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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS TO A SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH IMPULSES

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ABSTRACT. In this work, we obtain necessary and sufficient conditions for oscillation of solutions of second-order neutral impulsive differential system

$$\begin{cases} \Big(r(t)\big(z'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) = 0, & t \geq t_0,\, t \neq \lambda_k, \\ \Delta\Big(r(\lambda_k)\big(z'(\lambda_k)\big)^{\gamma}\Big) + \sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k)) = 0, & k = 1,2,3,\dots, \end{cases}$$

where $z(t) = x(t) + p(t)x(\tau(t))$. Under the assumption $\int_0^{\infty} \big(r(\eta)\big)^{-1/\gamma} d\eta = \infty$, we consider two cases when $\gamma > \alpha_i$ and $\gamma < \alpha_i$. Our main tool is Lebesgue's Dominated Convergence theorem. Examples are given to illustrate our main results and we state an open problem.

1. INTRODUCTION

In this article we consider the neutral impulsive differential system

$$\text{(1.1)} \qquad \begin{cases} \Big(r(t)\big(z'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) = 0, & t \geq t_0,\, t \neq \lambda_k, \\ \Delta\Big(r(\lambda_k)\big(z'(\lambda_k)\big)^{\gamma}\Big) + \sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k)) = 0, & k = 1,2,3,\dots, \end{cases}$$

where

$$z(t) = x(t) + p(t)x(\tau(t)), \quad \Delta x(a) = \lim_{s\to a^+} x(s) - \lim_{s\to a^-} x(s),$$

the functions $p, q_i, h_i, r, \sigma_i, \tau$ are continuous that satisfy the conditions stated below and assume that the sequence $\{\lambda_k\}$ satisfies $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k < \cdots$ as $k \to \infty$

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and γ and α_i are the quotient of two odd positive integers and λ_k's are fixed moment of impulsive effects..

(A1) $\sigma_i \in C([0,\infty),\mathbb{R}_+)$, $\tau \in C^2([0,\infty),\mathbb{R}_+)$, $\sigma_i(t) < t$, $\tau(t) < t$, $\lim_{t\to\infty}\sigma_i(t) = \infty$, $\lim_{t\to\infty}\tau(t) = \infty$.

(A2) $r \in C^1([0,\infty),\mathbb{R}_+)$, $q_i, h_i \in C([0,\infty),\mathbb{R}_+)$, $0 < r(t)$, $0 \le q_i(t)$, $0 \le h_i(t)$ for all $t \ge 0$ and $i = 1,2,\dots,m$, $\sum q_i(t)$ is not identically zero in any interval $[b,\infty)$.

(A3) $\int_0^\infty r^{-1/\gamma}(s)\,ds = \infty$ and let $\Pi(t) = \int_0^t r^{-1/\gamma}(\eta)\,d\eta$.

(A4) $-1 < -p_0 \le p(t) \le 0$ for $t \ge t_0$.

(A5) There exists a differentiable function $\sigma_0(t)$ such that $0 < \sigma_0(t) = \min\{\sigma_i(t) : t \ge t^*\}$ and $\sigma_0'(t) \ge \alpha$ for $t \ge t^*$, $\alpha > 0$, $i = 1,2,\dots,m$.

The main feature of this article is having conditions that are both necessary and sufficient for the oscillation of all solutions to (1.1). Sufficient conditions for the oscillation and nonoscillation of all solutions to the first and second order neutral impulsive differential systems are provided in [12–15, 18–22]. The necessary and sufficient conditions for oscillation of all solutions to the first order neutral impulsive differential systems are discussed in [20, 21]. In this work, our main aim is to present the necessary and sufficient conditions for oscillation of all solutions of (1.1).

In 2011, Dimitrova and Donev [13–15] have considered the first order impulsive differential system of the form

$$\begin{cases} \Big(x(t) + p(t)x(\tau(t))\Big)' + q(t)x(\sigma(t)) = 0, & t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(x(\lambda_k) + p(\lambda_k)x(\tau(\lambda_k))\Big) + q(\lambda_k)x(\sigma(\lambda_k)) = 0, & k \in \mathbb{N}, \end{cases} \tag{1.2}$$

and established several sufficient conditions for oscillation of the solutions of (1.2).

In 2014, Tripathy [19] have established sufficient conditions for oscillation of all solutions of

$$\begin{cases} \Big(x(t) + p(t)x(t-\tau)\Big)' + q(t)f\Big(x(t-\sigma)\Big) = 0, & t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(x(\lambda_k) + p(\lambda_k)x(\tau(\lambda_k-\tau))\Big) + q(\lambda_k)f\Big(x(\sigma(\lambda_k-\sigma))\Big) = 0, & k \in \mathbb{N}. \end{cases} \tag{1.3}$$

In 2015, Tripathy and Santra [20] obtained the necessary and sufficient conditions for oscillatory and asymptotic behavior of solutions of

$$\begin{cases} \Big(x(t) + p(t)x(t-\tau)\Big)' + q(t)f\Big(x(t-\sigma)\Big) = g(t), & t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(x(\lambda_k) + p(\lambda_k)x(\lambda_k-\tau)\Big) + q(\lambda_k)f\Big(x(\lambda_k-\sigma)\Big) = h(\lambda_k), & k \in \mathbb{N}. \end{cases}$$

In 2016, Tripathy, Santra and Pinelas [21] obtained necessary and sufficient conditions of (1.3). In the subsequent year, Tripathy and Santra [22] established sufficient conditions for oscillation and existence of positive solutions of

$$\begin{cases} \Big(r(t)\Big(x(t) + p(t)x(t-\tau)\Big)'\Big)' + q(t)f\Big(x(t-\sigma)\Big) = 0, & t \ne \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(r(\lambda_k)\Big(x(\lambda_k) + p(\lambda_k)x(\lambda_k-\tau)\Big)'\Big) + q(\lambda_k)f\Big(x(\lambda_k-\sigma)\Big) = 0, & k \in \mathbb{N}. \end{cases}$$

In 2018, Santra [18] established sufficient conditions for oscillations of solutions of

$$\begin{cases} \Big(r(t)\Big(x(t)+p(t)x(\tau(t))\Big)'\Big)' + q(t)f\Big(x(\sigma(t))\Big) = 0, \quad t \neq \lambda_k,\ k \in \mathbb{N}, \\ \Delta\Big(r(\lambda_k)\Big(x(\lambda_k)+p(\lambda_k)x(\tau(\lambda_k))\Big)'\Big) + q(\lambda_k)f\Big(x(\sigma(\lambda_k))\Big) = 0, \quad k \in \mathbb{N}. \end{cases}$$

By a solution x we mean a function differentiable on $[t_0, \infty)$ such that $z(t)$ and $z'(t)$ are differentiable for $t \neq t_k$, and $z(t)$ is left continuous at λ_k and has right limit at λ_k, and x satisfies (1.1). We restrict our attention to solutions for which $\sup_{t \geq b} |x(t)| > 0$ for every $b \geq 0$. A solution is called oscillatory it has arbitrarily large zeros; otherwise is non-oscillatory.

To define a particular solution, we need an initial function $\phi(t)$ which is twice differentiable for t in the interval

$$\min\Big\{\inf\{\tau(t) : t_0 \leq t\},\ \inf\{\sigma_i(t) : t_0 \leq t, i = 1, 2, \dots, m\}\Big\} \leq t.$$

Then a solution is obtained using the method of steps: When replacing $x(\tau(t))$ by $\phi(\tau(t))$, and $x(\sigma_i(t))$ by $\phi(\sigma_i(t))$ in (1.1), we obtain a second-order differential equation. We solve this equation taking into account discrete equation of (1.1), say on an interval $[t_0, t_1]$. Then repeat the process starting at $t = t_1$.

2. Necessary and Sufficient Conditions

Lemma 2.1. *Assume that* (A1)-(A4) *hold for* $t \geq t_0$. *If* x *is an eventually positive solution of* (1.1), *then* z *satisfies any one of the following two cases:*

(i) $z(t) < 0$, $z'(t) > 0$, $\Big(r(z')^\gamma\Big)'(t) \leq 0$;

(ii) $z(t) > 0$, $z'(t) > 0$, $\Big(r(z')^\gamma\Big)'(t) \leq 0$,

for all sufficiently large t.

Proof. Let x be an eventually positive solution. Then by (A1) there exists a t^* such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geq t^*$ and $i = 1, 2, \dots, m$. From (1.1) it follows that

$$\begin{aligned} \Big(r(t)\Big(z'(t)\Big)^\gamma\Big)' &= -\sum_{i=1}^m q_i(t)x^{\alpha_i}(\sigma_i(t)) \leq 0, \quad \text{for } t \neq \lambda_k, \\ \Delta\Big(r(\lambda_k)\Big(z'(\lambda_k)\Big)^\gamma\Big) &= -\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k)) \leq 0, \quad \text{for } k = 1, 2, \dots \end{aligned} \tag{2.1}$$

Therefore, $r(t)\Big(z'(t)\Big)^\gamma$ is non-increasing for $t \geq t^*$, including jumps of discontinuity. Next we show the $r(t)\Big(z'(t)\Big)^\gamma$ is positive. By contradiction assume that $r(t)\Big(z'(t)\Big)^\gamma \leq 0$ at a certain time $t \geq t^*$. Using that $\sum q_i$ is not identically zero on any interval $[b, \infty)$, and by (2.1), there exists $t_2 \geq t^*$ such that

$$r(t)\Big(z'(t)\Big)^\gamma \leq r(t_2)\Big(z'(t_2)\Big)^\gamma < 0, \quad \text{for all } t \geq t_2.$$

Recall that γ is the quotient of two positive odd integers. Then

$$z'(t) \leq \left(\frac{r(t_2)}{r(t)}\right)^{1/\gamma} z'(t_2), \quad \text{for } t \geq t_2.$$

Since $r(\lambda_k)\big(z'(\lambda_k)\big)^\gamma \leq r(t_2)\big(z'(t_2)\big)^\gamma < 0$ for all $\lambda_k \geq t_2$. Integrating from t_2 to t, we have

$$\begin{aligned} z(t) &\leq z(t_2) + \sum_{t_2 \leq \lambda_k < \infty} z'(\lambda_k) + \big(r(t_2)\big)^{1/\gamma} z'(t_2)\big(\Pi(t) - \Pi(t_2)\big) \\ &\leq z(t_2) + \big(r(t_2)\big)^{1/\gamma} z'(t_2)\big(\Pi(t) - \Pi(t_2)\big) \to -\infty, \end{aligned}$$

as $t \to \infty$ due to (A3). Now, we consider the following two possibilities.

If x is unbounded, then there exists a sequence $\{\eta_k\} \to \infty$ such that $x(\eta_k) = \sup\{x(\eta) : \eta \leq \eta_k\}$. By $\tau(\eta_k) \leq \eta_k$, we have $x(\tau(\eta_k)) \leq x(\eta_k)$ and hence

$$z(\eta_k) = x(\eta_k) + p(\eta_k)x(\tau(\eta_k)) \geq (1 + p(\eta_k))x(\eta_k) \geq (1 - p_0))x(\eta_k) \geq 0,$$

which contradicts $\lim_{k\to\infty} z(t) = -\infty$. Recall that $\{\lambda_k\}$ are the sequence of points for $t \geq \lambda_k$, then by similar argument we can show that $z(\lambda_k) \geq 0$ to get a contradiction to $\lim_{k\to\infty} z(t) = -\infty$. Therefore, $r(t)\big(z'(t)\big)^\gamma > 0$ for all $t \geq t^*$.

If x is bounded, then z is also bounded, which is a contradiction to $\lim_{k\to\infty} z(t) = -\infty$.

From $r(t)\big(z'(t)\big)^\gamma > 0$ and $r(t) > 0$, it follows that $z'(t) > 0$. Then there is $t_1 \geq t^*$ such that z satisfies only one of two cases (i) and (ii). This completes the proof. □

Lemma 2.2. *Assume that* (A1)-(A4) *hold. If x is an eventually positive solution of* (1.1)*, then any one of following two cases exists:*

(1) *if z satisfies* (i), $\lim_{t\to\infty} x(t) = 0$*;*
(2) *if z satisfies* (ii)*, there exist $t_1 \geq t_0$ and $\delta > 0$ such that*

$$0 < z(t) \leq \delta\Pi(t), \tag{2.2}$$

$$\big(\Pi(t) - \Pi(t_1)\big)\Big[\int_t^\infty \sum_{i=1}^m q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,d\zeta + \sum_{\lambda_k \geq t}\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]^{1/\gamma} \leq z(t) \leq x(t), \tag{2.3}$$

for all $t \geq t_1$.

Proof. Let x be an eventually positive solution. Then by (A1) there exists a t^* such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geq t^*$ and $i = 1, 2, \ldots, m$. Then Lemma 2.1 holds and we have following two possible cases.

Case 1. Let z satisfies (i) for all $t \geq t_1$. Note that $\lim_{t\to\infty} z(t)$ exists and by (A1), $\limsup_{t\to\infty} x(t) = \limsup_{t\to\infty} x(\tau(t))$. Then $0 > z(t) \geq x(t) - p_0x(\tau(t))$ implies

$$0 \geq \lim_{t\to\infty} z(t) \geq \lim_{t\to\infty} \Big[x(t) - p_0x(\tau(t))\Big] \geq (1 - p_0)\limsup_{t\to\infty} x(t).$$

Since $(1-p_0) > 0$, it follows that $\limsup_{t\to\infty} x(t) = 0$, hence $\lim_{t\to\infty} x(t) = 0$ for $t \neq \lambda_k$, $k \in \mathbb{N}$. We may note that $\{x(\lambda_k - 0)\}_{k\in\mathbb{N}}$ and $\{x(\lambda_k + 0)\}_{k\in\mathbb{N}}$ are sequences of real numbers and because of continuity of x

$$\lim_{k\to\infty} x(\lambda_k - 0) = 0 = \lim_{k\to\infty} x(\lambda_k + 0)$$

due to $\liminf_{t\to\infty} x(t) = 0 = \limsup_{t\to\infty} x(t)$. Hence, $\lim_{t\to\infty} x(t) = 0$ for all t and λ_k, $k \in \mathbb{N}$.

Case 2. Let z satisfies (ii) for all $t \geq t_1$. Note that $x(t) \geq z(t)$ and z is positive and increasing so x cannot converge to zero. From $r(t)\Big(z'(t)\Big)^\gamma$ being non-increasing, there exists a constant $\delta > 0$ and $t \geq t_1$ such that $\Big(r(t)\Big)^{1/\gamma} z'(t) \leq \delta$ and hence $z(t) \leq \delta\Pi(t)$ for $t \geq t_1$.

Since $r(t)\Big(z'(t)\Big)^\gamma$ is positive and non-increasing, $\lim_{t\to\infty} r(t)\Big(z'(t)\Big)^\gamma$ exists and is non-negative. Integrating (1.1) from t to a, we have

$$r(a)\Big(z'(a)\Big)^\gamma - r(t)\Big(z'(t)\Big)^\gamma = -\int_t^a \sum_{i=1}^m q_i(\eta) x^{\alpha_i}(\sigma_i(\eta))\, d\eta + \sum_{t\leq\lambda_k<a} \Delta\Big(r(\lambda_k) z'(\lambda_k)\Big)^\gamma.$$

Computing the limit as $a \to \infty$

$$r(t)\Big(z'(t)\Big)^\gamma \geq \int_t^\infty \sum_{i=1}^m q_i(\eta) x^{\alpha_i}(\sigma_i(\eta))\, d\eta + \sum_{\lambda_k\geq t} \sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k)). \tag{2.4}$$

Then

$$z'(t) \geq \Big[\frac{1}{r(t)}\Big[\int_t^\infty \sum_{i=1}^m q_i(\eta) x^{\alpha_i}(\sigma_i(\eta))\, d\eta + \sum_{t\leq\lambda_k} \sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\Big]^{1/\gamma}.$$

Since $z(t_1) > 0$, integrating the above inequality yields

$$z(t) \geq \int_{t_1}^t \Big[\frac{1}{r(\eta)}\Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\, d\zeta + \sum_{\eta\leq\lambda_k} \sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\Big]^{1/\gamma} d\eta.$$

Since the integrand is positive, we can increase the lower limit of integration from η to t, and then use the definition of $\Pi(t)$, to obtain

$$z(t) \geq \Big(\Pi(t) - \Pi(t_1)\Big)\Big[\int_t^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\, d\zeta + \sum_{t\leq\lambda_k} \sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k))\Big]^{1/\gamma},$$

which yields (2.3). □

2.1. **The Case $\alpha_i < \gamma$.** In this subsection, we assume that there exists a constant β_1, the quotient of two positive odd integers such that $0 < \alpha_i < \beta_1 < \gamma$.

Theorem 2.1. *Under assumptions* (A1)-(A4)*, each solution of* (1.1) *is either oscillatory or converge to zero if and only if*

$$\int_0^\infty \sum_{i=1}^m q_i(\eta) \Pi^{\alpha_i}(\sigma_i(\eta))\, d\eta + \sum_{k=1}^\infty \sum_{i=1}^m h_i(\lambda_k) \Pi^{\alpha_i}(\sigma_i(\lambda_k)) = \infty. \tag{2.5}$$

Proof. We prove the sufficiency by contradiction. Initially, we assume that a solution x is eventually positive which does not converge to zero. So, Lemma 2.1 holds and z satisfies any one of two cases (i) and (ii). In Lemma 2.2, Case 1 leads to $\lim_{t\to\infty} x(t) = 0$ which is a contradiction.

For Case 2, we can find a $t_1 > 0$ such that

$$x(t) \geq z(t) \geq \Big(\Pi(t) - \Pi(t_1)\Big) w^{1/\gamma}(t) \geq 0, \quad \text{for } t \geq t_1,$$

where

$$w(t) = \int_t^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\, d\zeta + \sum_{\lambda_k \geq t} \sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k)) \geq 0.$$

As $\lim_{t\to\infty} \Pi(t) = \infty$, there exists $t_2 \geq t_1$, such that $\Pi(t) - \Pi(t_1) \geq \frac{1}{2}\Pi(t)$ for $t \geq t_2$ and hence

$$z(t) \geq \frac{1}{2}\Pi(t) w^{1/\gamma}(t). \tag{2.6}$$

Note that w is left continuous at λ_k,

$$\begin{aligned} w'(t) &= -\sum_{i=1}^m q_i(t) x^{\alpha_i}(\sigma_i(t)), \quad \text{for } t \neq \lambda_k, \\ \Delta w(\lambda_k) &= -\sum_{i=1}^m h_i(\lambda_k) x^{\alpha_i}(\sigma_i(\lambda_k)) \leq 0. \end{aligned}$$

Thus w is non-negative and non-increasing for $t \geq t_2$. Using (2.2), $\alpha_i - \beta_1 < 0$ and (2.6), we have

$$\begin{aligned} x^{\alpha_i}(t) &\geq z^{\alpha_i - \beta_1}(t) z^{\beta_1}(t) \geq (\delta \Pi(t))^{\alpha_i - \beta_1} z^{\beta_1}(t) \\ &\geq \Big(\delta \Pi(t)\Big)^{\alpha_i - \beta_1} \Big(\frac{\Pi(t) w^{1/\gamma}(t)}{2}\Big)^{\beta_1} = \frac{\delta^{\alpha_i - \beta_1}}{2^{\beta_1}} \Pi^{\alpha_i}(t) w^{\beta_1/\gamma}(t), \quad \text{for } t \geq t_2. \end{aligned}$$

Since w is non-increasing, $\frac{\beta_1}{\gamma} > 0$, and $\sigma_i(\eta) < \eta$, it follows that

$$x^{\alpha_i}(\sigma_i(\eta)) \geq \frac{\delta^{\alpha_i - \beta_1}}{2^{\beta_1}} \Pi^{\alpha_i}(\sigma_i(\eta)) w^{\beta_1/\gamma}(\sigma_i(\eta)) \geq \frac{\delta^{\alpha_i - \beta_1}}{2^{\beta_1}} \Pi^{\alpha_i}(\sigma_i(\eta)) w^{\beta_1/\gamma}(\eta). \tag{2.7}$$

Now, we have

$$\Big(w^{1-\beta_1/\gamma}(t)\Big)' = \Big(1 - \frac{\beta_1}{\gamma}\Big) w^{-\beta_1/\gamma}(t) \Big(-\sum_{i=1}^m q_i(t) x^{\alpha_i}(\sigma_i(t))\Big), \quad \text{for } t \neq \lambda_k. \tag{2.8}$$

To estimate the discontinuities of $w^{1-\beta_1/\gamma}$ we use a Taylor polynomial of order 1 for the function $h(x) = x^{1-\beta_1/\gamma}$, with $0 < \beta_1 < \gamma$ about $x = a$

$$b^{1-\beta_1/\gamma} - a^{1-\beta_1/\gamma} \leq \Big(1 - \frac{\beta_1}{\gamma}\Big) a^{-\beta_1/\gamma}(b - a).$$

Then $\Delta w^{1-\beta_1/\gamma}(\lambda_k) \leq \left(1-\frac{\beta_1}{\gamma}\right) w^{-\beta_1/\gamma}(\lambda_k)\Delta w(\lambda_k)$. Integrating (2.8) from t_2 to t, we have

(2.9)

$$\begin{aligned}
w^{1-\beta_1/\gamma}(t_2) \geq& \Big(1-\frac{\beta_1}{\gamma}\Big)\Big[-\int_{t_2}^t w^{-\beta_1/\gamma}(\eta)w'(\eta)\,d\eta - \sum_{t_2\leq\lambda_k<t} w^{-\beta_1/\gamma}(\lambda_k)\Delta w(\lambda_k)\Big] \\
=& \Big(1-\frac{\beta_1}{\gamma}\Big)\Big[\int_{t_2}^t w^{-\beta_1/\gamma}(\eta)\Big(\sum_{i=1}^m q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\Big)\,d\eta \\
&+ \sum_{t_2\leq\lambda_k<t} w^{-\beta_1/\gamma}(\lambda_k)\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big] \\
\geq& \frac{1-\frac{\beta_1}{\gamma}}{2^{\beta_1}\delta^{(\beta_1-\alpha_i)}}\Big[\int_{t_2}^t \sum_{i=1}^m q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta)))\,d\eta + \sum_{t_2\leq\lambda_k<t}\sum_{i=1}^m h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k)))\Big],
\end{aligned}$$

which contradicts (2.5) as $t\to\infty$ and completes the proof of sufficiency for eventually positive solutions.

For an eventually negative solution x, we introduce the variables $y=-x$ so that we can apply the above process for the solution y.

Next we show the necessity part by a contrapositive argument. Let (2.5) do not hold. Then it is possible to find $t_1>0$ such that

$$\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\Pi^{\alpha_i}(\sigma_i(\zeta))\,d\zeta + \sum_{\lambda_k\geq\eta}\sum_{i=1}^m h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k)) \leq \frac{\epsilon}{\delta^{\alpha_i}}, \tag{2.10}$$

for all $\eta\geq t_1$ and $\delta,\epsilon>0$ satisfying the relation

$$(2\epsilon)^{1/\gamma} = (1-p_0)\delta, \tag{2.11}$$

so that $0<\epsilon^{1/\gamma}\leq(1-p_0)\delta/2^{1/\gamma}<\delta$. Define the set of continuous functions

$$M=\{x\in C([0,\infty)) : \epsilon^{1/\gamma}\big(\Pi(t)-\Pi(t_1)\big)\leq x(t)\leq\delta\big(\Pi(t)-\Pi(t_1)\big),\ t\geq t_1\},$$

and define an operator Φ on M by

$$(\Phi x)(t)=\begin{cases} 0, & \text{if } t\leq t_1,\\ -p(t)x(\tau(t)) + \displaystyle\int_{t_1}^t \Big[\frac{1}{r(\eta)}\Big[\epsilon + \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,d\zeta & \\ + \displaystyle\sum_{\lambda_k\geq\eta}\sum_{i=1}^m h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\Big]\Big]^{1/\gamma} d\eta, & \text{if } t>t_1. \end{cases}$$

We need to show that if x is a fixed point of Φ, i.e., $\Phi x = x$, then x is a solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. For $x \in M$, we have $0 \leq \epsilon^{1/\gamma}\Big(\Pi(t)-\Pi(t_1)\Big) \leq x(t)$ and by (A2) and (A3) we have

$$(\Phi x)(t) \geq 0 + \int_{t_1}^{t} \left[\frac{1}{r(\eta)}[\epsilon + 0 + 0]\right]^{1/\gamma} d\eta = \epsilon^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big).$$

Now we estimate $(\Phi x)(t)$ from above. For x in M, by definition of the set M we have $x^{\alpha_i}(\sigma_i(\eta)) \leq (\delta\Pi(\sigma_i(\eta)))^{\alpha_i}$. Therefore, by (2.10),

$$\begin{aligned}(\Phi x)(t) \leq & p_0\delta(\Pi(t) - \Pi(t_1)) + \int_{t_1}^{t} \left[\frac{1}{r(\eta)}\Big[\epsilon + \delta^{\alpha_i}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\Pi^{\alpha_i}(\sigma_i(\zeta))\, d\zeta\right.\\ &\left.+ \delta^{\alpha_i}\sum_{\lambda_k \geq \eta}\sum_{i=1}^{m} h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k))\Big]\right]^{1/\gamma} d\eta\\ \leq & p_0\delta\Big(\Pi(t) - \Pi(t_1)\Big) + (2\epsilon)^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big) = \delta\Big(\Pi(t) - \Pi(t_1)\Big).\end{aligned}$$

Therefore, Φ maps M to M.

To find a fixed point for Φ in M, let us define a sequence of functions in M by the recurrence relation

$$\begin{aligned} u_0(t) =& 0, \quad \text{for } t = 0,\\ u_1(t) =& (\Phi u_0)(t) = \begin{cases} 0, & \text{if } t < t_1,\\ \epsilon^{1/\gamma}\Big(\Pi(t) - \Pi(t_1)\Big), & \text{if } t \geq t_1,\end{cases}\\ u_{n+1}(t) =& (\Phi u_n)(t), \quad \text{for } n \geq 1,\ t \geq t_1.\end{aligned}$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using mathematical induction, we can show that $u_{n+1}(t) \geq u_n(t)$. Therefore, the sequence $\{u_n\}$ converges pointwise to a function u. Using the Lebesgue Dominated Convergence Theorem, we can show that u is a fixed point of Φ in M. This shows under assumption (2.10), there a non-oscillatory solution that does not converge to zero. □

Corollary 2.1. Under the assumptions of Theorem 2.1, every unbounded solution of (1.1) is oscillatory if and only if (2.5) holds.

Proof. The proof of the corollary is an immediate consequence of Theorem 2.1. □

2.2. **The Case $\alpha_i > \gamma$.** In this subsection, we assume that there exists a constant β_2, the quotient of two positive odd integers such that $\gamma < \beta_2 < \alpha_i$.

Theorem 2.2. *Under assumptions* (A1)-(A5) *and $r(t)$ is non-decreasing, every solution of* (1.1) *is either oscillatory or converges to zero if and only if*

$$\int_0^{\infty} \left[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\, d\zeta + \sum_{k=1}^{\infty}\sum_{i=1}^{m} h_i(\lambda_k)\Big]\right]^{1/\gamma} d\eta = \infty. \tag{2.12}$$

Proof. We prove the sufficiency by contradiction. Initially, we assume that x is an eventually positive solution not converging to zero. So, Lemma 2.1 holds and z satisfies any one of two cases (i) and (ii). In Lemma 2.2, Case 1 leads to $\lim_{t\to\infty} x(t) = 0$, which is a contradiction.

For Case 2, $z(t) > 0$ is non-decreasing for $t \geq t_1$ and

$$x^{\alpha_i}(t) \geq z^{\alpha_i}(t) \geq z^{\alpha_i-\beta_2}(t) z^{\beta_2}(t) \geq z^{\alpha_i-\beta_2}(t_1) z^{\beta_2}(t)$$

implies that

$$x^{\alpha_i}(\sigma_i(t)) \geq z^{\alpha_i-\beta_2}(t_1) z^{\beta_2}(\sigma_i(t)), \quad \text{for } t \geq t_2 > t_1. \tag{2.13}$$

Using (2.4), (2.13) and $\sigma_i(t) \geq \sigma_0(t)$, we have

$$r(t)\Big(z'(t)\Big)^{\gamma} \geq z^{\alpha_i-\beta_2}(t_1)\left[\int_t^{\infty} \sum_{i=1}^{m} q_i(\eta)\, d\eta + \sum_{\lambda_k \geq t} \sum_{i=1}^{m} h_i(\lambda_k)\right] z^{\beta_2}(\sigma_0(t)), \tag{2.14}$$

for $t \geq t_2$. Being $r(t)\Big(z'(t)\Big)^{\gamma}$ non-increasing and $\sigma_0(t) \leq t$, we have

$$r(\sigma_0(t))\Big(z'(\sigma_0(t))\Big)^{\gamma} \geq r(t)\Big(z'(t)\Big)^{\gamma}.$$

Using the last inequality in (2.14) and then dividing by $z^{\beta_2/\gamma}(\sigma_0(t)) > 0$, we get

$$\frac{z'(\sigma_0((t))}{z^{\beta_2/\gamma}(\sigma_0(t))} \geq \left[\frac{z^{\alpha_i-\beta_2}(t_1)}{r(\sigma_0(t))}\left[\int_t^{\infty} \sum_{i=1}^{m} q_i(\eta)\, d\eta + \sum_{\lambda_k \geq t} \sum_{i=1}^{m} h_i(\lambda_k)\right]\right]^{1/\gamma},$$

for $t \geq t_2$. Multiplying the left-hand side by $\sigma_0'(t)/\alpha \geq 1$ and integrating from t_2 to t, we find

$$\begin{aligned} \frac{1}{\alpha}\int_{t_2}^{t} \frac{z'(\sigma_0(\eta))\sigma_0'(\eta)}{z^{\beta_2/\gamma}(\sigma_0(\eta))}\, d\eta \geq{}& z^{(\alpha_i-\beta_2)/\gamma}(t_1)\int_{t_2}^{t}\left[\frac{1}{r(\sigma_0(\eta))}\left[\int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta)\, d\zeta\right.\right. \\ &\left.\left. + \sum_{\eta \leq \lambda_k} \sum_{i=1}^{m} h_i(\lambda_k)\right]\right]^{1/\gamma} d\eta, \quad \text{for } t \geq t_2. \end{aligned} \tag{2.15}$$

Since $\gamma < \beta_2$, $r(\sigma_0(\eta)) \leq r(\eta)$ and

$$\frac{1}{\alpha(1-\beta_2/\gamma)}\left[z^{1-\beta_2/\gamma}(\sigma_0(\eta))\right]_{\eta=t_2}^{t} \leq \frac{1}{\alpha(\beta_2/\gamma-1)} z^{1-\beta_2/\gamma}(\sigma_0(t_2)),$$

then (2.15) becomes

$$\int_{t_2}^{t}\left[\frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta)\, d\zeta + \sum_{\eta \leq \lambda_k} \sum_{i=1}^{m} h_i(\lambda_k)\right]\right]^{1/\gamma} d\eta < \infty,$$

which is a contradiction to (2.12). This contradiction implies that the solution x cannot be eventually positive. The case with an eventually negative solution is proved.

To prove the necessity part, we assume that (2.12) does not hold. For given $\epsilon = \Big(2/(1-p_0)\Big)^{-\alpha_i/\gamma} > 0$, we can find a $t_1 > 0$ such that

$$\int_{t_1}^{\infty}\left[\frac{1}{r(\eta)}\left[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,d\zeta + \sum_{\lambda_k\geq s}\sum_{i=1}^{m} h_i(\lambda_k)\right]\right]^{1/\gamma} d\eta < \epsilon. \tag{2.16}$$

Consider

$$M = \left\{x \in C([0,\infty)) : 1 \leq x(t) \leq \frac{2}{1-p_0} \text{ for } t \geq t_1\right\}.$$

Define the operator

$$(\Phi x)(t) = \begin{cases} 0, & \text{if } t < t_1,\\ 1 - p(t)x(\tau(t)) \\ +\displaystyle\int_{t_1}^{t}\left[\frac{1}{r(\eta)}\left[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,d\zeta\right.\right. \\ \left.\left. +\displaystyle\sum_{\lambda_k\geq\eta}\sum_{i=1}^{m} h_i(\lambda_k)x^{\alpha_i}(\sigma_i(\lambda_k))\right]\right]^{1/\gamma} d\eta, & \text{if } t \geq t_1. \end{cases}$$

Indeed, $\Phi x = x$ implies that x is a solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. Let $x \in M$. Then $1 \leq x$ implies that $(\Phi x)(t) \geq 1$, on $[t_1,\infty)$. Estimating $(\Phi x)(t)$ from above. Let $x \in M$. Then $x \leq 2/(1-p_0)$ and thus

$$\begin{aligned}(\Phi x)(t) \leq& 1 - p(t)\frac{2}{1-p_0} + \int_{t_1}^{t}\left[\frac{1}{r(\eta)}\left[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\Big(\frac{2}{1-p_0}\Big)^{\alpha_i} d\zeta\right.\right.\\ &\left.\left. + \sum_{\lambda_k\geq\eta}\sum_{i=1}^{m} h_i(\lambda_k)\Big(\frac{2}{1-p_0}\Big)^{\alpha_i}\right]\right]^{1/\gamma} d\eta.\end{aligned}$$

Since $\sigma_0(\eta) \leq \eta$ and $r(\cdot)$ is non-decreasing, we can replace $r(\eta)$ by $r(\sigma_0(\eta))$ and the above inequality is still valid. By (2.16) and the definition of ϵ, we have

$$(\Phi x)(t) \leq 1 + \frac{2p_0}{1-p_0} + \Big(2/(1-p_0)\Big)^{\alpha_i/\gamma}\epsilon = 1 + \frac{2p_0}{1-p_0} + 1 = \frac{2}{1-p_0}.$$

Therefore, Φ maps M to M.

To find a fixed point for Φ in M, we define a sequence of functions by the recurrence relation

$$\begin{aligned} u_0(t) =& 0, \quad \text{for } t = 0,\\ u_1(t) =& (\Phi u_0)(t) = 1, \quad \text{for } t \geq t_1,\\ u_{n+1}(t) =& (\Phi u_n)(t), \quad \text{for } n \geq 1,\ t \geq t_1. \end{aligned}$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using that f is non-decreasing and mathematical induction, we can prove that $u_{n+1}(t) \geq u_n(t)$. Therefore, $\{u_n\}$

converges pointwise to a function u in M. Then u is a fixed point of Φ and a positive solution to (1.1) that does not converge to zero. $\square$

Corollary 2.2. Under the assumptions of Theorem 2.2, every unbounded solution of (1.1) is oscillatory if and only if (2.12) hold.

Example 2.1. Consider the neutral differential equation

(2.17)
$$\begin{cases} \left(e^{-t}\left(\left(x(t)-e^{-t}x(\tau(t))\right)'\right)^{11/3}\right)' + \frac{1}{t+1}(x(t-2))^{1/3} + \frac{1}{t+2}(x(t-1))^{5/3} = 0, \\ \left(e^{-k}\left(\left(x(k)-e^{-k}x(\tau(k))\right)'\right)^{11/3}\right)' + \frac{1}{t+4}(x(k-2))^{1/3} + \frac{1}{t+5}(x(k-1))^{5/3} = 0. \end{cases}$$

Here $\gamma = 11/3$, $r(t) = e^{-t}$, $-1 < p(t) = -e^{-t} \le 0$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $\lambda_k = k$ for $k \in \mathbb{N}$, $\Pi(t) = \int_0^t e^{11s/3}\, ds = \frac{3}{11}\left(e^{11t/3}-1\right)$, $\alpha_1 = 1/3$ and $\alpha_2 = 5/3$. For $\beta_1 = 7/3$, we have $0 < \max\{\alpha_1, \alpha_2\} < \beta_1 < \gamma$, and $u^{\alpha_i-\beta_1} = u^{-2}$ and $u^{\alpha_2-\beta_1} = u^{-2/3}$ which both are decreasing functions. To check (2.5) we have

$$\begin{aligned} &\int_0^\infty \sum_{i=1}^m q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta))\, d\eta + \sum_{k=1}^\infty \sum_{i=1}^m h_i(\lambda_k)\Pi^{\alpha_i}(\sigma_i(\lambda_k)) \\ &\ge \int_0^\infty \sum_{i=1}^m q_i(s)\Pi^{\alpha_i}(\sigma_i(\eta)))\, d\eta \\ &\ge \int_0^\infty q_1(\eta)\Pi^{\alpha_i}(\sigma_1(\eta)))\, d\eta \\ &= \int_0^\infty \frac{1}{\eta+1}\left(\frac{3}{11}\left(e^{5(\eta-2)/3}-1\right)\right)^{1/3} d\eta = \infty, \end{aligned}$$

since the integral approaches $+\infty$ as $\eta \to +\infty$. So, all the conditions of Theorem 2.1 hold, and therefore, each solution of (2.17) is oscillatory or converges to zero.

Example 2.2. Consider the neutral differential equation

(2.18)
$$\begin{cases} \left(\left(\left(x(t)-e^{-t}x(\tau(t))\right)'\right)^{1/3}\right)' + t(x(t-2))^{7/3} + (t+1)(x(t-1))^{11/3} = 0, \\ \left(\left(\left(x(2^k)-e^{-2^k}x(\tau(2^k))\right)'\right)^{1/3}\right)' + \frac{t}{2}(x(2^k-2))^{7/3} + \frac{t}{3}(x(2^k-1))^{11/3} = 0. \end{cases}$$

Here $\gamma = 1/3$, $r(t) = 1$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $\alpha_1 = 7/3$ and $\alpha_2 = 11/3$. For $\beta_2 = 5/3$, we have $\min\{\alpha_1, \alpha_2\} > \beta_2 > \gamma$ and $u^{\alpha_1-\beta_2} = u^{2/3}$ and $u^{\alpha_2-\beta_2} = u^2$, which both are increasing functions. To check (2.12) we have

$$\int_{t_1}^{\infty}\Big[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,d\zeta+\sum_{\lambda_k\geq\eta}\sum_{i=1}^{m} h_i(\lambda_k)\Big]\Big]^{1/\gamma} d\eta$$
$$\geq \int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,d\zeta\Big]\Big]^{1/\gamma} d\eta$$
$$\geq \int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\Big[\int_{\eta}^{\infty} q_1(\zeta)\,d\zeta\Big]\Big]^{1/\gamma} d\eta \geq \int_{2}^{\infty}\Big[\int_{\eta}^{\infty}\zeta\,d\zeta\Big]^{3} d\eta=\infty.$$

So, all the conditions of of Theorem 2.2 hold. Thus, all solution of (2.18) is oscillatory or converges to zero.

Remark 2.1. Based on this work and [13–15, 18–22] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation (1.1) for $p > 0$ and $-\infty < p \leq -1$.

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