1^k|^2\,dx\\
&\le \int_\Omega |\nabla\Delta\dot y_2^k|^2 g_2'(\Delta\dot y_2^k)\,dx+\frac{c'}{\tau_0}\|\nabla a\|^2C'.
\end{aligned}\tag{3.18}$$

Combining (3.16)–(3.18), into (3.15) and integrating over $(0,t)$, we obtain

$$\begin{aligned}
&F^k(t)+\int_0^t\int_\Omega|\nabla\Delta\dot y_1^k(s)|^2g_1'(\Delta\dot y_1^k(s))\,dx\,dt+\int_0^t\int_\Omega|\nabla\Delta\dot y_2^k(s)|^2g_2'(\Delta\dot y_2^k(s))\,dx\,dt\\
&\le B^k(0)\le C_2,\quad \text{for all } t\in[0,t_k),
\end{aligned}$$

where C_2 independent of k and

$$\begin{aligned}
F^k(t)&=\int_\Omega\left(|\Delta\dot y_1^k|^2+|\Delta\dot y_2^k|^2+|\Delta^2y_1^k|^2\right)dx+\frac12\int_\Omega|\nabla\Delta y_2^k|^2\,dx,\\
B^k(0)&=\int_\Omega\left(|\Delta y_1^{1k}|^2+|\Delta y_2^{1k}|^2+|\Delta^2 y_1^{0k}|^2\right)dx+2\|a\|^2C'+\frac12\int_\Omega|\nabla\Delta y_2^{0k}|^2\,dx\\
&\quad+\left(\frac{c'}{\tau_0}\|\nabla a\|^2C'+\frac{c'}{\tau_0}\|\nabla a\|^2C'\right)T.
\end{aligned}$$

Therefore, we conclude that

$$y_1^k \text{ is bounded in } L^\infty(0,T;H^4(\Omega)),\tag{3.19}$$

$$y_2^k \text{ is bounded in } L^\infty(0,T;H^3_\Delta(\Omega)),\tag{3.20}$$

$$\dot y_1^k \text{ is bounded in } L^\infty(0,T;H^2(\Omega)),\tag{3.21}$$

$$\dot y_2^k \text{ is bounded in } L^\infty(0,T;H^2(\Omega)).\tag{3.22}$$

The third estimate. Assume that $t < T$ and let $0 < \xi < T - t$ and

$$y_i^{k\xi}(x,t) = y_i^k(x, t+\xi), \quad i = 1, 2.$$

So, $U_1^{k,\xi}(x,t) = y_1^k(x,t+\xi) - y_1^k(x,t)$, solves the differential equation

$$\left(\ddot{U}_1^{k,\xi} + \Delta^2 U_1^{k,\xi} - a(x)\Delta y_2^{k,\xi} - (g_1(\Delta \dot{y}_1^{k\xi}) - g_1(\Delta \dot{y}_1^k)), w_1^j\right) = 0, \quad \text{for all } w_1^j \in V^k, \tag{3.23}$$

and the set

$$U_2^{k,\xi}(x,t) = y_2^k(x,t+\xi) - y_2^k(x,t).$$

$U_2^{k,\xi}$ solves the differential equation

$$\left(\ddot{U}_2^{k,\xi} - \Delta U_2^{k,\xi} - a(x)\Delta U_1^{k,\xi} - (g_2(\Delta \dot{y}_2^{k\xi}) - g_2(\Delta \dot{y}_2^k)), w_2^j\right) = 0, \quad \text{for all } w_2^j \in V^k. \tag{3.24}$$

Choosing $w_1^j = -\Delta \dot{y_1}^{k\xi}$ in (3.23) and $w_2^j = \Delta \dot{U_2}^{k\xi}$ in (3.24), and using the fact that g_i is nondecreasing, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(|\nabla \dot{U_1}^{k\xi}(x,t)|^2 + |\nabla \dot{U_2}^{k\xi}(x,t)|^2 + |\nabla \Delta U_1^{k\xi}(x,t)|^2 + |\Delta U_2^{k\xi}(x,t)|^2\right) dx \\
&+ 2\frac{d}{dt}\int_\Omega a(x)\Delta U_2^{k\xi}(x,t)\Delta U_1^{k\xi}(x,t)\, dx \le 0, \quad \text{for all } t \ge 0.
\end{aligned}$$

Integrating over $[0,t]$, we get

$$\begin{aligned}
&\int_\Omega \left(|\nabla \dot{U_1}^{k\xi}(t)|^2 + |\nabla \dot{U_2}^{k\xi}(t)|^2\right) dx + (1 - c'\|a\|)\int_\Omega \left(|\nabla \Delta U_1^{k\xi}(t)|^2 + |\Delta U_2^{k\xi}(t)|^2\right) dx \\
\le & C_2 \int_\Omega \left(|\nabla \dot{U_1}^{k\xi}(0)|^2 + |\nabla \dot{U_2}^{k\xi}(0)|^2 + |\nabla \Delta U_1^{k\xi}(0)|^2 + |\Delta U_2^{k\xi}(0)|^2\right) dx,
\end{aligned}$$

where C_2 is a positive constant depending only on $\|a\|$ and c'. By dividing by ξ^2, and pass to the limit when $\xi \to 0$, we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla \ddot{y}_1^k(t)|^2 + |\nabla \ddot{y}_2^k(t)|^2 + |\nabla \Delta \dot{y}_1^k(t)|^2 + |\Delta \dot{y}_2^k(t)|^2\right) dx \\
\le & C_2' \int_\Omega \left(|\nabla \ddot{y}_1^k(0)|^2 + |\nabla \ddot{y}_2^k(0)|^2 + |\nabla \Delta y_1^{1k}|^2 + |\Delta y_2^{1k}|^2\right) dx.
\end{aligned}$$

Now we estimate $\|\nabla \ddot{y}_i^k(0)\|$. Choosing $v = -\Delta \ddot{y}_i^k$ in (3.1) and substitute $t = 0$, we obtain

$$\|\nabla \ddot{y}_1^k(0)\|^2 = \int_\Omega \nabla \ddot{y}_1^k(0)\nabla\left(-\Delta^2 y_1^{0k} - a(x)y_2^{0k} + g_1(\Delta y_1^{1k})\right) dx$$

and

$$\|\nabla \ddot{y}_2^k(0)\|^2 = \int_\Omega \nabla \ddot{y}_2^k(0)\nabla\left(\Delta y_2^{0k} - a(x)y_1^{0k} + g_2(\Delta y_2^{1k})\right) dx.$$

By Cauchy-Schwarz inequality, we obtain

$$\|\nabla \ddot{y}_1^k(0)\| \le \left(\int_\Omega \left|\nabla\left(-\Delta^2 y_1^{0k} - a(x)y_2^{0k} + g_1(\Delta y_1^{1k})\right)\right|^2 dx\right)^{\frac{1}{2}}$$

and

$$\|\nabla \ddot{y}_2^k(0)\| \le \left(\int_\Omega \left|\nabla\left(\Delta y_2^{0k} - a(x)y_1^{0k} + g_2(\Delta y_2^{1k})\right)\right|^2 dx\right)^{\frac{1}{2}}.$$

(3.6) and (3.7) yields

$$(\ddot{y}_1^k(0), \ddot{y}_2^k(0)) \text{ are bounded in } W \times W. \tag{3.25}$$

And by (3.4), (3.5) and (3.25) we deduce

$$\int_\Omega \Big(|\nabla \ddot{y}_1^k(t)|^2 + |\nabla \ddot{y}_2^k(t)|^2 + |\nabla\Delta \dot{y}_1^k(t)|^2 + |\Delta \dot{y}_2^k(t)|^2\Big)\,dx \le C_3, \quad \text{for all } t \ge 0,$$

where C_3 is a positive constant independent of $k \in \mathbb{N}$. Therefore, we deduce

$$\dot{y}_1^k \text{ is bounded in } L^\infty(0,T;H^3_\Delta(\Omega)), \tag{3.26}$$

$$\dot{y}_2^k \text{ is bounded in } L^\infty(0,T;H^2(\Omega)), \tag{3.27}$$

$$\ddot{y}_1^k \text{ is bounded in } L^\infty(0,T;H^1_0(\Omega)), \tag{3.28}$$

$$\ddot{y}_2^k \text{ is bounded in } L^\infty(0,T;H^1_0(\Omega)). \tag{3.29}$$

Applying Dunford-Pettis and Banach-Alaoglu-Bourbaki theorems, we conclude from (3.10)–(3.14), (3.19)–(3.22) and (3.26)–(3.29) that there exists a subsequence $\{y_i^m\}$ of $\{y_i^k\}$ such that

$$(y_1^m, y_2^m) \rightharpoonup (y_1, y_2), \quad \text{weak-star in } L^\infty(0,T;\widetilde{V}), \tag{3.30}$$

$$(\dot{y}_1^m, \dot{y}_2^m) \rightharpoonup (y_1', y_2'), \quad \text{weak-star in } L^\infty(0,T;V), \tag{3.31}$$

$$(\ddot{y}_1^m, \ddot{y}_2^m) \rightharpoonup (y_1'', y_2''), \quad \text{weak-star in } L^\infty(0,T;W), \tag{3.32}$$

$$(\dot{y}_1^m, \dot{y}_2^m) \to (y_1', y_2'), \quad \text{almost everywhere in } \Omega \times [0,+\infty), \tag{3.33}$$

$$g_i(\Delta \dot{y}_i^m) \rightharpoonup \chi_i, \quad \text{weak-star in } L^2(\mathcal{A}). \tag{3.34}$$

As (y_1^m, y_2^m) is bounded in $L^\infty(0,T;\widetilde{V})$ by (3.30) and the injection of $\widetilde{V}$ in H is compact, we have

$$(y_1^m, y_2^m) \to (y_1, y_2), \quad \text{strong in } L^2(0,T;H). \tag{3.35}$$

On the other hand, using (3.30), (3.32) and (3.35), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\ddot{y}_1^m(x,t) + \Delta^2 y_1^m(x,t) - a(x)\Delta y_2^m(x,t)\Big)\, w\, dxdt \\ \to &\int_0^T \int_\Omega \Big(y_1''(x,t) + \Delta^2 y_1(x,t) - a(x)\Delta y_2(x,t)\Big)\, w\, dxdt \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\ddot{y}_2^m(x,t) - \Delta y_2^m(x,t) - a(x)\Delta y_1^m(x,t)\Big) w\, dxdt \\ \to &\int_0^T \int_\Omega \Big(y_2''(x,t) - \Delta y_2(x,t) - a(x)\Delta y_1(x,t)\Big) w\, dxdt, \end{aligned} \tag{3.37}$$

for all $w \in L^2(0,T;L^2(\Omega))$.

It remains to prove the convergence

$$\int_0^T \int_\Omega g_i(\Delta \dot{y}_i^m)\; w\, dxdt \to \int_0^T \int_\Omega g_i(\Delta y_i')\; w\, dxdt,$$

when $m \to +\infty$. To finish the proof we shall use the following lemma.

Lemma 3.1. *Let $g_i(\Delta y_i') \in L^1(\mathcal{A})$ and $\|g_i(\Delta y_i')\|_{L^1(\mathcal{A})} \le K$, where K is a constant independent of t. Then $g_i(\Delta \dot{y}_i^m) \to g_i(\Delta y_i')$ in $L^1(\mathcal{A})$.*

Proof. Let $g(\Delta y') \in L^1(\mathcal{A})$. Since g_i is continuous, we deduce from (3.33)

$$g_i(\Delta \dot{y}_i^k) \rightarrow g_i(\Delta y_i'), \quad \text{almost everywhere in } \mathcal{A}, \tag{3.38}$$
$$\Delta \dot{y}_i^m g_i(\Delta \dot{y}_i^m) \rightarrow \Delta y_i' g_i(\Delta y_i'), \quad \text{almost everywhere in } \mathcal{A}.$$

Also, by (3.13) and Fatou's lemma, we have

$$\int_0^T \int_\Omega \Delta y_i'(x,t) g_i(\Delta y_i'(x,t))\, dxdt \le K_1, \quad \text{for } T > 0. \tag{3.39}$$

Now, we can estimate $\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dxdt$. By using Cauchy-Schwarz inequality and (2.3), we have the following.

1. If $|\Delta y_i'| \ge 1$, then

$$\begin{aligned} \int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dx\, dt &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|^2\, dx\, dt \right)^{1/2} \\ &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega \Delta y_i' g_i(\Delta y_i'(x,t))\, dx\, dt \right)^{1/2} \\ &\le K_2. \end{aligned}$$

2. If $|\Delta y_i'| < 1$, then

$$\begin{aligned} \int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dx\, dt &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|^2\, dx\, dt \right)^{1/2} \\ &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|^{\frac{2}{p+1}}\, dx\, dt \right)^{1/2} \\ &\le c|\mathcal{A}|^{(3p+1)/2(p+1)} \left(\int_0^T \int_\Omega \Delta y_i' g_i(\Delta y_i'(x,t))\, dx\, dt \right)^{1/(p+1)} \\ &\le K_3, \quad \text{for } T > 0. \end{aligned}$$

Then

$$\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dxdt \le K, \quad \text{for } T > 0.$$

And let $E \subset \Omega \times [0,T]$ and $|E|$ is the measure of E and the set

$$E_1 = \left\{ (x,t) \in E : |g_i(\Delta \dot{y}_i^m(x,t))| \le \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \backslash E_1.$$

If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g_i(s)| \ge r\}$, then

$$\int_E |g_i(\Delta \dot{y}_i^m)|\, dxdt \le c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |\Delta \dot{y}_i^m g_i(\Delta \dot{y}_i^m)|\, dxdt.$$

By applying (3.13), we deduce

$$\sup_m \int_E g_i(\Delta \dot{y}_i^m)\, dxdt \to 0, \quad \text{when } |E| \to 0.$$

From Vitali's convergence theory, we deduce

$$g_i(\Delta \dot{y}_i^m) \to g_i(\Delta y_i'), \quad \text{in } L^1(\mathcal{A}).$$

Proof of lemma is completed. □

End of proof of Theorem 3.1**.** Now (3.34) implies that

$$g_i(\Delta \dot{y}_i^m) \rightharpoonup g_i(\Delta y_i'), \quad \text{weak-star in } L^2([0,T]\times\Omega).$$

We deduce, for all $v \in L^2([0,T] \times L^2(\Omega)$, that

$$\int_0^T \int_\Omega g_i(\Delta \dot{y}_i^m) w \, dxdt \to \int_0^T \int_\Omega g_i(\Delta y_i') w \, dxdt.$$

Finally, for all $w \in L^2([0,T] \times L^2(\Omega))$:

$$\int_0^T \int_\Omega \Big(y_1''(x,t) + \Delta^2 y_1(x,t) - a(x)\Delta y_2(x,t) - g_1(\Delta y_1'(x,t)) \Big) w \, dxdt = 0$$

and

$$\int_0^T \int_\Omega \Big(y_2''(x,t) - \Delta y_2(x,t) - a(x)\Delta y_1(x,t) - g_2(\Delta y_2'(x,t)) \Big) w \, dxdt = 0.$$

Therefore, (y_1, y_2) are a solutions for the problem (1.1).

This concludes the proof of Theorem 3.1. □

4. Asymptotic Behavior

In this section, we prove stability result for the energy of the solution of system (1.1), by using the multiplier technique.

Theorem 4.1. *Let* $(y_1^0, y_2^0) \in \widetilde{V}$ *and* $(y_1^1, y_2^1) \in V$*. Assume that* (2.1)–(2.4) *hold. The energy of system* (1.1)*, given by* (2.5) *decay estimate:*

$$E(t) \le Ct^{-2/(p-1)}, \quad \text{for all } t > 0 \text{ if } p > 1, \tag{4.1}$$

and

$$E(t) \le C' E(0) e^{-wt}, \quad \text{for all } t > 0 \text{ if } p = 1, \tag{4.2}$$

where C *is a positive constant only depending on* $E(0)$ *and* C', w *are positive constants independent of the initial data.*

Proof. This proof is established in two steps.

Step 1. Multiplying the first equation of (1.1) by $-E^\mu \Delta y_1$, we obtain

$$\begin{aligned}
0 = & \int_S^T -E^\mu \int_\Omega \Delta y_1 \Big(y_1'' + \Delta^2 y_1 - a(x)\Delta y_2 + g_1(\Delta y_1') \Big) dxdt \\
= & -\Big[E^\mu \int_\Omega y_1' \Delta y_1 \, dx \Big]_S^T + \mu \int_S^T E' E^{\mu-1} \int_\Omega \Delta y_1 y_1' \, dxdt \\
& - 2 \int_S^T E^\mu \int_\Omega |\nabla y_1'|^2 \, dxdt + \int_S^T E^\mu \int_\Omega \Big(|\nabla y_1'|^2 + |\nabla \Delta y_1|^2 \Big) \, dxdt \\
& + \int_S^T E^\mu \int_\Omega a(x) \Delta y_1 \Delta y_2 \, dxdt + \int_S^T E^\mu \int_\Omega \Delta y_1 g_1(\Delta y_1') \, dxdt.
\end{aligned}$$

Step 2. Multiplying the second equation of (1.1) by $-E^\mu \Delta y_2$, we obtain

$$\begin{aligned}
0 =& \int_S^T -E^\mu \int_\Omega \Delta y_2 \Big(y_2'' + \Delta y_2 - a(x)\Delta y_1 + g_2(\Delta y_2')\Big)\, dxdt \\
=& -\Big[E^\mu \int_\Omega y_2' \Delta y_2 \, dx\Big]_S^T + \mu \int_S^T E' E^{\mu-1} \int_\Omega \Delta y_2 y_2' \, dxdt \\
&- 2\int_S^T E^\mu \int_\Omega |\nabla y_2'|^2 \, dxdt + \int_S^T E^\mu \int_\Omega \Big(|\nabla y_2'|^2 + |\Delta y_2|^2)\Big)\, dxdt \\
&+ \int_S^T E^\mu \int_\Omega a(x) \Delta y_2 \Delta y_1 \, dxdt + \int_S^T E^\mu \int_\Omega \Delta y_2 g_2(\Delta y_2')\, dxdt.
\end{aligned}$$

By their sum, we obtain

$$\begin{aligned}
\int_S^T E^{\mu+1}\, dt \le& \Big[E^\mu \int_\Omega \Big(y_1'\Delta y_1 + y_2'\Delta y_2\Big) dx\Big]_S^T \\
&- \mu \int_S^T E' E^{\mu-1} \int_\Omega \Big(\Delta y_1 y_1' + \Delta y_2 y_2'\Big) dxdt \\
&+ 2\int_S^T E^\mu \int_\Omega \Big(|\nabla y_1'|^2 + |\nabla y_2'|^2\Big) dxdt \\
&- \int_S^T E^\mu \int_\Omega \Big(\Delta y_1 g_1(\Delta y_1') + \Delta y_2 g_2(\Delta y_2')\Big) dxdt.
\end{aligned} \tag{4.3}$$

Since E is non-increasing, we find that

$$\begin{aligned}
\Big[E^\mu \int_\Omega (y_1'\Delta y_1 + y_2'\Delta y_2)\, dx\Big]_S^T \le& c E^{\mu+1}(S), \\
\mu \Big| \int_S^T E' E^{\mu-1} \int_\Omega (\Delta y_1 y_1' + \Delta y_2 y_2')\, dxdt \Big| \le& c E^{\mu+1}(S).
\end{aligned}$$

Using these estimates, we conclude from (4.3) that

$$\begin{aligned}
\int_S^T E^{\mu+1}\, dt \le& C E^{\mu+1}(S) + 2\int_S^T E^\mu \int_\Omega \Big(|\nabla y_1'|^2 + |\nabla y_2'|^2\Big)\, dxdt \\
&+ \int_S^T E^\mu \int_\Omega \Big(|\Delta y_1||g_1(\Delta y_1')| + |\Delta y_2||g_2(\Delta y_2')|\Big) dxdt.
\end{aligned} \tag{4.4}$$

Now, we estimate the terms of the right-hand side of the inequality (4.4), see Komornik [7].

We consider the following partition of Ω

$$\Omega^+ = \{x \in \Omega : |\Delta y_i'| \ge 1\}, \quad \Omega^- = \{x \in \Omega : |\Delta y_i'| < 1\}.$$

By using Sobolev embedding and Young's inequality, we obtain

(4.5)
$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^+} |\Delta y_1||g_1(\Delta y_1')|\,dxdt + \int_S^T E^\mu \int_{\Omega^+} |\nabla y_1'|^2\,dxdt \\
\leq&\varepsilon \int_S^T E^\mu \int_{\Omega^+} |\Delta y_1|^2\,dxdt + C(\varepsilon)\int_S^T E^\mu \int_{\Omega^+} |g_1(\Delta y_1')|^2\,dx\,dt + c\int_S^T E^\mu \int_{\Omega^+} |\Delta y_1'|^2 \\
\leq&\varepsilon c' \int_S^T E^\mu \int_\Omega |\nabla \Delta y_1|^2\,dxdt + \left(C(\varepsilon)c_2 + \frac{c}{c_1}\right)\int_S^T E^\mu \int_\Omega \Delta y_1' g_1(\Delta y_1')\,dxdt \\
\leq&\varepsilon C \int_S^T E^{\mu+1}\,dt + C_1(\varepsilon)\int_S^T E^\mu(-E')\,dt \\
\leq&\varepsilon C \int_S^T E^{\mu+1}\,dt + C_1(\varepsilon,\mu)E^{\mu+1}(S).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^+} |\Delta y_2||g_2(\Delta y_2')|\,dxdt + \int_S^T E^\mu \int_{\Omega^+} |\nabla y_2'|^2\,dxdt \\
\leq&\varepsilon C \int_S^T E^{\mu+1}\,dt + C_2(\varepsilon,\mu)E^{\mu+1}(S).
\end{aligned} \tag{4.6}$$

Summing (4.5) and (4.6), we obtain

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^+} \Big(|\Delta y_1||g_1(\Delta y_1')| + |\Delta y_2||g_2(\Delta y_2')|\Big)dxdt \\
&+ \int_S^T E^\mu \int_{\Omega^+} \left(|\nabla y_1'|^2 + |\nabla y_2'|^2\right)dxdt \\
\leq&\varepsilon C \int_S^T E^{\mu+1}\,dt + C(\varepsilon,\mu)E^{\mu+1}(S)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^-} \Big(|\Delta y_1||g_1(\Delta y_1')| + |\nabla y_1'|^2\Big)dxdt \\
\leq&\varepsilon' c' \int_S^T E^\mu \int_\Omega |\nabla \Delta y_1|^2\,dxdt + C(\varepsilon')\int_S^T E^\mu \int_\Omega \Big(|\Delta y_1'|^2 + |g_1(\Delta y_1')|^2\Big)dxdt \\
\leq&\varepsilon' c' \int_S^T E^{\mu+1}\,dt + C(\varepsilon')\int_S^T E^\mu \int_\Omega (\Delta y_1' g_1(\Delta y_1'))^{\frac{2}{p+1}}\,dxdt \\
\leq&\varepsilon' C \int_S^T E^{\mu+1}\,dt + C(\varepsilon',p)\int_S^T E^\mu \left(\int_\Omega \Delta y_1' g_1(\Delta y_1')\,dx\right)^{\frac{2}{p+1}}\,dt.
\end{aligned} \tag{4.8}$$

Similarly, we have

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^-} \Big(|\Delta y_2||g_2(\Delta y_2')| + |\nabla y_2'|^2\Big)dxdt \\
\leq&\varepsilon' C \int_S^T E^{\mu+1}\,dt + C(\varepsilon',p)\int_S^T E^\mu \left(\int_\Omega \Delta y_2' g_2(\Delta y_2')\,dx\right)^{\frac{2}{p+1}}\,dt.
\end{aligned} \tag{4.9}$$

Summing (4.8) and (4.9), we obtain

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^-} \Big(|\Delta y_1||g_1(\Delta y_1')| + |\Delta y_2||g_2(\Delta y_2')|\Big) dxdt \\
&+ \int_S^T E^\mu \int_{\Omega^-} \Big(|\nabla y_1'|^2 + |\nabla y_2'|^2\Big) dxdt \\
\le& \varepsilon_0 C \int_S^T E^{\mu+1}\, dt + C(\varepsilon_0, p) \int_S^T E^\mu \left(-E'\right)^{\frac{2}{p+1}}\, dt \\
&\le \varepsilon_0 C \int_S^T E^{\mu+1}\, dt + \varepsilon_1 \int_S^T E^{\mu\frac{p+1}{p-1}}\, dt + C\left(\varepsilon_1, p\right) E(S).
\end{aligned} \tag{4.10}$$

Comblining (4.7) and (4.10) in (4.4), we find

$$\int_\Omega E^{\mu+1}\, dt \le CE(S) + C' E^{\mu+1}(S) + \varepsilon C \int_S^T E^{\mu+1}\, dt + \varepsilon_1 \int_S^T E^{\mu\frac{p+1}{p-1}}\, dt.$$

We choose μ such that $\mu\frac{p+1}{p-1} = \mu+1$, so, $\mu = \frac{p-1}{2}$, and choosing ε and ε_1 small enough, we obtain

$$\int_\Omega E^{\mu+1}\, dt \le C' E(S) + C' E^\mu(0) E(S),$$

where C' is positive constant independent of $E(0)$. Hence, the estimates (4.1) and (4.2) follow by applying the following result of Martinez.

Lemma 4.1. *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and assume that there are two constants $\mu \ge 0$, $\omega > 0$ such that*

$$\int_t^{+\infty} E(s)^{\mu+1}\, ds \le \omega E(0)^\mu E(t), \quad \textit{for all } t \ge 0.$$

Then, we have for every $t > 0$

$$\begin{cases} E(t) \le E(0) \left(\dfrac{1+\mu}{1+\omega\mu t}\right)^{-\frac{1}{\mu}}, & \textit{if } \mu > 0, \\ E(t) \le E(0) e^{1-\omega t}, & \textit{if } \mu = 0. \end{cases}$$

For a short proof of this lemma we refer to [12].

This completes the proof of Theorem 4.1. □

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References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, Inc., New York, San Francisco, London, 1975.

[2] F. Alabau-Boussouira, *Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems,* Appl. Math. Optim. **51** (2005), 61–105. `https://doi.org/10.1007/s00245`

[3] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd Ed., Graduate Texts in Math. **60**, Springer, New York, 1989. https://doi.org/10.1007/978-1-4757-2063-1

[4] M. Bahlil and F. Baowei, *Global existence and energy decay of solutions to a coupled wave and Petrovsky system with nonlinear dissipations and source terms*, Mediterr. J. Math. (2020), 27 pages. https://doi.org/10.1007/s00009-020-1497-5

[5] A. Guesmia, *Energy decay for a damped nonlinear coupled system*, J. Math. Anal. Appl. **239**(1) (1999), 38–48. https://doi.org/10.1006/jmaa.1999.6534

[6] A. Haraux, *Two Remarks on Dissipative Hyperbolic Problems*, Pitman, Boston, MA, 1985, 161–179.

[7] V. Komornik, *Well-posedness and decay estimates for a Petrovsky system by a semigroup approach*, Acta Sci. Math. (Szeged) **60** (1995), 451–466.

[8] I. Lasiecka and D. Toundykov, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Anal. **64**(8) (2006), 1757–1797. https://doi.org/10.1016/j.na.2005.07.024

[9] I. Lasiecka, *Stabilization of wave and plate-like equation with nonlinear dissipation on the boundary*, J. Differential Equations **79**(2) (1989), 340–381. https://doi.org/10.1016/0022-0396(89)90107-1

[10] J. L. Lions, *Quelques Méthodes De Résolution Des Problémes Aux Limites Nonlinéaires*, Dund Gautier-Villars, Paris, 1969.

[11] W. J. Liu and E. Zuazua, *Decay rates for dissipative wave equations*, Ric. Mat. **48** (1999), 61–75.

[12] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems*, ESAIM Control Optim. Calc. Var. **4** (1999), 419–444. https://doi.org/10.1051/cocv:1999116

[13] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), 273–303. https://doi.org/10.1007/BF02761595

[14] D. H. Sattinger, *On global solutions for nonlinear hyperbolic equations*, Arch. Ration. Mech. Anal. **30** (1968), 148–172. ttps://doi.org/10.1007/BF00250942

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