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LIST COLORING UNDER SOME GRAPH OPERATIONS

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Abstract. The list coloring of a graph $G = G(V, E)$ is to color each vertex $v \in V(G)$ from its color set $L(v)$. If any two adjacent vertices have different colors, then G is properly colored. The aim of this paper is to study the list coloring of some graph operations.

1. Introduction

Throughout this paper, our notations are standard and can be taken from the famous book of West [16]. The set of all positive integers is denoted by $\mathbb{N}$, and for a set X, the power set of X is denoted by $P(X)$. All graphs are assumed to be simple and finite, and if G is such a graph, then its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively.

The graph coloring is an important concept in modern graph theory with many applications in computer science. A function $\alpha : V(G) \to \mathbb{N}$ is called a coloring for G. The coloring α is said to be proper, if for each edge $uv \in E(G)$, $\alpha(u) \neq \alpha(v)$. If the coloring α uses only the colors $[k] = \{1, 2, \dots, k\}$, then α is called a k-coloring for G, and if such a proper k-coloring exists, then the graph G is said to be k-colorable. The smallest possible number k for which the graph G is k-colorable is the chromatic number of G and is denoted by $\chi(G)$.

The list coloring of graphs is a generalization of the classical notion of graph coloring, which was introduced independently by Erdős, Rubin and Taylor [7] and Vizing [15]. In the list coloring of a graph G, a list $L(v)$ of colors is assigned to each vertex $v \in V(G)$, and we have to find a proper coloring c for G in such a way that $c(v) \in L(v)$, for any vertex v in G. Concretely, we assume that there is a

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function $L : V(G) \to P(\mathbb{N})$ that assigns a set of colors to each vertex of G. A coloring $c : V(G) \to \mathbb{N}$ is called an L-coloring if for all $v \in V(G)$, $c(v) \in L(v)$. This coloring is said to be proper if $c(u) \neq c(v)$, when $uv \in E(G)$. The graph G is called L-colorable if such an L-coloring exists. This graph is k-choosable if it is L-colorable for every assignment L that satisfies $|L(v)| \geq k$, for all $v \in V(G)$. The list chromatic number $\chi_L(G)$ of G is the smallest k such that G is k-choosable. In [9], Isaak showed that the list chromatic number of the Cartesian product K_2 and K_n is equal to $n^2 + \left\lceil \frac{5n}{3} \right\rceil$. One year later, Axenovich [2] proved that if each vertex $x \in V(G) \setminus P$ is assigned a list of colors of size Δ and each vertex $x \in P$ is assigned a list of colors of size 1, then it is possible to color $V(G)$ such that adjacent vertices receive different colors and each vertex has a color from its list, where G is a non-complete graph with maximum degree $\Delta \geq 3$ and P is a subset of vertices with pairwise distance $d(P)$ between them at least 8. After that, in 2009, Rackham [12] studied on the list coloring of K_Δ-free graphs. We encourage potential readers to consult the interesting thesis of Lastrina [10] and Tuza's survey [14] for more information on this topic.

By a well-known result of Nordhaus and Gaddum [11], if G is an n-vertex graph, then $\chi(G) + \chi(\overline{G}) \leq n + 1$, where $\overline{G}$ is the complement of a graph G.

Erdös, Rubin and Taylor [7] extended this inequality to the list coloring of graphs and proved that for every n-vertex graph G, $\chi_L(G) + \chi_L(\overline{G}) \leq n + 1$. Thus, it is natural to study the list coloring of graphs under some other graph operations, which is the main topic of this paper.

Suppose $\{G_i = (V_i, E_i)\}_{i=1}^N$ is a family of graphs having a root vertex 0. Following Barrière, Comellas, Dalfó, Fiol, and Mitjana [3, 4], the hierarchical product $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is the graph with vertices as N-tuples $x_N \ldots x_3x_2x_1$, for $x_i \in V_i$, and edges defined as follows:

$$x_N \ldots x_3x_2x_1 \sim \begin{cases} x_N \ldots x_3x_2y_1 & \text{if } y_1 \sim x_1 \text{ in } G_1, \\ x_N \ldots x_3y_2x_1, & \text{if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\ x_N \ldots y_3x_2x_1, & \text{if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\ \quad\vdots & \quad\vdots \\ y_N \ldots x_3x_2x_1, & \text{if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = x_2 = \cdots = x_{N-1} = 0. \end{cases}$$

In [13], Tavakoli, Rahbarnia and Ashrafi obtained exact formulas for some graph invariants under the hierarchical product, and some applications in chemistry were presented by Arezoomand and Taeri in [1].

Suppose G is a connected graph. Following Cvetković, Doob, Sachs, Yan, Yang and Yeh [6, 17], we define four types of graphs resulting from edge subdivision.

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.
(b) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle.
(c) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

(d) The graph $T(G)$ of a graph G has a vertex for each edge and vertex of G and an edge in $T(G)$ for every edge-edge, vertex-edge, and vertex-vertex adjacency in G.

The graphs $S(G)$ and $T(G)$ are called the subdivision and total graphs of G, respectively.

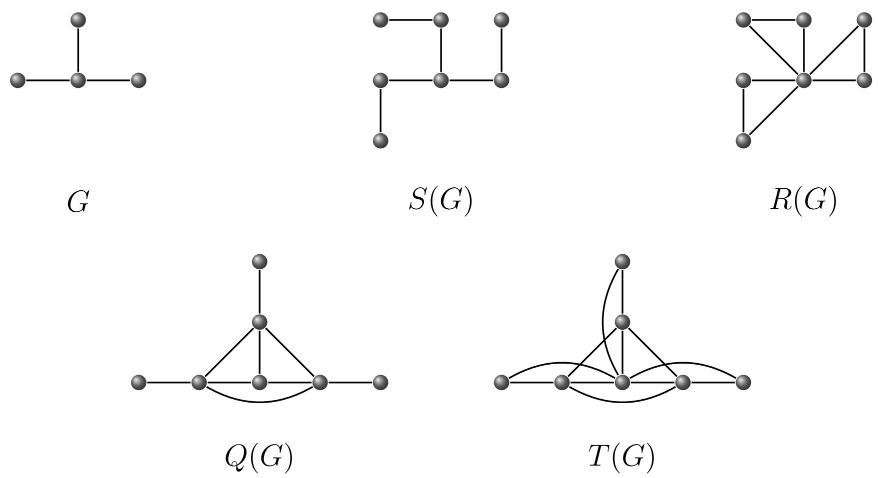


FIGURE 1. Subdivision graphs of G.

Let G and H be two graphs. The corona product GoH is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$, see Yeh and Gutman [19]. In Yarahmadi and Ashrafi [18], the authors obtained exact formulas for some graph invariant under the corona product of graphs. The edge corona product of two graphs G and H, $G\diamondsuit H$, is obtained in a similar way by taking one copy of G and $|E(G)|$ copies of H and joining each end vertices of the i-th edge of G to every vertex in the i-th copy of H, see Chithra, Germina, Sudev, Hou and Shiu [5, 8]. If the graphs G and H have disjoint vertex sets, then $G + H$ will be the graph obtained from G and H by connecting all vertices of G with all vertices of H.

2. Main Results

Suppose G is a simple graph. The suspension of a graph G is another graph G' constructed from G by adding a new vertex u and connecting u to all vertices of G.

2.1. **Relationship between the coloring and the list coloring of graphs.**

It is clear that the list chromatic number $\chi_L(G)$ of a graph G is at least its chromatic number $\chi(G)$, but it can be strictly larger, in other words $\chi(G) < \chi_L(G)$. We consider the following cases for showing the difference between the list coloring and the coloring of a given graph G.

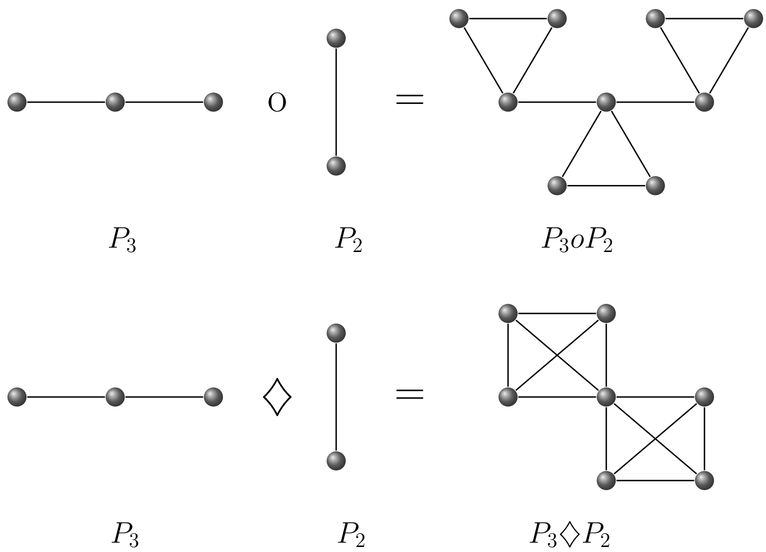


FIGURE 2. The corona and edge corona products of two graphs P_3 and P_2.

- Suppose $\chi_L(G) - \chi(G) = 1$. In this case, if we color the graph with lists of length $\chi(G)$, then in each coloring of this graph there will be at least one vertex x such that all adjacent vertices of x can be colored, and there is no edge that its end vertices cannot be colored.
- Suppose $\chi_L(G) - \chi(G) = 2$. In this case, if we color the graph with lists of length $\chi(G)$, then in each coloring of this graph there will be at least two vertices x and y such that $xy \in E(G)$, all adjacent vertices of x, y can be colored, and there is no triangle in G that its vertices cannot be colored.

Note that the above statements cannot be generalized to the case that $\chi_L(G) - \chi(G) > 2$. To show this, we define $r = \binom{2k-1}{k}$. Then, the complete bipartite graph $K_{r,r}$ is not k-choosable and so $\chi_L(K_{r,r}) > k$. If G has a list coloring of length m in such a way that we can find a coloring in which there is a k-vertex graph without a possible color, then $\chi_L(G) = m + k$. Finally, if the graph G can be colored with lists of length $\chi_L(G) - 1$ then there will be lists of length $\chi_L(G) - 1$, in which for every coloring of these lists there exists a vertex that all its adjacent vertices are colored and there is no edge that its end vertices cannot be colored.

2.2. **List chromatic numbers of the suspension graph and the corona product.** The aim of this section is to compute the list chromatic number of the suspension graph and the corona product of graphs. We start this section by the following crucial result:

Theorem 2.1. *Let G be a graph with $G' = G + K_1$. Then $\chi_L(G') = \chi_L(G)$ or $\chi_L(G) + 1$.*

Proof. Let $V(K_1) = \{u\}$. It is clear that $\chi(G') = \chi(G) + 1$. Suppose $\chi_L(G) = \chi(G)$. Then, $\chi_L(G) + 1 = \chi(G) + 1 = \chi(G') \le \chi_L(G')$. We claim that $\chi_L(G') = \chi_L(G) + 1$. To prove it, we assign lists of length $\chi_L(G) + 1$ to the vertices of G'. We color u with a color t in $L(u)$. In the worst case, $t \in \bigcap_{v \in V(G')} L(v)$ and since G has a coloring with lists of length $\chi_L(G)$, we will find an appropriate coloring for G'.

We now assume that $\chi_L(G) = \chi(G)+1$. Since $\chi(G') \le \chi_L(G')$, $\chi(G)+1 = \chi_L(G) \le \chi_L(G')$. For the list coloring of G' we have the following two cases.

(a) After coloring of G with lists of length $\chi_L(G) - 1$, we will have at most two vertices without a possible color: $\chi_L(G') = \chi_L(G)$. We assign lists of length $\chi_L(G)$ to all vertices of G'. We first consider the case that we cannot color only one vertex of G'. There are two cases for $L(u)$ as follows.

a.1 There is a color $a \in L(u)$ such that for each $v \neq a \in V(G')$, $a \notin L(v)$. In this case, we assign a to the vertex u. By our hypothesis, the problem is changed to the list coloring of G by $\chi_L(G)$ colors, which is possible by definition.

a.2 For each color $a \in L(u)$, there exists a vertex $u \neq v \in V(G')$ such that $a \in L(v)$. Suppose $V(G) = \{v_1, \dots, v_n\}$ and assign a list L_i to each vertex v_i for $1 \le i \le n$. We consider the following two cases.

(i) $L(u) \subseteq \bigcap_{i=1}^{n} L_i$. In this case, all vertices have the same list of colors. Since $\chi_L(G) = \chi(G) + 1 = \chi(G')$, the vertices of G can be colored with $\chi(G)$ colors and it remains a color for u. Hence, $\chi_L(G') = \chi_L(G)$.

(ii) $L(u) \nsubseteq \bigcap_{i=1}^{n} L_i$. In this case, there exist a color $a \in L(u)$ and an integer i for $1 \le i \le n$, such that $a \in L_i$ and $a \notin \bigcap_{j=1}^{n} L_j$. We assign the color a to the vertex u and remove a from the list of other vertices. This shows that there exists a list L_j such that $a \notin L_j$. Therefore, the length of some lists is $\chi(G)$ or $\chi(G) + 1$. By the hypothesis, there is only one vertex without a feasible color when a list has length $\chi_L(G) - 1$. It is clear that, in all cases, we will have an appropriate coloring for the graph.

We now assume that after the coloring of the graph with lists of length $\chi_L(G) - 1$ there are two vertices without assigning a color. If we have a color $a \in L(u)$ such that $a \notin \cup_{v \neq u} L(v)$, then by a similar argument as above, we will have an appropriate coloring for the graph. So, we can assume that every color in $L(u)$ will appear in at least one list of colors. We have again the following two cases.

(i) $L(u) \subseteq \bigcap_{i=1}^{n} L_i$. A similar argument as above shows that we have an appropriate coloring of the graph.

(ii) $L(u) \nsubseteq \bigcap_{i=1}^{n} L_i$. In this case, there exist a color $a \in L(u)$ and an integer i, for $1 \le i \le n$, such that $a \in L_i$ and $a \notin \bigcap_{j=1}^{n} L_j$. We prove that it is possible to find an appropriate coloring with lists of length $\chi_L(G)$. To do this, we show that there exists at least one color c in $L(u)$, such that c is outside of at least two other lists. On the contrary, we assume that there is at most one list $L(v)$ with $c \notin L(v)$. If c is outside of all the other lists, then clearly we will find an appropriate coloring for the graph. Hence, we can assume that there is a unique v such that $c \notin L(v)$. Therefore, all lists except one of them are equal and we have an appropriate coloring with lists of length $\chi_L(G) - 1$, which is impossible. Therefore, G' can be colored with lists of length $\chi_L(G)$.

(b) After the coloring of G with lists of length $\chi_L(G) - 1$, we will have more than two vertices without a possible color: In this case, we will prove $\chi_L(G') = \chi_L(G) + 1$. Suppose $\chi_L(G) - \chi(G) = m$. We prove that the graph G' does not have a list coloring with lists of length $\chi_L(G) - 1$. We assign lists of length $\chi_L(G) - 1$ in such a way that there is no appropriate coloring for the graph. Consider the $\chi_L(G)$ copies of the graph G with the same lists and add a_1 to all lists of the first copy of G, a_2 to all lists of the second copy of G, $\dots$, $a_{\chi_L(G)}$ to all lists of the $\chi_L(G)$-copy of G. We also assign the list $\{1, 2, \dots, \chi_L(G)\}$ to the vertex u. Note that by assigning each of a_i to the vertex u, we will not have an appropriate coloring for the i-th copy of G. Thus, we cannot find a feasible coloring for the graph. Therefore, an appropriate coloring of G' needs lists of length $\chi_L(G) + 1$, see Figure 3.

This completes the proof. $\square$

Lemma 2.1. *Suppose G is a graph containing disjoint subgraphs $G_1, \dots, G_{\chi_L(G)}$ such that for each subgraph we can find lists of length $\chi_L(G) - 1$ in which at least one vertex does not have a color. If u is an isolated vertex, then $\chi_L(G') = \chi_L(G) + 1$.*

Proof. On the contrary, we assume that $\chi_L(G') = \chi_L(G)$. We define the lists of the graph G' as follows:

- assign lists of length $\chi_L(G) - 1$ to the vertices of G_1 from the set $\{2, 3, \dots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of G_2 from the set $\{1, 3, \dots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of G_i from the set $\{1, 2, \dots, i - 1, i + 1, \dots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of $G_{\chi_L(G)}$ from the set $\{1, 2, \dots, \chi_L(G) - 1, \chi_L(G) + 1\}$.

We now add the color i to all lists corresponding to the subgraph G_i for $1 \leq i \leq \chi_L(G)$, and assign the set $\{1, 2, \dots, \chi_L(G)\}$ to the vertex u. If we assign a color, say i, to the vertex u, then the subgraph G_i cannot be colored, and so G does not have an appropriate coloring, a contradiction. Thus, $\chi_L(G') = \chi_L(G) + 1$. $\square$

Corollary 2.1. *Suppose G and H are two graphs. Then,*

$$\chi_L(GoH) \begin{cases} = \max\{\chi_L(G), \chi_L(H)\}, & \chi_L(H) \neq \chi(H) \textit{ and in the coloring of } H \\ & \textit{with lists of length } \chi_L(H) - 1 \textit{ at most} \\ & \textit{two vertices cannot be colored,} \\ \leq \max\{\chi_L(G), \chi_L(H) + 1\}, & \textit{otherwise.} \end{cases}$$

2.3. **List chromatic number of the edge corona product.** Suppose G is a simple graph, $e = uv$ and $u, v \notin V(G)$. Let $G'' = G + K_2$, where $V(K_2) = \{u, v\}$ and $E(K_2) = \{e\}$. It is easy to see that $G'' = G' + K_1$, where $V(K_1) = \{v\}$ with

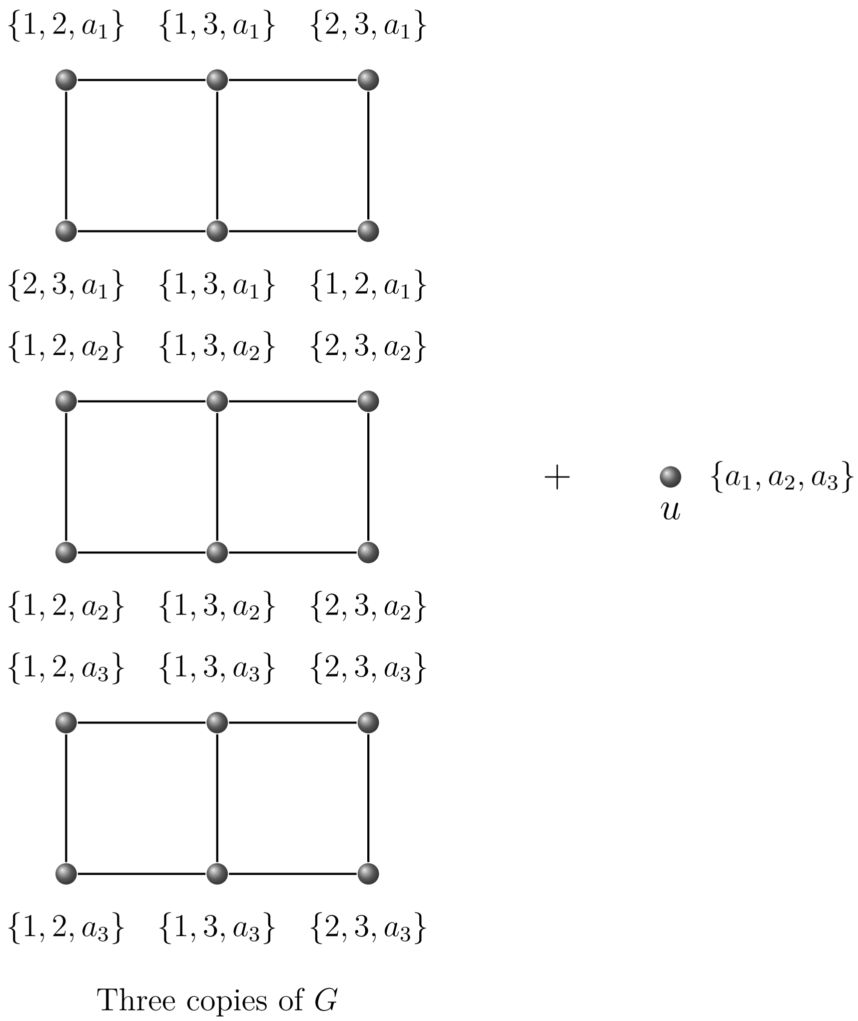


FIGURE 3. Adding the vertex u to a graph G that after coloring with lists of length $\chi_L(G)-1$, the vertex u will be without an assigned color.

$G' = G + K_1$, where $V(K_1) = \{u\}$. It is clear that $\chi(G'') = \chi(G) + 2$. By Corollary 2.1,

$$\chi_L(G'')\begin{cases} = \chi_L(G'), & \chi_L(G') \neq \chi(G') \text{ and in the coloring of the graph with} \\ & \text{llists of ength } \chi_L(G') - 1 \text{ at most two vertices cannot} \\ & \text{be colored,} \\ \leq \chi_L(G'') + 1, & \text{otherwise.} \end{cases}$$

We now apply this inequality to prove the following lemma.

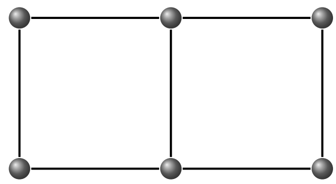
Lemma 2.2. *The list chromatic number of G'' is given by the following formula:*

$$\chi_L(G'') = \begin{cases} \chi_L(G), & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists length } \chi_L(G)-1 \text{ of exactly one vertex cannot be colored,} \\ \chi_L(G)+1, & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists length } \chi_L(G)-1 \text{ of exactly two vertices cannot be colored,} \\ \chi_L(G)+2, & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Suppose G and H are two graphs. The list chromatic number of $G\diamondsuit H$ is given by the following formula:*

$$\chi_L(G\diamondsuit H) \begin{cases} = \max\{\chi(G), \chi(H)\}, & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists of length } \chi_L(G)-1 \text{exactly one vertex cannot be colored,} \\ \leq \max\{\chi(G), \chi(H)+1\}, & \chi_L(G) \neq \chi(G) \text{ and in the coloring of the graph with lists of length } \chi_L(G)-1 \text{ exactly two vertices cannot be colored,} \\ \leq \max\{\chi(G), \chi(H)+2\}, & \text{otherwise.} \end{cases}$$

2.4. **List chromatic number of the join of two graphs.** The aim of this subsection is to investigate under which conditions $\chi_L(G+H) = \chi_L(G) + \chi_L(H)$. If $\chi_L(G) = \chi(G)$ and $\chi_L(H) = \chi(H)$, then $\chi(G+H) = \chi(G) + \chi(H)$, and so $\chi_L(G+H) = \chi_L(G) + \chi_L(H)$. On the other hand, if one of G or H is a complete graph, then by Corollary 2.1, $\chi_L(G+H) = \chi_L(G) + \chi_L(H)$. In Figures 4 and 5, some examples are given, which show that the quantities $\chi_L(G+H)$ and $\chi_L(G) + \chi_L(H)$ can be non-equal.

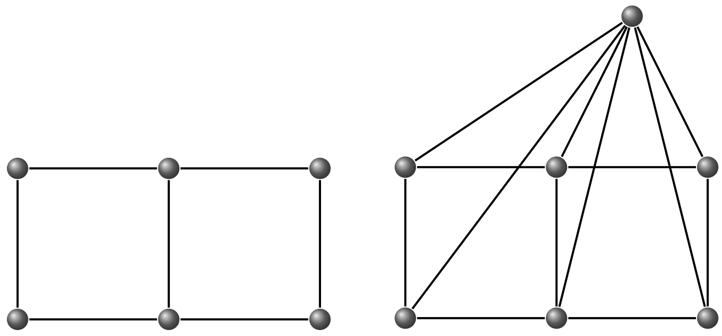


G

FIGURE 4. Graphs G and $H \cong G$ that $\chi_L(G+H) \neq \chi_L(G) + \chi_L(H)$.

Theorem 2.3. *Suppose G and H are graphs such that the following holds.*

- $\chi_L(H) \leq \chi_L(G)$ *(or* $\chi_L(G) \leq \chi_L(H)$*).*
- *The graph G (H) has subgraphs $G_1, \ldots, G_{\chi_L(G)+1}$ ($H_1, \ldots, H_{\chi_L(H)+1}$) such that for each subgraph G_i for $1 \leq i \leq \chi_L(G)+1$, (or H_i for $1 \leq i \leq \chi_L(H)+1$) there exist lists of length $\chi_L(G)+1$ (or $\chi_L(H)+1$) in such a way that in each subgraph there exists at least one vertex that cannot be colored.*



G $\qquad$ $G+H$

FIGURE 5. Graphs G and $H \cong K_1$ with $\chi_L(G+H) \neq \chi_L(G) + \chi_L(H)$.

Then, $\chi_L(G+H) = \chi_L(G) + \chi_L(H)$.

Proof. On the contrary, we assume that $\chi_L(G+H) = \chi_L(G) + \chi_L(H) - 1$. We assign lists of length $\chi_L(H) - 1$ to the graph H in such a way that H does not have an appropriate coloring related to these lists. Similarly to Lemma 2.1, we assign lists to the subgraphs $G_1, \ldots, G_{\chi_L(G)+1}$ as follows:

- assign lists of length $\chi_L(G) - 1$ to the vertices of G_1 from the set $\{2, 3, \ldots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of G_2 from the set $\{1, 3, \ldots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of G_i from the set $\{1, 2, \ldots, i - 1, i + 1, \ldots, \chi_L(G) + 1\}$;
- assign lists of length $\chi_L(G) - 1$ to the vertices of $G_{\chi_L(G)+1}$ from the set $\{1, 2, \ldots, \chi_L(G) - 1, \chi_L(G) + 1\}$.

By our hypothesis, there exists a vertex $x_i \in V(G_i)$, for $1 \leq i \leq \chi_L(G) + 1$, such that in the process of the coloring for vertices of H_i, x_i cannot be colored. We now add the color i to all lists corresponding to the subgraph G_i, for $1 \leq i \leq \chi_L(G) + 1$. We also assign the lists of the graph H to the subgraphs of G in such a way that we assign different lists to at least two vertices of a given subgraph, and at least three lists of each subgraphs are different. Note that the smallest subgraph with these properties has at least six vertices. Next, we assign lists of length $\chi_L(G)$ from the set $\{1, 2, \ldots, \chi_L(G) + 1\}$ to the vertices of H such that at least two vertices of the graph have different lists and if $|V(H)| \geq 3$, then at least three lists of vertices in H are different. We assign numbers to the lists of G and letters to the lists of H. Our main proof will consider the following three separate cases.

(a) *In the coloring of H we use only letters.* By our hypothesis, there will be one vertex that cannot be colored, and we assign the number i to this vertex. So, the subgraph G_i cannot be colored, as desired.

(b) *In the coloring of H we use only numbers.* In this case, we will have a list of letters for a subgraph (vertices of G which are colored with numbers) and since $\chi_L(H) - 1 \leq \chi_L(H) \leq \chi_L(G)$, the graph cannot be colored.

(c) *In the coloring of H we use a combination of letters and numbers.* By our hypothesis, there will be one vertex that cannot be colored, and we assign the number i to this vertex. So, the subgraph G_i cannot be colored, as desired. In this case, we use numbers instead of letters. For example, we use 1 as a. Again, we will have a vertex that cannot be colored by letters and the number 1. We assign the number i to this vertex. Consider a list L in G_i containing number 1. If $a \notin L$, then the graph obviously cannot be colored. If $a \in L$, then we lead to a contradiction with our substitution. So, the graph cannot be colored. In the case that more than one letter is substituted by a number, we lead to a similar contradiction, and so the graph cannot be colored.

This proves that $\chi_L(G + H) = \chi_L(G) + \chi_L(H)$. □

2.5. **List chromatic number of the subdivision graphs.** In this subsection, the list chromatic number of four types of edge subdivision of a graph G containing $R(G), S(G), Q(G)$ and $T(G)$ are computed.

Theorem 2.4. $\chi_L(R(G)) = \max\{\chi_L(G), 3\}$.

Proof. The subdivision graph $R(G)$ is isomorphic to the edge corona product of G and H, where $H = K_1$. Since $\chi_L(H) = \chi(H) = 1$, by Theorem 2.2, $\chi_L(R(G)) = \max\{\chi_L(G), 3\}$. □

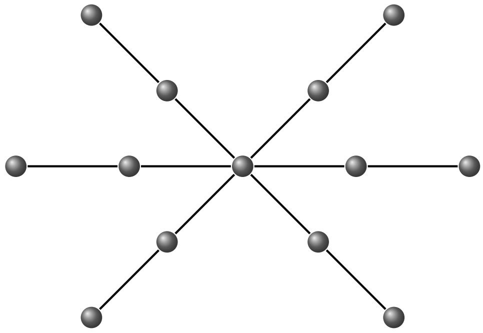
Theorem 2.5. *Suppose G has at least one edge. Then $\chi_L(S(G)) = 2$ or 3 and all cases can occur.*

Proof. Suppose $|V(G)| = n$, $|E(G)| = m$ and $V(G) = \{v_1, \ldots, v_n\}$. In the graph $S(G)$, the additional vertices of each edge of G are labeled by $u_1, \ldots, u_m$. It is clear that all cycles of $S(G)$ have even length and so $S(G)$ is a bipartite graph with bipartite classes (U_1, U_2), where $U_1 = V(G)$ and $U_2 = \{u_1, \ldots, u_m\}$. Therefore, $\chi_L(S(G)) \geq \chi(S(G)) = 2$. In Figures 6 and 7, two graphs G_1 and G_2 are presented, such that $\chi_L(S(G_1)) = 2$ and $\chi_L(S(G_2)) = 2$.

To complete the proof, we assign a color to all vertices of $V(G)$ and the other vertices can be colored with two other colors. This proves that $\chi_L(S(G)) \leq 3$, which completes the proof. □

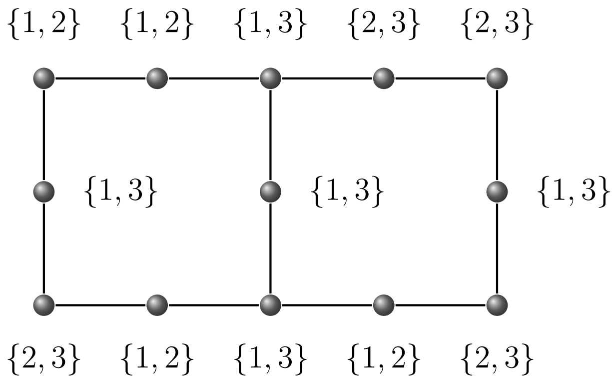
Theorem 2.6. $\chi_L(Q(G)) = \Delta(G) + 1$.

Proof. We use the labeling of the vertices in $S(G)$ given in the proof of Theorem 2.5 for the graph $Q(G)$. By definition of $Q(G)$, each vertex v_i together with all vertices u_j adjacent to v_i constitutes a complete graph of order $\deg(v_i) + 1$ and each u_i is a



G_1

FIGURE 6. The graph G_1 with $\chi_L(S(G_1)) = 2$.



G_2

FIGURE 7. The graph G_2 with $\chi_L(S(G_2)) = 3$.

common vertex of exactly two complete subgraphs. So, the graph $Q(G)$ has $|V(G)|$ such complete graphs. It is obvious that for each triangle in G, some of the vertices in $A = \{u_i \mid 1 \leq i \leq m\}$ induces a triangle in $Q(G)$ and in the other case, the vertices in A can not construct a triangle in $Q(G)$. Since G has a vertex of degree $\Delta(G)$, $Q(G)$ has a complete subgraph of order $\Delta(G) + 1$, and so $\chi(Q(G)) \geq \Delta(G) + 1$. We will prove that it is possible to color the graph $Q(G)$ by lists of length $\Delta(G) + 1$. To prove it, we assign lists of length $\Delta(G) + 1$ to all vertices of $Q(G)$. Since $Q(G)$ can be constructed from complete graphs of minimum order 3 and maximum order $\Delta(G) + 1$, each vertex of $V(G)$ is a vertex of exactly one complete graph, each vertex u_i is a common vertex of exactly two complete subgraphs, and each complete graph of order n has n distinct colorings with lists of length n, the graph $Q(G)$ has an appropriate coloring. This proves the theorem. $\square$

Theorem 2.7. $\Delta(G) + 1 \leq \chi_L(T(G)) \leq \Delta(G) + 2$.

Proof. Since the graphs G and $Q(G)$ are subgraphs of $T(G)$, $\max\{\chi_L(G), \chi_L(Q(G))\} \leq \chi_L(T(G))$. On the other hand, $\chi_L(G) \geq \Delta(G)+1$ and so $\Delta(G)+1 \leq \chi_L(T(G))$. To prove $\chi_L(T(G)) \leq \Delta(G)+2$, we assign the lists of length $\Delta(G)+2$ to each vertex of the graph. We first color all vertices of G. Since each vertex of $A = \{u_i \mid 1 \leq i \leq m\}$ are adjacent to two vertices of G, which are adjacent in $T(G)$, the length of lists corresponding to vertices in A is at least $\Delta(G)$. Therefore, $\chi_L(T(G)) \leq \Delta(G)+2$. $\square$

2.6. **List chromatic number of the hierarchical product of graphs.** In this section, the list chromatic number of the hierarchical product of graphs is computed. We first compute this number for the case of two graphs.

Theorem 2.8. *The list chromatic number of the hierarchical product of two graphs G and H is given by the following formula:*

$$\chi_L(G \sqcap H) = \begin{cases} 3, & \chi_L(G) = \chi_L(H) = 2,\ G \text{ has a cycle of even length and the root is a vertex of an even cycle,} \\ 2, & \chi_L(G) = \chi_L(H) = 2,\ G \text{ does not have an even cycle or } G \text{ has an even cycle but the root is not a vertex of an even cycle,} \\ \max\{\chi_L(G), \chi_L(H)\}, & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that $\chi(G \sqcap H) = \max\{\chi(G), \chi(H)\}$. Moreover, if $\chi_L(G) = \chi_L(H) = 2$, G has a cycle of even length and the root is a vertex of an even cycle, then $\chi_L(G \sqcap H) = 3$, see Figure 8. If $\chi_L(G) = \chi_L(H) = 2$, G does not have an even cycle or G has an even cycle, but the root is not a vertex of an even cycle, then $\chi_L(G \sqcap H) = 2$. On the other hand, if $\chi_L(G) > \chi_L(H)$, then clearly the graph $G \sqcap H$ can be colored by lists of length $\chi_L(G)$ and if $\chi_L(G) < \chi_L(H)$, then the graph $G \sqcap H$ can be colored by lists of length $\chi_L(H)$. So, it is enough to consider the case that $\chi_L(G) = \chi_L(H)$. In this case, we first color the graph G by $\chi_L(G)$ colors. In this coloring, for the coloring of each vertex in G, a vertex in H will be colored and if $\chi_L(H) \geq 3$, then the graph will have an appropriate coloring. $\square$

Corollary 2.2. *Suppose $G_1, G_2, \ldots, G_k$ are k simple graphs. Then,*

$$\chi_L(G_k \sqcap \cdots \sqcap G_2 \sqcap G_1) = \begin{cases} 3, & \chi_L(G_1) = \cdots = \chi_L(G_k) = 2,\ G_k \text{ has an even cycle and the root is a vertex of an even cycle,} \\ 2, & \chi_L(G_1) = \cdots = \chi_L(G_k) = 2, \text{ the root is not a vertex of an even cycle or } G_k \text{ does not have a cycle of even length,} \\ \max\{\chi_L(G_1), \ldots, \chi_L(G_k)\}, & \text{otherwise.} \end{cases}$$

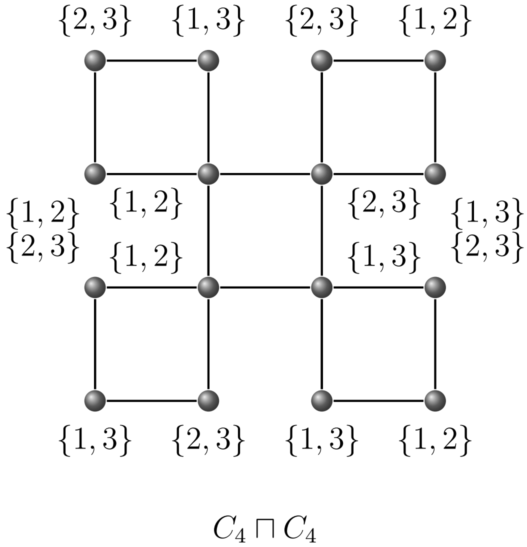


FIGURE 8. The hierarchical product of C_4 and C_4 with the list chromatic number 3.

Proof. We proceed by induction. In Theorem 2.8, we proved the case of $k = 2$. Suppose $k = m - 1$ and $H = G_k \sqcap \cdots \sqcap G_2 \sqcap G_1$. To prove the case of $k = m$, we first assume that $\chi_L(G_1) = \cdots = \chi_L(G_k) = 2$. Then, the following four cases can occur.

(a) *Let G_m be a tree and there are no even cycles in other graphs.* Since the other $m - 1$ graphs do not have even cycles, the graph H does not have an even cycle, and so $\chi_L(G_m \sqcap H) = 2$.

(b) *Let G_m be a tree and there exists at least one even cycle in the other graphs.* Since in the other $m - 1$ graphs we have at least one even cycle, the graph H has an even cycle. If $\chi_L(H) \geq 3$, then $\chi_L(G_m \sqcap H) = \max\{2, 3\} = 3$. If $\chi_L(H) = 2$, then $\chi_L(G_m \sqcap H) = 2$, as desired.

(c) *G_m has an even cycle and there are no even cycles in other graphs.* A similar argument as in the first case shows that $\chi_L(G_m \sqcap H) = 2$.

(d) *G_m has an even cycle and there exists at least one even cycle in the other graphs.* In this case, the graph H has at least one even cycle. If $\chi_L(H) \geq 3$, then $\chi_L(G_m \sqcap H) = \max\{2, 3\} = 3$. Suppose $\chi_L(H) = 2$. If the root vertex is in a cycle, then $\chi_L(H) = 3$, and otherwise $\chi_L(H) = 2$.

Next we assume that there exists i such that $\chi_L(G_i) > 2$. Then

$$\max_{1 \leq i \leq m} \{\chi_L(G_i)\} = \max \left\{ \chi_L(G_m), \max_{1 \leq i \leq m-1} \{\chi_L(G_i)\} \right\} = \max\{\chi_L(G_m), \chi_L(H)\}.$$

This shows that the problem for the case of $k = m$ can be reduced to the case of $k = 2$ such that one of the graphs has the list chromatic number greater than 2. By induction hypothesis, this is feasible, and so the proof is complete. $\square$

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