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ROUGH STATISTICAL CONVERGENCE FOR DIFFERENCE SEQUENCES

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Abstract. As known, difference sequences have their own characteristics. In this paper, we study the concept of rough statistical convergence for difference sequences in a finite dimensional normed space. At the same time, we examine some properties of the set $st-\lim^r_{\Delta x_i} = \left\{x_* \in X : \Delta x_i \xrightarrow{r} x_*\right\}$, which is called as r-statistical limit set of the difference sequence (Δx_i).

1. Introduction and Background

In this study, since the concept of rough statistical convergence will be studied for difference sequences, it is important to give some literature knowledge about difference sequences. Kizmaz [19] defined the concept of difference sequence such that $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$, where $x = (x_i)$ is a real sequence for all $i \in \mathbb{N}$ (the set of all natural numbers). In this paper, he also defined $c_0(\Delta) = \{x = (x_i) : \Delta x \in c_0\}$, $c(\Delta) = \{x = (x_i) : \Delta x \in c\}$ and $l_\infty(\Delta) = \{x = (x_i) : \Delta x \in l_\infty\}$ spaces, where l_∞, c and c_0 are bounded, convergent and null sequence spaces, respectively. Furthermore, he investigated relations between these spaces and obtained $c_0(\Delta) \subseteq c(\Delta) \subseteq l_\infty(\Delta)$.

After this study, which can be considered as a base about difference sequences, Et [11], Et and Çolak [12], Başarır [5], Et and Nuray [15], Gümüş and Nuray [18], Aydın and Başar [1], Bektaş et al. [6], Et and Esi [14], Savaş [23] and many others researched various properties of this concept. Et and Çolak [12] generalized Kızmaz's results for generalized difference sequences.

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One of the other basic concepts of this study is the concept of statistical convergence. Statistical convergence was defined by Fast [16] and Steinhaus' [25], independently. Schoenberg's work [24] for this kind of convergence can be shown as one of the important studies in summability theory. Since the concept of statistical convergence has been applied to many fields by many researchers, a wide area of use has emerged. Some of these areas are number theory [10], measure theory [20], trigonometric series [30] and summability theory [17].

Statistical convergence has recently been studied by Ulusu and Nuray [27, 29] and Ulusu and Dündar [28] for set sequences.

The concept of density is quite wide and is defined in many different ways such as natural density (asymptotic density), uniform density, density of rational and real numbers, density of ratio sets. Natural density will also form the basis of statistical convergence. Let $K \subseteq \mathbb{N}$ be a subset of $\mathbb{N}$. $d(K) := \lim_n \frac{1}{n}\sum_{j=1}^{n}\chi_K(j)$ is said to be natural density of K whenever the limit exists, where χ_K is the characteristic function of K. According to the definition of statistical convergence, sets with natural density zero will be important for us. In more detail we can say that, if K is a finite set, then it is clear that $d(K) = 0$. Another notation that we will use during our studies will be the notation that a P feature is provided for almost all $i \in \mathbb{N}$. If a sequence $x = (x_i)$ provides any P property for all other elements except the elements with zero natural density then the sequence is called "provides the P property for almost all i" and is abbreviated by writing $(a.a.i.)$. Now, it is possible to give the definition of statistical convergence as follows.

Definition 1.1 ([16])**.** Let $x = (x_i)$ be a real or complex sequence. x is statistically convergent to L if

$$\lim_n \frac{1}{n}\left|\{i \leq n : |x_i - L| \geq \varepsilon\}\right| = 0,$$

for each $\varepsilon > 0$ or equivalently

$$|x_i - L| < \varepsilon \quad (a.a.i).$$

This is indicated by $st\text{-}\lim x = L$. So, it is easy to say that each sequence that convergent is also statistical convergent.

Basarir [5] defined the concept of Δ-statistical convergence as follows.

Definition 1.2 ([5])**.** Let $x = (x_i)$ be a real sequence and $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$. For each $\varepsilon > 0$ if

$$\lim_n \frac{1}{n}\left|\{i \leq n : |\Delta x_i - L| \geq \varepsilon\}\right| = 0,$$

or equivalently

$$|\Delta x_i - L| < \varepsilon \quad (a.a.i),$$

then x is Δ-statistically convergent to L. The set of all Δ-statistically convergent sequences is denoted by $S(\Delta)$.

The concept of rough convergence is based on the idea of defining a new convergence type by extending the radius of convergence of a non-convergent but bounded sequence. Rough convergence is defined by Phu [21] in finite dimensional normed spaces. This concept was later extended by Phu [22] to infinite dimensional normed spaces. The definition of rough convergence in a finite dimensional normed space can be given as follows.

Definition 1.3 ([21]). Let $(X, \|\cdot\|)$ be a normed linear space and r be a non-negative real number. Then the sequence $x = (x_i)$ in X is said to be rough convergent (or r-convergent) to x_*, if for any $\varepsilon > 0$, there exists an $i_\varepsilon \in \mathbb{N}$ such that

$$\|x_i - x_*\| < r + \varepsilon,$$

for all $i \geq i_\varepsilon$. This expression means that

$$\limsup \|x_i - x_*\| < r,$$

and r is called as roughness degree. In this definition, we say that x_* is an r-limit point of the sequence (x_i) and it is denoted by $x_i \xrightarrow{r} x_*$.

Let (x_i) be a rough convergent sequence in a finite dimensional normed space $(X, \|\cdot\|)$ and r be a non-negative real number. For each $r > 0$ we obtain a different x_* point. So, this point which is called as the r-limit point of the sequence may not be unique. Therefore, a set of these points can be mentioned. This set is called as the set of r-limit points and is indicated by $\lim_{x_i}^r$. As seen, the topological and analytical features of this set are very important. The r-limit points set of the sequence (x_i) is defined by

$$\lim_{x_i}^r = \left\{ x_* \in X : x_i \xrightarrow{r} x_* \right\}.$$

Following Phu's definition [21], Aytar [2] described rough statistically convergent sequences as follows.

Definition 1.4 ([2]). Let $(X, \|\cdot\|)$ be a normed linear space and r be a non-negative real number. The sequence $x = (x_i)$ in X is said to be rough statistically convergent (or r-statistically convergent) to x_*, if the set

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$. This expression means that

$$st - \limsup \|x_i - x_*\| \leq r,$$

and it is denoted by $x_i \xrightarrow{rst} x_*$.

Aytar [3, 4] also studied with rough limit set and rough cluster points. After these studies, Demir [7] and Demir and Gümüş [8] studied the concept of rough convergence for difference sequences and proved some basic theorems. On the other hand, Dündar and Çakan [9] define rough $\mathcal{I}$-convergence.

2. Our Aim

The idea of rough statistical convergence has developed a new perspective for non-convergent sequences. Applying this new perspective to difference sequences, which are known with their own properties, will produce very interesting results.

3. Main Results

In this part we investigate the concept of rough statistical convergence for difference sequences in $(\mathbb{R}^n, \|\cdot\|)$ space, where $\mathbb{R}^n$ is real n-dimensional normed space and we prove some important theorems.

Definition 3.1. Let $(\mathbb{R}^n, \|\cdot\|)$ be the real n-dimensional normed space and r be a non-negative real number. A difference sequence $\Delta x = (\Delta x_i)$ in $\mathbb{R}^n$ is said to be rough statistically convergent (or r-statistically convergent) to x_*, provided that the set

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$ or equivalently

$$st - \limsup \|\Delta x_i - x_*\| \leq r.$$

In this case we write $\Delta x_i \overset{r-st}{\to} x_*$.

The set of all r-st-limit points of a difference sequence Δx is indicated by

$$st - \lim^r_{\Delta x_i} = \left\{ x_* \in \mathbb{R}^n : \Delta x_i \overset{r-st}{\to} x_* \right\}.$$

The notation r denotes the degree of roughness and it is easy to see that if $r = 0$, then statistical convergence is obtained.

The following example gives us an example of a difference sequence which is not statistically convergent but r-statistically convergent.

Example 3.1. Let the difference sequence $\Delta y = (\Delta y_i)$ be a statistically convergent to y_* and cannot be measured exactly. Additionally, let $\Delta x = (\Delta x_i)$ be a sequence that provides the property $\|\Delta x_i - \Delta y_i\| \leq r$ $(a.a.i.)$. Then the sets

$$\{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\}$$

and

$$\{i \in \mathbb{N} : \|\Delta x_i - \Delta y_i\| \geq \varepsilon\},$$

have natural density zero for any $\varepsilon > 0$. According to these informations we can not say that Δx is statistically convergent. But we know that

$$\{i \in \mathbb{N} : \|\Delta x_i - y_*\| \geq r + \varepsilon\} \subseteq \{i \in \mathbb{N} : \|\Delta y_i - y_*\| \geq \varepsilon\}$$

and this relation gives us that the natural density of the set on the left will be zero. So, the sequence Δx is r-statistically convergent.

For the set of all $r-st$-limit points of Δx, if $st-\lim^r_{\Delta x_i} \neq \emptyset$, then $st-\lim^r_{\Delta x_i} = [st-\limsup \Delta x - r, st-\liminf \Delta x + r]$. On the other hand, we know that if Δx is unbounded, then the set of r-limit points is empty, i.e., $\lim^r_{\Delta x_i} = \emptyset$. Whereas this sequence might be rough statistically convergent. The following example explains this situation.

Example 3.2. Let

$$\Delta x_i = \begin{cases} (-1)^i, & \text{if } i = k^2, \\ i, & \text{otherwise,} \end{cases}$$

i.e.,

$$(\Delta x_i) = (-1, 2, 3, 1, 5, 6, 7, 8, -1, \dots).$$

Then

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\} = \{1, 4, 9, 16, \dots\}$$

and this set has natural density zero. So, we obtain

$$st-\lim^r_{\Delta x_i} = \begin{cases} \emptyset, & \text{if } r < 1, \\ [1-r, r-1], & \text{otherwise.} \end{cases}$$

Corollary 3.1. *$st-\lim^r_{\Delta x_i} \neq \emptyset$ does not imply $\lim^r_{\Delta x_i} \neq \emptyset$, but $\lim^r_{\Delta x_i} \neq \emptyset$ implies $st-\lim^r_{\Delta x_i} \neq \emptyset$. Therefore,*

$$\lim^r_{\Delta x_i} \subseteq st-\lim^r_{\Delta x_i}$$

and

$$\operatorname{diam}(\lim^r_{\Delta x_i}) \subseteq \operatorname{diam}\left(st-\lim^r_{\Delta x_i}\right).$$

Theorem 3.1. *For any difference sequence $\Delta x = (\Delta x_i)$, diameter of $st-\lim^r_{\Delta x_i}$ is not greater than $2r$. Generally, there is no smaller bound.*

Proof. Suppose that $\operatorname{diam}\left(st-\lim^r_{\Delta x_i}\right) > 2r$. Then there exist $y, z \in st-\lim^r_{\Delta x_i}$ such that

$$d := \|y - z\| > 2r.$$

Take an arbitrary $\varepsilon \in \left(0, \frac{d}{2} - r\right)$. Define A_1 and A_2 sets such that

$$A_1 := \{i \in \mathbb{N} : \|\Delta x_i - y\| \geq r + \varepsilon\}$$

and

$$A_2 := \{i \in \mathbb{N} : \|\Delta x_i - z\| \geq r + \varepsilon\}.$$

Because $y, z \in st-\lim^r_{\Delta x_i}$, we have $d(A_1) = 0$, $d(A_2) = 0$ and from the properties of natural density, $d(A_1^c \cap A_2^c) = 1$. So,

$$\|y - z\| \leq \|\Delta x_i - y\| + \|\Delta x_i - z\| < 2(r + \varepsilon) < 2r + 2\left(\frac{d}{2} - r\right) = d = \|y - z\|,$$

for all $i \in A_1^c \cap A_2^c$. This is a contradiction. Therefore, $\operatorname{diam}\left(st-\lim^r_{\Delta x_i}\right) \leq 2r$.

Now, let's show that there is generally no smaller bound. For this, we show that $st-\lim^r_{\Delta x_i}=\bar{B}_r(x_*)$. We know that diam $\left(\bar{B}_r(x_*)\right)=2r$ for

$$\bar{B}_r(x_*):=\{y\in X:\|x_*-y\|\leq r\}\,.$$

Choose a difference sequence (Δx_i), with $st-\lim\Delta x=x_*$. For each $\varepsilon>0$ we have

$$d(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq\varepsilon\})=0.$$

Then

$$\|\Delta x_i-y\|\leq\|\Delta x_i-x_*\|+\|x_*-y\|\leq\|\Delta x_i-x_*\|+r,$$

for each $y\in\bar{B}_r(x_*)$. In this case,

$$\|\Delta x_i-y\|<r+\varepsilon,$$

for each $i\in\{i\in\mathbb{N}:\|\Delta x_i-x_*\|<\varepsilon\}$. At the same time, we know that

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|<\varepsilon\}\right)=1$$

and so, $y\in st-\lim^r_{\Delta x_i}$. Then we have $st-\lim^r_{\Delta x_i}=\bar{B}_r(x_*)$. □

Theorem 3.2. *For a bounded sequence* (Δx_i), *there is a non-negative real number* r *such that* $st-\lim^r_{\Delta x_i}\neq 0$.

The question of whether the converse of the above theorem is also valid is a question that can immediately come to mind. The answer is no. But if the sequence is statistically bounded, the converse is valid. The theorem that gives this case is below.

Theorem 3.3. (Δx_i) *is statistically bounded if and only if there exists a non-negative real number* r *such that* $st-\lim^r_{\Delta x_i}\neq 0$.

Proof. First, let's show that $st-\lim^r_{\Delta x_i}\neq 0$, when Δx is statistically bounded. From the definition of statistically boundedness, there exists a positive real number M such that

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i\|\geq M\}\right)=0.$$

Let's define $r':=\sup\{\|\Delta x_i\|:i\in K^c\}$, where $K=\{i\in\mathbb{N}:\|\Delta x_i\|\geq M\}$. Then $st-\lim^{r'}_{\Delta x_i}$ contains the origin of $\mathbb{R}^n$ and $st-\lim^{r'}_{\Delta x_i}\neq\emptyset$.

Now, assume that $st-\lim^{r'}_{\Delta x_i}\neq\emptyset$ for some $r\geq 0$. Then we have an x_* such that $x_*\in st-\lim^{r'}_{\Delta x_i}$. In that case

$$d\left(\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\geq r+\varepsilon\}\right)=0,$$

for each $\varepsilon>0$. So, we can say that almost all Δx_i are contained in some ball with any radius greater than r and Δx_i is statistically bounded. □

In rough convergence, we know that when $\left(\Delta x_{i_j}\right)$ is a subset of (Δx_i)

$$\lim^r_{\Delta x_i}\subseteq\lim^r_{\Delta x_{i_j}}.$$

In the case of rough statistical convergence, the subsequence must be non-thin to satisfy this condition.

Example 3.3. Let $\Delta x_i := \begin{cases} i, & \text{if } i = k^2, \\ 0, & \text{otherwise}, \end{cases}$ is a difference sequence of real numbers. Then $\left(\Delta x_{i_j}\right) := (1, 4, 9, 16, \dots)$ is a subsequence of (Δx_i). We have $st - \lim^r_{\Delta x_i} = [-r, r]$ and $st - \lim^r_{\Delta x_{i_j}} = \emptyset$.

Definition 3.2. $\left(\Delta x_{i_j}\right)$ is a non-thin subsequence of (Δx_i) provided that the set B does not have natural density zero where $B = \{i_j : j \in \mathbb{N}\}$.

Theorem 3.4. *If* $\left(\Delta x_{i_j}\right)$ *is a non-thin subsequence of* (Δx_i), *then* $st - \lim^r_{\Delta x_i} \subseteq st - \lim^r_{\Delta x_{i_j}}$.

Theorem 3.5. $st - \lim^r_{\Delta x_i}$ *is closed.*

Proof. For this proof, we use one of the well-known theorems of functional analysis. According to this theorem, "For a convergent sequence $\Delta y_i \to y_*$, when $\Delta y \in st - \lim^r_{\Delta x_i}$ (at the same time $y_* \in st - \lim^r_{\Delta x_i}$), then $st - \lim^r_{\Delta x_i}$ is closed". If $st - \lim^r_{\Delta x_i} = \emptyset$, then the proof is trivial. Suppose that $st - \lim^r_{\Delta x_i} \neq \emptyset$. Then we have a sequence $(\Delta y_i) \subseteq st - \lim^r_{\Delta x_i}$ such that $\Delta y_i \to y_*$. From the definition of convergence, for each $\varepsilon > 0$ there exists $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that $\|\Delta y_i - y_*\| < \frac{\varepsilon}{2}$ for all $i > i_{\frac{\varepsilon}{2}}$. Choose an $i_0 \in \mathbb{N}$ such that $i_0 > i_{\frac{\varepsilon}{2}}$. Then $\|\Delta y_{i_0} - y_*\| < \frac{\varepsilon}{2}$.

On the other hand, since $\Delta y_i \subseteq st - \lim^r_{\Delta x_i}$, we have $y_{i_0} \in st - \lim^r_{\Delta x_i}$, i.e.,

$$d\left(\left\{i \in N : \|\Delta x_i - y_{i_0}\| \geq r + \frac{\varepsilon}{2}\right\}\right) = 0.$$

Now, we need to show following inclusion

$$\{i \in \mathbb{N} : \|\Delta x_i - y_*\| < r + \varepsilon\} \supseteq \left\{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\right\}.$$

Let $k \in \left\{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\right\}$. Then $\|\Delta x_k - y_{i_0}\| < r + \frac{\varepsilon}{2}$ and hence

$$\|\Delta x_k - y_*\| \leq \|\Delta x_k - y_{i_0}\| + \|y_{i_0} - y_*\| < r + \varepsilon.$$

It means $k \in \{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \varepsilon\}$ and we have the proof. □

Theorem 3.6. $st - \lim^r_{\Delta x_i}$ *is convex.*

Proof. Suppose that $y_0, y_1 \in st - \lim^r_{\Delta x_i}$ and let $\varepsilon > 0$ be given. Define the sets

$$K_1 := \{i \in \mathbb{N} : \|\Delta x_i - y_0\| \geq r + \varepsilon\}$$

and

$$K_2 := \{i \in \mathbb{N} : \|\Delta x_i - y_1\| \geq r + \varepsilon\}.$$

We know that $d(K_1) = d(K_2) = 0$ and $d(K_1^c \cap K_2^c) = 1$ from the assumption. Then we have

$$\|\Delta x_i - [(1-\lambda) y_0 + \lambda y_1]\| = \|(1-\lambda)(\Delta x_i - y_0) + \lambda(\Delta x_i - y_1)\| < r + \varepsilon,$$

for each $i \in K_1^c \cap K_2^c$ and each $\lambda \in [0, 1]$. We get

$$d(\{i \in \mathbb{N} : \|\Delta x_i - [(1-\lambda) y_0 + \lambda y_1]\| \geq r + \varepsilon\}) = 0,$$

this means $[(1-\lambda)\,y_0+\lambda y_1] \in st-\lim^r_{\Delta x_i}$ and so $st-\lim^r_{\Delta x_i}$ is convex. $\square$

Theorem 3.7. *The sequence (Δx_i) is r-statistically convergent to x_* if only if there exists a difference sequence $\Delta y = (\Delta y_i)$ such that $st-\lim \Delta y = x_*$ and $\|\Delta x_i - \Delta y_i\| \leq r$ for each $i \in \mathbb{N}$.*

Proof. For the necessity part, suppose that (Δx_i) is r-statistically convergent to x_*. From the definition

$$st-\limsup \|\Delta x_i - x_*\| \leq r.$$

Let's define the sequence (Δy_i) as follows:

$$\Delta y_i := \begin{cases} x_*, & \text{if } \|\Delta x_i - x_*\| \leq r, \\ \Delta x_i + r\dfrac{x_* - \Delta x_i}{\|x_* - \Delta x_i\|}, & \text{otherwise.} \end{cases}$$

Then it is easy to see that

$$\|\Delta y_i - x_*\| = \begin{cases} 0, & \text{if } \|\Delta x_i - x_*\| \leq r, \\ \|\Delta x_i - x_*\| - r, & \text{otherwise,} \end{cases}$$

and $\|\Delta x_i - \Delta y_i\| \leq r$ for each $i \in \mathbb{N}$.

For the sufficiency, suppose that $st-\lim \Delta y = x_*$ and $\|\Delta x_i - \Delta y_i\| \leq r$ for each $i \in \mathbb{N}$. From the definiton of statistical convergence, for each $\varepsilon > 0$ we get

$$d\left(\{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\}\right) = 0.$$

We know that

$$\{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\} \supseteq \{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\},$$

and we have

$$d\left(\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}\right) = 0.$$

$\square$

In order to prove the next theorem, we will need the following lemma, which is related to statistical cluster points.

Lemma 3.1. *Let $\Gamma_{\Delta x}$ be the set of all statistical cluster points of Δx and c be an arbitrary element of this set. For all $x_* \in st-\lim^r_{\Delta x_i}$ we have $\|x_* - c\| \leq r$.*

Proof. Let's accept the contrary of the lemma and find the contradiction. Assume that there exist a point $c \in \Gamma_{\Delta x}$ and $x_* \in st-\lim^r_{\Delta x_i}$ such that $\|x_* - c\| > r$. Define $\varepsilon = \frac{\|x_* - c\| - r}{3}$. In that case,

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\} \supseteq \{i \in \mathbb{N} : \|\Delta x_i - c\| < \varepsilon\}.$$

From the fact that $c \in \Gamma_{\Delta x}$, we know that the natural density of the set

$$\{i \in \mathbb{N} : \|\Delta x_i - c\| < \varepsilon\}$$

is not zero. So, by using the inclusion above, we obtain

$$d\left(\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}\right) \neq 0,$$

and this completes the proof. $\square$

Theorem 3.8. *For a difference sequence* $\Delta x = (\Delta x_i)$, $\Delta x_i \overset{r-st}{\to} x_*$ *if and only if* $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*)$.

Proof. In Theorem 3.1, we proved the necessity part. So, we need to prove if $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*)$, then $\Delta x_i \overset{r-st}{\to} x_*$. We know that if the statistical cluster point of a statistically bounded sequence is unique, then the sequence is statistically convergent to this point.

In that case, if $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*) \neq \emptyset$, then (Δx_i) is statistically bounded. Let (Δx_i) sequence has two different statistical cluster points, such as x_* and x'_*. Then the point

$$\bar{x}_* := x_* + \frac{r}{\|x_* - x'_*\|}\left(x_* - x'_*\right),$$

satisfies

$$\|\bar{x}_* - x'_*\| = \left(\frac{r}{\|x_* - x'_*\|} + 1\right)\|x_* - x'_*\| = r + \|x_* - x'_*\| > r.$$

From the previous lemma, $\bar{x}_* \notin st - \lim^r_{\Delta x_i}$ but this contradicts the fact that $\|\bar{x}_* - x_*\| = r$ and $st - \lim^r_{\Delta x_i} = \bar{B}_r(x_*)$. This means that x_* is the unique statistical cluster point of Δx. So, Δx is statistically convergent to x_*. □

4. Conclusions and Future Developments

In our paper, we obtain some different results by defining the concept of rough statistical convergence for difference sequences. Later on, we investigate some properties of r-statistical limit point set of a difference sequence. In addition, it may be of interest to investigate similar results for generalized difference sequences.

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