

## ON IMPRIMITIVITY HILBERT BIMODULES OVER COMMUTATIVE $H^*$ -ALGEBRAS

M. KHANEHGIR<sup>1\*</sup>, M. MORADIAN Khibary<sup>2</sup>, AND Z. NIAZI MOGHANI<sup>1</sup>

**ABSTRACT.** In this paper, we introduce the notion of imprimitivity Hilbert  $H^*$ -bimodule and describe some properties of it. Moreover, we show that if  $\mathcal{A}$  and  $\mathcal{B}$  are proper and commutative  $H^*$ -algebras,  ${}_{\mathcal{A}}E_{\mathcal{B}}$  is a Hilbert  $H^*$ -bimodule and  $e_1$  is a minimal projection in  $\mathcal{A}$  with  ${}_{\mathcal{A}}[x|x] = e_1$  for some  $x \in \mathcal{A}$ , then  $[x|x]_{\mathcal{B}}$  is a minimal projection in  $\mathcal{B}$ , too. Furthermore, the existence of orthonormal bases for such spaces is studied.

### 1. INTRODUCTION AND PRELIMINARIES

An  $H^*$ -algebra, introduced by Ambrose [1] is a complex Banach algebra  $\mathcal{A}$  satisfying the following conditions:

- (i)  $\mathcal{A}$  is a Hilbert space under an inner product  $\langle \cdot, \cdot \rangle$ ;
- (ii) for each  $a$  in  $\mathcal{A}$  there is an element  $a^*$  in  $\mathcal{A}$ , the so-called adjoint of  $a$ , such that  $\langle ab, c \rangle = \langle b, a^*c \rangle$  and  $\langle ab, c \rangle = \langle a, cb^* \rangle$ , for all  $b, c \in \mathcal{A}$ .

Recall that  $\mathcal{A}_0 = \{a \in \mathcal{A} : a\mathcal{A} = \{0\}\} = \{a \in \mathcal{A} : \mathcal{A}a = \{0\}\}$  is called the annihilator ideal of  $\mathcal{A}$ . A proper  $H^*$ -algebra is an  $H^*$ -algebra with zero annihilator ideal. Ambrose [1] proved that an  $H^*$ -algebra is proper if and only if every element has a unique adjoint. The trace-class  $\tau(\mathcal{A})$  of a proper  $H^*$ -algebra  $\mathcal{A}$  is defined by the set  $\tau(\mathcal{A}) = \{ab : a, b \in \mathcal{A}\}$ . It is known that  $\tau(\mathcal{A})$  is an ideal of  $\mathcal{A}$ , which is a Banach  $*$ -algebra under a suitable norm  $\tau_{\mathcal{A}}(\cdot)$ . The norm  $\tau_{\mathcal{A}}$  is related to the given norm  $\|\cdot\|$  on  $\mathcal{A}$  by  $\tau_{\mathcal{A}}(a^*a) = \|a\|^2$  for all  $a \in \mathcal{A}$ . The trace functional  $\text{tr}_{\mathcal{A}}$  on  $\tau(\mathcal{A})$  is defined by  $\text{tr}_{\mathcal{A}}(ab) = \langle a, b^* \rangle = \langle b, a^* \rangle = \text{tr}_{\mathcal{A}}(ba)$  for each  $a, b \in \mathcal{A}$ . In particular  $\text{tr}_{\mathcal{A}}(aa^*) = \text{tr}_{\mathcal{A}}(a^*a) = \|a\|^2$  for each  $a \in \mathcal{A}$ . A nonzero element  $e \in \mathcal{A}$  is called a projection, if it is self-adjoint and idempotent. In addition, if  $e\mathcal{A}e = \mathbb{C}e$  then,

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*Key words and phrases.*  $\mathcal{A}$ - $\mathcal{B}$ -bimodule,  $H^*$ -algebra, full Hilbert  $H^*$ -module, minimal projection.  
2010 *Mathematics Subject Classification.* Primary: 46H05. Secondary: 46C05.

*Received:* September 18, 2017.

*Accepted:* January 16, 2018.

it is called a minimal projection. Each simple  $H^*$ -algebra (that is, an  $H^*$ -algebra without nontrivial closed two-sided ideals) contains minimal projections. It is known that all minimal projections in a simple  $H^*$ -algebra have equal norms equal to  $\alpha$  for some  $\alpha \geq 1$  [2]. Two idempotents  $e$  and  $e'$  are doubly orthogonal if  $\langle e, e' \rangle = 0$  and  $ee' = e'e = 0$ . An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotents. Every proper  $H^*$ -algebra contains a maximal family of doubly orthogonal primitive self-adjoint idempotents [1]. Recall that in a commutative  $H^*$ -algebra an element is a primitive projection if and only if it is a minimal projection [7, Lemma 1.1]. There are many scholars have worked on  $H^*$ -algebras and developed the topic in several directions, see [1, 3, 8–10] and references cited therein.

**Proposition 1.1.** *Let  $\mathcal{A}$  be a proper commutative  $H^*$ -algebra. If  $e$  and  $e'$  are distinct minimal projections in  $\mathcal{A}$ , then they are doubly orthogonal.*

*Proof.* We are going to show that  $ee' = 0$ . If on the contrary  $ee' \neq 0$ , then commutativity of  $\mathcal{A}$  and minimality of the projections  $e$  and  $e'$ , imply that  $ee' = ee'e = \lambda_1 e = \lambda_2 e'$  for some nonzero and distinct scalars  $\lambda_1$  and  $\lambda_2$ . On the other hand, since  $e$ ,  $e'$  and  $ee'$  are idempotents, then  $(\lambda_1 e)^2 = \lambda_1 e = ee' = (\lambda_2 e')^2 = \lambda_2 e'$  and so  $\lambda_1 = \lambda_2 = 1$ , which gives  $e = e'$  a contradiction. Thus  $ee' = e'e = 0$  and therefore  $\langle e, e' \rangle = \text{tr}_{\mathcal{A}}(ee') = 0$ .  $\square$

An immediate consequence of the above proposition is the following result.

**Corollary 1.1.** *Each commutative  $H^*$ -algebra has a unique maximal family of doubly orthogonal minimal projections which contains all of its minimal projections.*

Let us recall the definition of a Hilbert  $H^*$ -module.

**Definition 1.1.** [2] A Hilbert  $H^*$ -module over a proper  $H^*$ -algebra  $\mathcal{A}$  is a left  $\mathcal{A}$ -module  $E$  on which there is a mapping  $[\cdot|\cdot] : E \times E \rightarrow \tau(\mathcal{A})$  (called  $\tau(\mathcal{A})$ -valued product), satisfying

- (i)  $[\alpha x|y] = \alpha[x|y]$ ;
- (ii)  $[x + y|z] = [x|z] + [y|z]$ ;
- (iii)  $[ax|y] = a[x|y]$ ;
- (iv)  $[x|y]^* = [y|x]$ ;
- (v) for each nonzero element  $x$  in  $E$  there is a nonzero element  $a$  in  $\mathcal{A}$  such that  $[x|x] = a^*a$ ;
- (vi)  $E$  is a Hilbert space under the inner product  $(x, y) = \text{tr}_{\mathcal{A}}([x|y])$ ;

for each  $\alpha \in \mathbb{C}$ ,  $x, y, z \in E$  and  $a \in \mathcal{A}$ .

The Hilbert  $H^*$ -module  $E$  is full [7] if the ideal  $[E|E] = \text{span}\{[x|y] : x, y \in E\}$ , is dense in  $\tau(\mathcal{A})$  under the norm  $\tau_{\mathcal{A}}(\cdot)$ .

*Example 1.1.* [2] Let  $H$  be an infinite dimensional Hilbert space and  $\mathcal{HS}(H)$  be the standard  $H^*$ -algebra of Hilbert-Schmidt operators on it. Let us denote by  $\Theta_{x,y}$  the

rank 1 operator on  $H$  defined by  $\Theta_{x,y}(z) = (z, y)x$ . It is well known that  $H$  may be regarded as a Hilbert  $H^*$ -module over  $\mathcal{HS}(H)$ . Given  $x \in H$  and  $T \in \mathcal{HS}(H)$ ,  $Tx$  is interpreted as the action of  $T$  and  $\mathcal{HS}(H)$ -valued product on  $H$  is defined by  $[x|y] = \Theta_{x,y}$ . Since  $\text{tr}_{\mathcal{HS}(H)}\Theta_{x,y} = (x, y)$ , then the resulting norm on  $H$  coincides with the original one.

For a Hilbert  $H^*$ -module  $E$  over a proper  $H^*$ -algebra  $\mathcal{A}$  the following relations between the two norms  $\|\cdot\|$  and  $\tau_{\mathcal{A}}$  hold (see [2]):

$$\begin{aligned} \|x\|^2 &= \text{tr}_{\mathcal{A}}([x|x]) = \tau_{\mathcal{A}}([x|x]), \quad \text{for all } x \in E, \\ \|[x|y]\| &\leq \tau_{\mathcal{A}}([x|y]) \leq \|x\|\|y\|, \quad \text{for all } x, y \in E, \\ \|ax\| &\leq \|a\|\|x\|, \quad \text{for all } a \in \mathcal{A}, x \in E. \end{aligned}$$

**Definition 1.2.** [2] An element  $u \in E$  is a basic element if there exists a minimal projection  $e \in \mathcal{A}$  (called the supporting projection) such that  $[u|u] = e$ . An orthonormal system in  $E$  is a family of basic elements  $\{u_{\lambda}\}$ ,  $\lambda \in \Lambda$  satisfying  $[u_{\lambda}|u_{\mu}] = 0$ , for all  $\lambda, \mu \in \Lambda$ ,  $\lambda \neq \mu$ . An orthonormal basis in  $E$  is an orthonormal system generating a dense submodule of  $E$ .

Note that if  $\{u_{\lambda}\}$  is an orthonormal basis for  $E$ , then for each  $x \in E$ ,  $x = \sum_{\lambda} [x|u_{\lambda}]u_{\lambda}$  (Fourier expansion) (see [2]). We recall that each Hilbert  $H^*$ -module  $E$  contains basic elements, orthonormal systems and orthonormal bases and moreover, all orthonormal bases for  $E$  have the same cardinal number called the hilbertian dimension of  $E$ .

**Lemma 1.1.** [2] Let  $E$  be a Hilbert module over an arbitrary  $H^*$ -algebra  $\mathcal{A}$ ,  $e \in \mathcal{A}$  be a projection (not necessarily minimal) and let  $x \in E$  be such that  $[x|x] = e$ . Then  $ex = x$ .

In the above lemma one observes that if  $[x|x] = \lambda e$  for some scalar  $\lambda$  and some projection  $e$  in  $\mathcal{A}$ , then

$$[ex - x|ex - x] = [ex|ex] - [ex|x] - [x|ex] + [x|x] = \lambda e^3 - \lambda e^2 - \lambda e^2 + \lambda e = 0,$$

which implies that  $ex = x$ . Let  $E$  be a Hilbert  $H^*$ -module over an  $H^*$ -algebra  $\mathcal{A}$  and let  $e \in \mathcal{A}$  be a minimal projection. Then  $E_e = \{x \in E : [x|x] = \lambda e, \lambda \geq 0\}$  is a closed subspace of the Hilbert space  $E$ . If  $\mathcal{A}$  is a simple  $H^*$ -algebra, then the subspace  $E_e$  generates a dense submodule in  $E$  (see [2]). For emphasizing its  $H^*$ -algebra, we denote  $E_e$  by  $({}_{\mathcal{A}}E)_e$  (or  $(E_{\mathcal{A}})_e$  in right module case). For more details on this issue see [2]. Also, for general facts about Hilbert  $H^*$ -modules we refer the interested reader to [2, 4, 7, 10, 11].

We introduce the notion of imprimitivity Hilbert  $H^*$ -bimodule and describe some properties of it. In this paper, we show that if  $\mathcal{A}$  and  $\mathcal{B}$  are proper and commutative  $H^*$ -algebras,  ${}_{\mathcal{A}}E_{\mathcal{B}}$  is an imprimitivity Hilbert  $H^*$ -bimodule and  $e_1$  is a minimal projection in  $\mathcal{A}$  with  ${}_{\mathcal{A}}[x|x] = e_1$  for some  $x \in \mathcal{A}$ , then  $[x|x]_{\mathcal{B}}$  is a minimal projection in  $\mathcal{B}$ , too. Furthermore, the existence of orthonormal bases for such spaces is studied.

2. MAIN RESULTS

In this section, we state the notions of Hilbert  $H^*$ -bimodule and imprimitivity Hilbert  $H^*$ -bimodule. We then investigate the existence of orthonormal bases for imprimitivity Hilbert bimodules over the commutative  $H^*$ -algebras. Before giving our results, we state two interesting facts related to Hilbert modules over the commutative  $H^*$ -algebras which will be used in the sequel.

**Proposition 2.1.** *Let  $E$  be a Hilbert module over a commutative  $H^*$ -algebra  $\mathcal{A}$ . If  $\{u_\lambda\}$ ,  $\lambda \in \Lambda$  is an orthonormal basis for  $E$  and  $x \in E$ , then  $x = \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda$  for some  $\mu_\lambda \in \mathbb{C}$ .*

*Proof.* Let  $\{e_i\}$ ,  $i \in I$ , be the maximal family of doubly orthogonal minimal projections in  $\mathcal{A}$  as Corollary 1.1. Let's also suppose that each  $u_\lambda$ ,  $\lambda \in \Lambda$ , has supporting projection  $e_{i_\lambda}$  for some  $i_\lambda \in I$ . Since  $x = \sum_{\lambda \in \Lambda} [x|u_\lambda]u_\lambda$ , then applying [1, Theorem 4.1] and by the commutativity of  $\mathcal{A}$ , we get  $[x|u_\lambda] = \sum_{i \in I} \mu_{\lambda,i} e_i$ , for each  $\lambda \in \Lambda$  and some scalars  $\mu_{\lambda,i}$ . Thus we have  $x = \sum_{\lambda \in \Lambda} \sum_{i \in I} \mu_{\lambda,i} e_i u_\lambda$ .

On the other hand,  $e_i u_\lambda = 0$  for all  $i \neq i_\lambda$ . Indeed, by applying Proposition 1.1 we conclude that  $[e_i u_\lambda | e_i u_\lambda] = e_i [u_\lambda | u_\lambda] = e_i e_{i_\lambda} = 0$  for each  $i \neq i_\lambda$ . Therefore,  $x = \sum_{\lambda \in \Lambda} \mu_{\lambda,i_\lambda} e_{i_\lambda} u_\lambda = \sum_{\lambda \in \Lambda} \mu_{\lambda,i_\lambda} u_\lambda$  by Lemma 1.1. □

**Proposition 2.2.** *Let  $E$  be a full Hilbert module over a commutative  $H^*$ -algebra  $\mathcal{A}$ ,  $e_0 \in \mathcal{A}$  be a minimal projection and  $\{u_\lambda\}$ ,  $\lambda \in \Lambda$ , be an orthonormal basis for  $E$ . Then  $e_0 = [u_{\lambda_0} | u_{\lambda_0}]$  for some  $\lambda_0 \in \Lambda$ .*

*Proof.* On the contrary, we suppose that

$$(2.1) \quad e_0 \neq [u_\lambda | u_\lambda],$$

for all  $\lambda \in \Lambda$ . By the fullness of  $E$  we get  $e_0 = \sum_{t \in J} [x_t | y_t]$ , for some index set  $J$  and  $x_t$  and  $y_t$  in  $E$ . Regarding to Proposition 2.1 it follows that  $x_t = \sum_{\lambda \in \Lambda} \mu_{t,\lambda} u_\lambda$  and  $y_t = \sum_{\lambda \in \Lambda} \mu'_{t,\lambda} u_\lambda$ , for each  $t \in J$  and some scalars  $\mu_{t,\lambda}$  and  $\mu'_{t,\lambda}$ . Therefore, we can write

$$(2.2) \quad e_0 = \sum_{t \in J} [x_t | y_t] = \sum_{t,\lambda} \mu_{t,\lambda} \overline{\mu'_{t,\lambda}} [u_\lambda | u_\lambda].$$

Accordingly, by (2.1) and (2.2) and applying Proposition 1.1 we observe that,

$$e_0 = e_0^2 = \sum_{t,\lambda} \mu_{t,\lambda} \overline{\mu'_{t,\lambda}} [u_\lambda | u_\lambda] e_0 = 0,$$

which gives a contradiction to the fact  $e_0 \neq 0$ . □

**Definition 2.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two proper  $H^*$ -algebras. By a Hilbert bimodule  ${}_A E_B$  we mean a left Hilbert  $\mathcal{A}$ -module with the  $\tau(\mathcal{A})$ -valued product  ${}_A [\cdot | \cdot] : E \times E \rightarrow \tau(\mathcal{A})$  and a right Hilbert  $\mathcal{B}$ -module with the  $\tau(\mathcal{B})$ -valued product  $[\cdot | \cdot]_B : E \times E \rightarrow \tau(\mathcal{B})$  such that

$$(i) \quad (ax)b = a(xb);$$

- (ii)  ${}_A[xb|y] = {}_A[x|yb^*]$ ;
- (iii)  $[x|ay]_{\mathcal{B}} = [a^*x|y]_{\mathcal{B}}$ ;

for all  $x, y \in {}_A E_{\mathcal{B}}$ ,  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ .

Further, Hilbert  $H^*$ -bimodule  ${}_A E_{\mathcal{B}}$  is called full, if it is full both as a left and as a right Hilbert module over  $\mathcal{A}$  and  $\mathcal{B}$ , respectively.

**Definition 2.2.** A Hilbert  $\mathcal{A}$ - $\mathcal{B}$ -bimodule  $E$  is called an imprimitivity bimodule if

$${}_A[x|y]z = x[y|z]_{\mathcal{B}},$$

where  $x, y, z \in {}_A E_{\mathcal{B}}$ .

*Example 2.1.* Suppose  $\mathcal{A}$  is a proper  $H^*$ -algebra. It is easy to verify that  $\mathcal{A}$  is a full Hilbert  $H^*$ -bimodule over  $\mathcal{A}$  with the maps  ${}_A[a_1|a_2] = a_1a_2^*$  and  $[a_1|a_2]_{\mathcal{A}} = a_1^*a_2$ ,  $a_1, a_2 \in \mathcal{A}$ .

We point out that each Hilbert  $H^*$ -bimodule  ${}_A E_{\mathcal{B}}$  is a Hilbert space under both inner products  ${}_A(x, y) = \text{tr}_A({}_A[x|y])$ ,  $(x, y)_{\mathcal{B}} = \text{tr}_{\mathcal{B}}([x|y]_{\mathcal{B}})$  and therefore it has two norms, usually different, as follows

$${}_A\|x\| = \text{tr}_A({}_A[x|x])^{\frac{1}{2}}, \quad \|x\|_{\mathcal{B}} = \text{tr}_{\mathcal{B}}([x|x]_{\mathcal{B}})^{\frac{1}{2}}, \quad x \in E.$$

We however have the following result in the particular case  $\mathcal{A} = \mathcal{B}$ .

**Theorem 2.1.** Let  $E$  be a Hilbert  $H^*$ -bimodule over an  $H^*$ -algebra  $\mathcal{A}$ , then

$${}_A\|xa\| \leq \|a\|_A\|x\|, \quad \|ax\|_A \leq \|a\|\|x\|_A,$$

for each  $a \in \mathcal{A}$  and  $x \in E$ .

*Proof.* We are going to show that  ${}_A\|xa\| \leq \|a\|_A\|x\|$ . Without loss of generality, we can assume that  $\|a\| \leq 1$ . Take  $b = aa^*$ . Then  $b - b^2 = h^2$  for some positive element  $h \in \tau(\mathcal{A})$  (see [5, p. 34]). Therefore we can write  $\text{tr}_A({}_A[x(b - b^2)|x]) = \text{tr}_A({}_A[xh|xh]) = {}_A\|xh\|^2 \geq 0$  and thus we have

$$(2.3) \quad \text{tr}_A({}_A[x|xb b^*]) = \text{tr}_A({}_A[xb^2|x]) \leq \text{tr}_A({}_A[xb|x]).$$

On using (2.3), we get

$$\begin{aligned} 0 &\leq \text{tr}_A({}_A[x - xb|x - xb]) \\ &= \text{tr}_A({}_A[x|x]) - \text{tr}_A({}_A[x|xb]) - \text{tr}_A({}_A[xb|x]) + \text{tr}_A({}_A[xb|xb]) \\ &= \text{tr}_A({}_A[x|x]) - \text{tr}_A({}_A[x|xb]) - \text{tr}_A({}_A[xb|x]) + \text{tr}_A({}_A[x|xb b^*]) \\ &\leq \text{tr}_A({}_A[x|x]) - \text{tr}_A({}_A[x|xb]) - \text{tr}_A({}_A[xb|x]) + \text{tr}_A({}_A[xb|x]). \end{aligned}$$

It enforces that  $\text{tr}_A({}_A[x|xb]) \leq \text{tr}_A({}_A[x|x])$ . Hence

$$\begin{aligned} {}_A\|xa\|^2 &= \text{tr}_A({}_A[xa|xa]) = \text{tr}_A({}_A[x|xaa^*]) = \text{tr}_A({}_A[x|xb]) \\ &\leq \text{tr}_A({}_A[x|x]) = {}_A\|x\|^2, \end{aligned}$$

as desired. The proof of the other part is similar and therefore, to avoid repetition we remove it. □

Now we are in a position to state and prove our main result.

**Theorem 2.2.** *Suppose that  $E$  is an imprimitivity Hilbert  $\mathcal{A}$ - $\mathcal{B}$ -bimodule over the commutative  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and  $x \in E$ . Then  ${}_A[x|x]$  is a minimal projection in  $\mathcal{A}$  if and only if  $[x|x]_{\mathcal{B}}$  is a minimal projection in  $\mathcal{B}$ . Furthermore,  $x$  is in the Hilbert space  $({}_A E)_e$  for some minimal projection  $e \in \mathcal{A}$  if and only if  $x$  is in the Hilbert space  $(E_{\mathcal{B}})_{e'}$  for some minimal projection  $e' \in \mathcal{B}$ .*

*Proof.* Consider  ${}_A[x|x] = e_1$  for some minimal projection  $e_1$  in  $\mathcal{A}$ . Then  $e_1 x = x$  by Lemma 1.1 and therefore it establishes

$$[x|x]_{\mathcal{B}} = [x|e_1 x]_{\mathcal{B}} = \left[ x \Big|_A [x|x] x \right]_{\mathcal{B}} = \left[ x|x[x|x]_{\mathcal{B}} \right]_{\mathcal{B}} = [x|x]_{\mathcal{B}}^2.$$

Since  $[x|x]_{\mathcal{B}} = b^*b$  for some nonzero  $b \in \mathcal{B}$ , then  $[x|x]_{\mathcal{B}}$  is a projection. It remains to prove that it is a minimal projection. For this purpose, let  $\{e'_j\}$ ,  $j \in J$  be the maximal family of minimal projections in  $\mathcal{B}$ . In view of [1, Lemma 4.1] and [7, Lemma 1.1],  $[x|x]_{\mathcal{B}} = \sum_{j \in J} t'_j e'_j$  for some nonnegative numbers  $t'_j$ ,  $j \in J$ . Put  $[x|x]_{\mathcal{B}} = \sum_{j \in J_0} t'_j e'_j$ , where  $J_0 = \{j \in J : t'_j \neq 0\}$ . Now, since  $[x|x]_{\mathcal{B}}$  is idempotent, so we get  $[x|x]_{\mathcal{B}} = \sum_{j \in J_0} e'_j$ . We claim that  $[x|x]_{\mathcal{B}} = e'_j$  for some  $j \in J_0$ . First, on the contrary suppose that  $[x|x]_{\mathcal{B}} = e'_{j_1} + e'_{j_2}$ , for distinct elements  $j_1, j_2 \in J_0$ . Applying again Lemma 1.1, for each  $a \in \mathcal{A}$ , we have

$$\begin{aligned} e_1 a &= {}_A[x|x]a \\ &= {}_A \left[ {}_A[x|x]x \Big|_A [x|x]x \right] a = {}_A \left[ x[x|x]_{\mathcal{B}} \Big| x[x|x]_{\mathcal{B}} \right] a \\ &= {}_A [x(e'_{j_1} + e'_{j_2}) | x(e'_{j_1} + e'_{j_2})] a = {}_A [xe'_{j_1} + xe'_{j_2} | xe'_{j_1} + xe'_{j_2}] a \\ &= {}_A [xe'_{j_1} | xe'_{j_1}] a + {}_A [xe'_{j_2} | xe'_{j_2}] a + {}_A [xe'_{j_2} | xe'_{j_1}] a + {}_A [xe'_{j_1} | xe'_{j_2}] a. \end{aligned}$$

The double orthogonality of  $e_j$  s' ensures that

$$(2.4) \quad e_1 a = {}_A [xe'_{j_1} | xe'_{j_1}] a + {}_A [xe'_{j_2} | xe'_{j_2}] a.$$

Assume that  $\{u_\lambda\}$ ,  $\lambda \in \Lambda$ , is an orthonormal basis for the right Hilbert  $\mathcal{B}$ -module  $E$ . According to Proposition 2.1 we observe that  $x = \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda$  for some scalars  $\mu_\lambda$ ,  $\lambda \in \Lambda$ , and therefore,  $e'_{j_1} + e'_{j_2} = [x|x]_{\mathcal{B}} = [\sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda | \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda]_{\mathcal{B}} = \sum_{\lambda \in \Lambda} |\mu_\lambda|^2 [u_\lambda | u_\lambda]_{\mathcal{B}}$ . So, there exists  $\lambda_1$  and  $\lambda_2$  in  $\Lambda$  such that  $[u_{\lambda_1} | u_{\lambda_1}]_{\mathcal{B}} = e'_{j_1}$ ,  $[u_{\lambda_2} | u_{\lambda_2}]_{\mathcal{B}} = e'_{j_2}$ . Regarding to (2.4) we derive that

$$\begin{aligned} e_1 a &= {}_A \left[ x[u_{\lambda_1} | u_{\lambda_1}]_{\mathcal{B}} \Big| x[u_{\lambda_1} | u_{\lambda_1}]_{\mathcal{B}} \right] a + {}_A \left[ x[u_{\lambda_2} | u_{\lambda_2}]_{\mathcal{B}} \Big| x[u_{\lambda_2} | u_{\lambda_2}]_{\mathcal{B}} \right] a \\ &= {}_A \left[ {}_A [x|u_{\lambda_1}]u_{\lambda_1} \Big|_A [x|u_{\lambda_1}]u_{\lambda_1} \right] a + {}_A \left[ {}_A [x|u_{\lambda_2}]u_{\lambda_2} \Big|_A [x|u_{\lambda_2}]u_{\lambda_2} \right] a \\ &= {}_A [x|u_{\lambda_1}]_A [x|u_{\lambda_1}]_A^* [u_{\lambda_1} | u_{\lambda_1}] a + {}_A [x|u_{\lambda_2}]_A [x|u_{\lambda_2}]_A^* [u_{\lambda_2} | u_{\lambda_2}] a. \end{aligned}$$

Both of statements in the right hand side of the above relation are nonzero. Indeed, we have  $[{}_A [x|u_{\lambda_1}]u_{\lambda_1} | {}_A [x|u_{\lambda_1}]u_{\lambda_1}]_{\mathcal{B}} = [x[u_{\lambda_1} | u_{\lambda_1}]_{\mathcal{B}} | x[u_{\lambda_1} | u_{\lambda_1}]_{\mathcal{B}}]_{\mathcal{B}} = [x|x]_{\mathcal{B}} [u_{\lambda_1} | u_{\lambda_1}]_{\mathcal{B}} = (e'_{j_1} + e'_{j_2})e'_{j_1} = e'_{j_1}$  and similarly  $[{}_A [x|u_{\lambda_2}]u_{\lambda_2} | {}_A [x|u_{\lambda_2}]u_{\lambda_2}]_{\mathcal{B}} = e'_{j_2}$ . Whence  ${}_A [x|u_{\lambda_1}]u_{\lambda_1}$  and  ${}_A [x|u_{\lambda_2}]u_{\lambda_2}$  and consequently  ${}_A [x|u_{\lambda_i}]u_{\lambda_i} | {}_A [x|u_{\lambda_i}]u_{\lambda_i}$ ,  $i = 1, 2$  are nonzero.

Next, put  ${}_A[x|u_{\lambda_1}] = g$ ,  ${}_A[u_{\lambda_1}|u_{\lambda_1}] = h^*h$ ,  ${}_A[x|u_{\lambda_2}] = g'$  and  ${}_A[u_{\lambda_2}|u_{\lambda_2}] = h'^*h'$ , for some  $g, g', h, h'$  in  $\mathcal{A}$ . Thus, we derive that

$$e_1a = gg^*h^*ha + g'g'^*h'^*h'a = (gh)(gh)^*a + (g'h')(g'h')^*a.$$

Take  $gh = k$  and  $g'h' = k'$ , so

$$(2.5) \quad e_1a = (kk^* + k'k'^*)a.$$

Let  $\{e_i\}$ ,  $i \in I$ , be the maximal family of minimal projections in  $\mathcal{A}$  containing  $e_1$ . Without loss of generality, we may assume that  $e_{i_1} = e_1$ , where  $i_1 \in I$ . If we put  $k = \sum_{i \in I} t_i e_i$  and  $k' = \sum_{i \in I} s_i e_i$ , then  $e_1 = kk^* + k'k'^* = \sum_{i \in I} |t_i|^2 e_i + \sum_{i \in I} |s_i|^2 e_i$ . On the other hand,

$$\begin{aligned} (gh^*h)(g'h'^*h') &= {}_A[x|u_{\lambda_1}]_A[u_{\lambda_1}|u_{\lambda_1}]_A[x|u_{\lambda_2}]_A[u_{\lambda_2}|u_{\lambda_2}] \\ &= {}_A \left[ \begin{array}{c|c} [x|u_{\lambda_1}] & u_{\lambda_1} \\ \hline & u_{\lambda_1} \end{array} \right]_A \left[ \begin{array}{c|c} [x|u_{\lambda_2}] & u_{\lambda_2} \\ \hline & u_{\lambda_2} \end{array} \right] \\ &= {}_A \left[ \begin{array}{c|c} x[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}} & u_{\lambda_1} \\ \hline & u_{\lambda_1} \end{array} \right]_A \left[ \begin{array}{c|c} x[u_{\lambda_2}|u_{\lambda_2}]_{\mathcal{B}} & u_{\lambda_2} \\ \hline & u_{\lambda_2} \end{array} \right] \\ &= {}_A[xe'_{j_1}|u_{\lambda_1}]_A[xe'_{j_2}|u_{\lambda_2}] = {}_A \left[ \begin{array}{c|c} [xe'_{j_1}|u_{\lambda_1}] & xe'_{j_2} \\ \hline & u_{\lambda_2} \end{array} \right] \\ &= {}_A \left[ \begin{array}{c|c} xe'_{j_1}[u_{\lambda_1}|xe'_{j_2}]_{\mathcal{B}} & u_{\lambda_2} \\ \hline & u_{\lambda_2} \end{array} \right] = {}_A \left[ \begin{array}{c|c} xe'_{j_1}e'_{j_2}[u_{\lambda_1}|x]_{\mathcal{B}} & u_{\lambda_2} \\ \hline & u_{\lambda_2} \end{array} \right] \\ &= {}_A[0|u_{\lambda_2}] = 0, \end{aligned}$$

which in turn implies that

$$(2.6) \quad kk^*k'k'^* = {}_A[x|u_{\lambda_1}]_A[x|u_{\lambda_1}]^*_A[u_{\lambda_1}|u_{\lambda_1}]_A[x|u_{\lambda_2}]_A[x|u_{\lambda_2}]^*_A[u_{\lambda_2}|u_{\lambda_2}] = 0.$$

Clearly,  $kk^* + k'k'^* = \sum_{i \in I} |t_i|^2 e_i + \sum_{i \in I} |s_i|^2 e_i$  have a nonzero scalar  $t_{i_2}$  or  $s_{i_2}$  for some  $i_2 \neq i_1$ . Otherwise,  $kk^* + k'k'^* = |t_{i_1}|^2 e_{i_1} + |s_{i_1}|^2 e_{i_1}$  and so  $kk^*k'k'^* = |t_{i_1} s_{i_1}|^2 e_{i_1}^2 = |t_{i_1} s_{i_1}|^2 e_{i_1} \neq 0$  which is in contradiction with (2.6).

On the other hand, if such  $t_{i_2}$  or  $s_{i_2}$  occurs in  $kk^* + k'k'^*$ , then substituting  $a$  with  $e_{i_2}$  in (2.5), we get  $e_1 e_{i_2} = (kk^* + k'k'^*) e_{i_2}$ . It leads to a contradiction, since the right hand side of this equality is greater than  $|t_{i_2}|^2 e_{i_2}$  or  $|s_{i_2}|^2 e_{i_2}$  or sum of them but the left hand side is equal to zero. Therefore  $[x|x]_{\mathcal{B}}$  cannot be of the form  $e'_{j_1} + e'_{j_2}$ . Repeating the above procedure, we realize that  $[x|x]_{\mathcal{B}}$  cannot be appear as the form  $e'_{j_1} + \dots + e'_{j_n}$  where  $n > 2$ . Hence  $[x|x]_{\mathcal{B}} = e'_j$  for some  $j \in J_0$  and so the claim holds.

Finally, if  $x \in ({}_A E)_e$  for some minimal projection  $e$  in  $\mathcal{A}$ , then  ${}_A[x|x] = \lambda e$  for some  $\lambda > 0$ . Therefore  ${}_A[(\sqrt{\lambda})^{-1}x|(\sqrt{\lambda})^{-1}x] = e$  and so using the first part  $[\sqrt{\lambda}x|\sqrt{\lambda}x]_{\mathcal{B}}$  is a minimal projection in  $\mathcal{B}$ , too. This completes the proof.  $\square$

**Theorem 2.3.** *Let  $E$  be an imprimitivity Hilbert  $H^*$ -bimodule over commutative  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . If  $x$  and  $y$  are two nonzero elements in  $E$  such that  ${}_A[x|x]$  and  ${}_A[y|y]$  are scalar multiplication of some minimal projections in  $\mathcal{A}$ , then the following four statements are equivalent:*

- (i)  $x, y$  are in Hilbert space  $({}_A E)_e$  for some minimal projection  $e$  in  $\mathcal{A}$ ;
- (ii)  $[x|y]_{\mathcal{B}} \neq 0$ ;
- (iii)  $x, y$  are in Hilbert space  $(E_{\mathcal{B}})_{e'}$  for some minimal projection  $e'$  in  $\mathcal{B}$ ;
- (iv)  ${}_A[x|y] \neq 0$ .

*Proof.* (i) $\Rightarrow$ (ii) Let us assume that  ${}_A[x|x] = \lambda e$  and  ${}_A[y|y] = \mu e$  for some positive scalars  $\lambda$  and  $\mu$ . According to Lemma 1.1 and imprimitivity of  $E$  we conclude that

$$[x|x]_{\mathcal{B}} = \left[ \frac{1}{\mu} [y|y]x|x \right]_{\mathcal{B}} = \frac{1}{\mu} [x|_A[y|y]x]_{\mathcal{B}} = \frac{1}{\mu} [x|y[y|x]_{\mathcal{B}}]_{\mathcal{B}} = \frac{1}{\mu} [x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}},$$

which implies that  $[x|y]_{\mathcal{B}} \neq 0$ .

(ii) $\Rightarrow$ (i) Suppose, on the contrary that,  ${}_A[x|x] = \lambda_1 e_1$  and  ${}_A[y|y] = \lambda_2 e_2$ , for distinct minimal projections  $e_1$  and  $e_2$  in  $\mathcal{A}$ . These conditions assure us  $e_1 x = x$  and  $e_2 y = y$ . Thus we get  $[x|y]_{\mathcal{B}} = [e_1 x|e_2 y]_{\mathcal{B}} = [e_1 e_2 x|y]_{\mathcal{B}} = [0|y]_{\mathcal{B}} = 0$  which contradicts assertion (ii).

(i) $\Rightarrow$ (iii) Put  ${}_A[x|x] = \lambda e$  and  ${}_A[y|y] = \mu e$  for some positive scalars  $\lambda$  and  $\mu$ . Applying a similar argument as before we observe that

$$[y|y]_{\mathcal{B}} = \left[ \frac{1}{\lambda} [x|x]y|y \right]_{\mathcal{B}} = \frac{1}{\lambda} [x[x|y]_{\mathcal{B}}|y]_{\mathcal{B}} = \frac{1}{\lambda} [x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}},$$

which let us conclude that  $[x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}} \neq 0$ . According to [1, Lemma 2.3],  $([x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}})^2 \neq 0$  and so  $[x|x]_{\mathcal{B}}[y|y]_{\mathcal{B}} = \frac{1}{\lambda\mu} ([x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}})^2 \neq 0$ . It enforces that  $x, y \in (E_{\mathcal{B}})_{e'}$  for some minimal projection  $e'$  in  $\mathcal{B}$ .

Implications (iii) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (iv) are proved in similar ways and so we omit them. □

**Corollary 2.1.** *Suppose that  $E$  is an imprimitivity Hilbert  $\mathcal{A}$ - $\mathcal{B}$ -bimodule over the commutative  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and also assume that  $\{u_{\lambda}\}$ ,  $\lambda \in \Lambda$  is an orthonormal system for Hilbert  $H^*$ -module  ${}_A E$ . Then  $\{u_{\lambda}\}$  is an orthonormal system for right Hilbert  $H^*$ -module  $E_{\mathcal{B}}$  if and only if each  $u_{\lambda}$ ,  $\lambda \in \Lambda$ , has its exclusive supporting projection in  $\mathcal{A}$ , it means that if  $\lambda_1, \lambda_2$  are distinct elements in  $\Lambda$  with  ${}_A[u_{\lambda_1}|u_{\lambda_1}] = e_{\lambda_1}$  and  ${}_A[u_{\lambda_2}|u_{\lambda_2}] = e_{\lambda_2}$  for some minimal projections  $e_{\lambda_1}$  and  $e_{\lambda_2}$  in  $\mathcal{A}$ , then  $e_{\lambda_1} \neq e_{\lambda_2}$ .*

*Proof.* Suppose that  $\{u_{\lambda}\}$  is an orthonormal system for Hilbert module  $E_{\mathcal{B}}$ . We assert that each  $u_{\lambda}$ ,  $\lambda \in \Lambda$ , has its exclusive supporting projection in  $\mathcal{A}$ . If not, then there are distinct elements  $u_{\mu}$  and  $u_{\nu}$  in  $\{u_{\lambda}\}$  with the same supporting projection  $e$  in  $\mathcal{A}$ . Whence  $u_{\mu}, u_{\nu} \in E_e$  and by Theorem 2.3 we have that  $[u_{\mu}|u_{\nu}]_{\mathcal{B}} \neq 0$ , which leads to a contradiction. So each  $u_{\lambda}$ ,  $\lambda \in \Lambda$ , has its exclusive supporting projection in  $\mathcal{A}$ . The reverse direction is a straightforward consequence of Theorems 2.2 and 2.3. □

Up to now we discussed the existence of basic elements and orthonormal systems for a particular class of Hilbert  $H^*$ -bimodules. We are interested to prove the existence of orthonormal bases in these space. We focus on this subject below.



**Theorem 2.4.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two commutative  $H^*$ -algebras and  ${}_A E_{\mathcal{B}}$  be an imprimitivity Hilbert  $H^*$ -bimodule. Let  $\{u_{\lambda}\}$ ,  $\lambda \in \Lambda$  and  $\{v_{\gamma}\}$ ,  $\gamma \in \Gamma$  be orthonormal bases in  ${}_A E$  and  $E_{\mathcal{B}}$ , respectively. Then the following conditions hold:*

- (i) *for each  $\lambda_0 \in \Lambda$  there is a unique  $v_{\gamma_0} \in \{v_{\gamma}\}$  and a scalar  $t_{\gamma_0}$ ,  $\gamma_0 \in \Gamma$ , with  $|t_{\gamma_0}| = 1$  in which  $u_{\lambda_0} = t_{\gamma_0} v_{\gamma_0}$ ;*
- (ii)  *$u_{\lambda_0}$  and  $v_{\gamma_0}$  have the same supporting projections in  $\mathcal{A}$  and also in  $\mathcal{B}$ ;*
- (iii) *there is a bijection between  $\Lambda$  and  $\Gamma$ .*

*Proof.* Suppose that  $\lambda_0$  is any arbitrary fixed element in  $\Lambda$ . Regarding Proposition 2.1,  $u_{\lambda_0} = \sum_{\gamma \in \Gamma'} t_{\gamma} v_{\gamma}$ , where  $\Gamma' = \{\gamma \in \Gamma : t_{\gamma} \neq 0\}$ . We claim that there is a unique  $v_{\gamma_0}$  in  $\{v_{\gamma}\}$  such that  $[u_{\lambda_0} | v_{\gamma_0}] \neq 0$ . First, note that for each  $\gamma' \in \Gamma'$ , we get

$$(2.7) \quad [u_{\lambda_0} | v_{\gamma'}]_{\mathcal{B}} = \left[ \sum_{\gamma \in \Gamma'} t_{\gamma} v_{\gamma} | v_{\gamma'} \right]_{\mathcal{B}} = t_{\gamma'} [v_{\gamma'} | v_{\gamma'}]_{\mathcal{B}} \neq 0.$$

Take  $\gamma'$  an arbitrary fixed element in  $\Gamma'$  and set  ${}_A [u_{\lambda_0} | u_{\lambda_0}] = e$ ,  ${}_A [v_{\gamma'} | v_{\gamma'}] = e_1$  for some minimal projections  $e$  and  $e_1$  in  $\mathcal{A}$ . Notice that using Theorem 2.2,  ${}_A [v_{\gamma'} | v_{\gamma'}]$  is a minimal projection in  $\mathcal{A}$ . From (2.7) and applying Theorem 2.3, it follows that  $e = e_1$ . Hence  $u_{\lambda_0}$  and  $v_{\gamma'}$  have the same supporting projection in  $\mathcal{A}$  and also in  $\mathcal{B}$ . Taking into account Corollary 2.1 and since  $\gamma' \in \Gamma'$  was arbitrary, we deduce that there is a unique  $\gamma_0 \in \Gamma$ . with  $t_{\gamma_0} \neq 0$  and  $t_{\gamma} = 0$ , for all  $\gamma \in \Gamma \setminus \{\gamma_0\}$ . In fact, suppose that there are two distinct elements  $\gamma_1$  and  $\gamma_2$  in  $\Gamma'$  in which both of  $t_{\gamma_1}$  and  $t_{\gamma_2}$  are nonzero, then using the similar argument as above we conclude that  $v_{\gamma_1}$  and  $v_{\gamma_2}$  have the same supporting projections in  $\mathcal{A}$  and also in  $\mathcal{B}$ . It enforces that  $[v_{\gamma_1} | v_{\gamma_2}]_{\mathcal{B}} \neq 0$ , which is a contradiction. Therefore  $u_{\lambda_0} = t_{\gamma_0} v_{\gamma_0}$  and the claim holds.

On the other hand, if  $[u_{\lambda_0} | u_{\lambda_0}]_{\mathcal{B}} = e'$  for some minimal projection  $e'$  in  $\mathcal{B}$ , then we have

$$e' = [u_{\lambda_0} | u_{\lambda_0}]_{\mathcal{B}} = [t_{\gamma_0} v_{\gamma_0} | t_{\gamma_0} v_{\gamma_0}]_{\mathcal{B}} = |t_{\gamma_0}|^2 [v_{\gamma_0} | v_{\gamma_0}]_{\mathcal{B}} = |t_{\gamma_0}|^2 e'.$$

It follows that  $|t_{\gamma_0}| = 1$ . It proves items (i) and (ii). For proving (iii) consider the mapping  $\phi : \Lambda \rightarrow \Gamma$ , which assigns each  $u_{\lambda_0}$  to  $v_{\gamma_0}$ , where  $\lambda_0 \in \Lambda$ ,  $\gamma_0 \in \Gamma$  and  $v_{\gamma_0}$  is chosen as the proof of the previous parts. It is readily verified that  $\phi$  is an injection.

Surjectivity of  $\phi$  follows from changing the roles of  $\{u_{\lambda}\}$  and  $\{v_{\gamma}\}$  in the proof of (i). □

**Corollary 2.2.** *Suppose that  $E$  is an imprimitivity Hilbert  $\mathcal{A}$ - $\mathcal{B}$ -bimodule over the commutative  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . Then  $\{u_{\lambda}\}$ ,  $\lambda \in \Lambda$ , is an orthonormal basis for the Hilbert  $H^*$ -module  ${}_A E$  if and only if it is an orthonormal basis for the Hilbert  $H^*$ -module  $E_{\mathcal{B}}$ .*

*Proof.* Let  $\{u_{\lambda}\}$ ,  $\lambda \in \Lambda$ , be an orthonormal basis in  ${}_A E$ . It is an immediate consequence of Theorems 2.2 and 2.3, that  $\{u_{\lambda}\}$ ,  $\lambda \in \Lambda$ , is an orthonormal system for  $E_{\mathcal{B}}$ . So it is enough to prove that  $\{u_{\lambda}\}$  generates a dense submodule for  $E_{\mathcal{B}}$ . Using Theorem 2.4, we may consider  $\{v_{\lambda}\}$ ,  $\lambda \in \Lambda$  to be an orthonormal basis for  $E_{\mathcal{B}}$  such

that  $u_\lambda = t_\lambda v_\lambda$  for each  $\lambda \in \Lambda$  and some scalar  $t_\lambda$  with  $|t_\lambda| = 1$ . Let us denote by  $\mathcal{F}$  the family of finite subsets of  $\Lambda$ . Now if  $x \in E$ , then  $x = \sum_{\lambda \in \Lambda} \mu'_\lambda v_\lambda$  and thus we have

$$(2.8) \quad \left\| x - \sum_{\lambda \in \Lambda'} \mu'_\lambda v_\lambda \right\|_{\mathcal{B}} = \left\| x - \sum_{\lambda \in \Lambda'} \frac{\mu'_\lambda}{t_\lambda} u_\lambda \right\|_{\mathcal{B}},$$

for each  $\Lambda' \in \mathcal{F}$ . On using (2.8) we conclude that  $\{u_\lambda\}$  generates a dense submodule of  $E_{\mathcal{B}}$ , too. □

In the light of the previous corollary, the following definition is reasonable.

**Definition 2.3.** Let  $E$  be an imprimitivity Hilbert  $\mathcal{A}$ - $\mathcal{B}$ -bimodule over the commutative  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and  $\{u_\lambda\}$ ,  $\lambda \in \Lambda$ , be an orthonormal basis for Hilbert  $H^*$ -module  ${}_A E$  (or  $E_{\mathcal{B}}$ ). Then we say  $\{u_\lambda\}$  is an orthonormal basis for Hilbert  $H^*$ -bimodule  ${}_A E_{\mathcal{B}}$ .

In the sequel, we investigate the relationship between two topologies induced by  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ .

In general suppose that  $H$  is a Hilbert space with both inner products  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_1$  and corresponding norms  $\|\cdot\|$  and  $\|\cdot\|_1$ , respectively. If  $\|x\|_1 \leq \beta \|x\|$  for each  $x \in H$  and some  $\beta > 0$ , then there is a positive operator  $K \in B(H)$  (w.r.t.  $\|\cdot\|$ ) such that  $K$  is injective and moreover  $\langle x, y \rangle_1 = \langle Kx, y \rangle$ , for all  $x, y$  in  $H$ . On the other hand,  $\|\cdot\|$  and  $\|\cdot\|_1$  give rise to the same topology if  $K$  has an inverse in  $B(H)$  (see [6, Page 162]). Accordingly, if  ${}_A E_{\mathcal{B}}$  is a Hilbert  $H^*$ -bimodule, then  ${}_A \|\cdot\|$  and  $\|\cdot\|_{\mathcal{B}}$  are equivalent if and only if there is a positive invertible operator  $K$  in  $B(E)$  (w.r.t.  ${}_A \|\cdot\|$ ) in which  $\langle x, y \rangle_{\mathcal{B}} = {}_A \langle Kx, y \rangle$ , for all  $x, y$  in  $E$ . Further, some more interesting results can be found in the case that  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are commutative.

**Proposition 2.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two commutative  $H^*$ -algebras and  ${}_A E_{\mathcal{B}}$  be an imprimitivity Hilbert  $H^*$ -bimodule. Assume that all minimal projections in  $\mathcal{A}$  and  $\mathcal{B}$  have norms equal to some  $\alpha \geq 1$ . Then  ${}_A \|x\| = \|x\|_{\mathcal{B}}$  for each  $x \in E$ .

*Proof.* Let  $\{e_i\}$ ,  $i \in I$ , be the family of all minimal projections in  $\mathcal{A}$  and  $\{u_\lambda\}$ ,  $\lambda \in \Lambda$ , be an orthonormal basis for  ${}_A E_{\mathcal{B}}$  with  ${}_A [u_\lambda | u_\lambda] = e_{i_\lambda}$  for each  $\lambda \in \Lambda$  and some  $i_\lambda \in I$ . Take  $x \in E$ , then  $x = \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda$  for some scalars  $\mu_\lambda$  ( $\lambda \in \Lambda$ ) and thus we have

$$\begin{aligned} {}_A \|x\|^2 &= \text{tr}_A({}_A [x|x]) = \text{tr}_A \left( {}_A \left[ \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda \middle| \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda \right] \right) \\ &= {}_A \left( \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda \middle| \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda \right) = \sum_{\lambda \in \Lambda} |\mu_\lambda|_A^2 (u_\lambda | u_\lambda) = \sum_{\lambda \in \Lambda} |\mu_\lambda|^2 {}_A \|e_{i_\lambda}\|^2 \\ &= \sum_{\lambda \in \Lambda} |\mu_\lambda|^2 \alpha^2. \end{aligned}$$

Since the representation of  $x = \sum_{\lambda \in \Lambda} \mu_\lambda u_\lambda$  is the same with respect to both of norms  ${}_A \|\cdot\|$  and  $\|\cdot\|_{\mathcal{B}}$ , then similar relations proves that  $\|x\|_{\mathcal{B}}^2 = \sum_{\lambda \in \Lambda} |\mu_\lambda|^2 \alpha^2$ . So we achieve our goal. □

**Proposition 2.4.** *Let  ${}_A E_B$  be an imprimitivity Hilbert  $H^*$ -bimodule over commutative  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and let  $x, y \in E$ . Then  ${}_A[x|y] = 0$  if and only if  $[x|y]_B = 0$ .*

*Proof.* In the forward direction, suppose that  ${}_A[x|y] = 0$ . Let  $\{u_\lambda\}$ ,  $\lambda \in \Lambda$  be an orthonormal basis for Hilbert  $H^*$ -bimodule  ${}_A E_B$ , then for some suitable scalars  $t_\lambda$  and  $s_\mu$  ( $\lambda, \mu \in \Lambda$ ),  $x = \sum_{\lambda \in \Lambda'} t_\lambda u_\lambda$  and  $y = \sum_{\mu \in \Lambda''} s_\mu u_\mu$ , where  $\Lambda' = \{\lambda \in \Lambda : t_\lambda \neq 0\}$  and  $\Lambda'' = \{\mu \in \Lambda : s_\mu \neq 0\}$ . These allow us to write  ${}_A[x|y] = \sum_{\lambda \in \Lambda'} \sum_{\mu \in \Lambda''} t_\lambda \overline{s_\mu} [u_\lambda|u_\mu] = 0$ , which in turn implies that  $\Lambda' \cap \Lambda'' = \emptyset$ . It follows from this reasoning and by applying Corollary 2.2, that  $[x|y]_B = \sum_{\lambda \in \Lambda'} \sum_{\mu \in \Lambda''} t_\lambda \overline{s_\mu} [u_\lambda|u_\mu]_B = 0$ . The inverse implication is shown similarly.  $\square$

In the sequel, we give an example to verify usefulness of our results.

*Example 2.2.* Let  $\mathcal{A}$  be the commutative real  $H^*$ -algebra  $\left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$  together with the usual operations of addition and scalar multiplication and endowed with componentwise multiplication. Adjoint and inner product are defined by

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix}^* = \begin{pmatrix} a & a \\ b & b \end{pmatrix},$$

and

$${}_A \left\langle \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \begin{pmatrix} c & c \\ d & d \end{pmatrix} \right\rangle = k(ac + bd),$$

where  $k$  is a positive number greater or equal to 1. Obviously,  $\tau(\mathcal{A}) = \mathcal{A}$  and linear functional  $\text{tr}_A : \tau(\mathcal{A}) \rightarrow \mathbb{R}$  defined by  $\text{tr}_A \left( \begin{pmatrix} a & a \\ b & b \end{pmatrix} \right) = k(a + b)$  is positive. Similarly,

consider the commutative real  $H^*$ -algebra  $\mathcal{B} = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$  together with the operations of addition, scalar multiplication, componentwise multiplication and adjoint which are defined as the similar way as  $\mathcal{A}$  and inner product is defined by

$$\left\langle \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} c & d \\ c & d \end{pmatrix} \right\rangle_B = p(ac + bd),$$

for some positive number  $p \geq 1$ . Evidently,  $\tau(\mathcal{B}) = \mathcal{B}$  and linear functional  $\text{tr}_B : \tau(\mathcal{B}) \rightarrow \mathbb{R}$  is defined by  $\text{tr}_B \left( \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right) = p(a + b)$  is positive. It is routine to verify that

$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$  and  $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$  are the sets of all minimal projections in  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Now, take  $E$  the space of all  $2 \times 2$  matrices  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ ,  $a, b \in \mathbb{R}$ , and define left module multiplication  $\cdot : \mathcal{A} \times E \rightarrow E$  and right module multiplication  $\cdot : E \times \mathcal{B} \rightarrow E$  by

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

and

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a & b \\ a & b \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix},$$

respectively. Also, define  $\tau(\mathcal{A})$ - and  $\tau(\mathcal{B})$ -valued inner products  ${}_{\mathcal{A}}[\cdot|\cdot] : E \times E \rightarrow \tau(\mathcal{A})$  and  $[\cdot|\cdot]_{\mathcal{B}} : E \times E \rightarrow \tau(\mathcal{B})$  by

$${}_{\mathcal{A}} \left[ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \right] = \begin{pmatrix} ac & ac \\ bd & bd \end{pmatrix}$$

and

$$\left[ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} ac & bd \\ ac & bd \end{pmatrix},$$

respectively. It is not hard to see that  $E$  is an imprimitivity  $\mathcal{A}$ - $\mathcal{B}$  Hilbert bimodule.

Next, we point that the set  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$  is an orthonormal basis for  ${}_{\mathcal{A}}E_{\mathcal{B}}$ . This holds since

$${}_{\mathcal{A}} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$${}_{\mathcal{A}} \left[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

and with the help of Corollaries 2.1 and 2.2 we get the desired result. Furthermore, assume that  $p = k$ , then all minimal projections in  $\mathcal{A}$  and  $\mathcal{B}$  have the same norm  $\sqrt{k}$ . Therefore,  ${}_{\mathcal{A}}\|\cdot\| = \|\cdot\|_{\mathcal{B}}$  by Proposition 2.3.

**Theorem 2.5.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two commutative  $H^*$ -algebras and  ${}_{\mathcal{A}}E_{\mathcal{B}}$  be a full Hilbert  $H^*$ -bimodule. Then  $H^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are isomorphic.*

*Proof.* Consider  $\{e_i\}, i \in I$ , is the family of all minimal projections in  $\mathcal{A}$ ,  $\{u_{\lambda}\}, \lambda \in \Lambda$  is an orthonormal basis for  ${}_{\mathcal{A}}E_{\mathcal{B}}$  and  $a$  is an arbitrary element in  $\mathcal{A}$ . By the commutativity of  $\mathcal{A}$ ,  $a = \sum_{i \in I} \mu_i e_i$ . According to Proposition 2.2, for each  $i \in I$  there exists  $\lambda_i \in \Lambda$  such that  $e_i = {}_{\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}]$ . Hence,  $a = \sum_{i \in I} \mu_i {}_{\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}]$ . Define a mapping  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  by  $\varphi(a) := \sum_{i \in I} \mu_i [u_{\lambda_i}|u_{\lambda_i}]_{\mathcal{B}}$ , where  $a = \sum_{i \in I} \mu_i {}_{\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}]$ . In view of Theorem 2.3, we observe that  $a = \sum_{i \in I} \mu_i {}_{\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}] = 0$  if and only if  $\varphi(a) = \sum_{i \in I} \mu_i [u_{\lambda_i}|u_{\lambda_i}]_{\mathcal{B}} = 0$ . This shows that  $\varphi$  is well defined and injective. It is easy to verify that  $\varphi$  is a morphism, i.e.,  $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$ ,  $\varphi(a_1 a_2) = \varphi(a_1) \varphi(a_2)$  and  $\varphi(a^*) = \varphi(a)^*$ , for all  $a_1, a_2, a$  in  $\mathcal{A}$ . The surjectivity of  $\varphi$  is evident. This is somewhat similar to the situation discussed for constructing  $\varphi$ . Therefore,  $\varphi$  is an isomorphism.  $\square$

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<sup>1</sup>DEPARTMENT OF MATHEMATICS,  
MASHHAD BRANCH, ISLAMIC AZAD UNIVERSITY,  
MASHHAD, IRAN  
*Email address:* [khaneghir@mshdiau.ac.ir](mailto:khaneghir@mshdiau.ac.ir)  
*Email address:* [zahra-nm79@yahoo.com](mailto:zahra-nm79@yahoo.com)

<sup>2</sup>DEPARTMENT OF PURE MATHEMATICS,  
FARHANGIAN UNIVERSITY,  
MASHHAD, IRAN  
*Email address:* [MMkh926@gmail.com](mailto:MMkh926@gmail.com)

\*CORRESPONDING AUTHOR