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EXISTENCE RESULTS FOR A FRACTIONAL DIFFERENTIAL INCLUSION OF ARBITRARY ORDER WITH THREE-POINT BOUNDARY CONDITIONS

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Abstract. This paper studies existence of solutions for a new class of fractional differential inclusions of arbitrary order with three-point fractional integral boundary conditions. Our results are based on Bohnenblust-Karlin's fixed point theorem.

1. Introduction

Fractional differential equations are being used in various fields of science and engineering such as control system, electrochemistry, viscoelasticity, electromagnetics, physics, biophysics, fitting of experimental data, blood flow phenomena, electrical circuits, biology, porous media etc. [11, 12, 18]. Due to these features, models of fractional order become more practical and realistic than the models of integer-order.

A generalization of differential inequalities and equations are known as differential inclusions. Some recent development on fractional differential equations and inclusions can be found in [2, 4–6, 8–10, 14–17, 20, 22, 23]. Interesting and important applications of differential inclusions are in problems arising from stochastic processes, optimal control theory, economics and so on. If the velocity of a dynamical system cannot be uniquely determined by the state of the system, then such a system can be modeled as a differential inclusion.

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In [14], Benchohra and Hamidi studied the boundary value problem for fractional differential inclusions given by

$$\begin{cases} {}^cD^{\alpha}w(\xi) \in Z(\xi, w(\xi)), \\ w(0) = w_0, \end{cases}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2]$ and $Z : [0, \infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map with compact and convex values.

Ntouyas [20] investigated the existence of solutions for fractional order differential inclusions of the form

$$\begin{cases} {}^cD^{q}w(\xi) \in Z(\xi, w(\xi)), & 0 < \xi < 1, \\ w(0) = 0, w(1) = \alpha J^p w(\nu), & 0 < \nu < 1, \end{cases}$$

where ${}^cD^{q}$ is the Caputo fractional derivative of order $q \in (1, 2]$, J^p is the Riemann-Liouville fractional integral of order p, $Z : [0, 1) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map.

In this paper, we consider the multi-valued version of [21]. We study existence results for solutions of the following fractional differential inclusion

$$\begin{cases} {}^cD^{\beta_2}w(\xi) \in Z(\xi, w(\xi)), & \xi \in [0, 1], \\ w(\nu) = w'(0) = w''(0) = \cdots = w^{n-2}(0) = 0, & I^{\beta_1}w(1) = 0, \end{cases} \tag{1.1}$$

where $\beta_1 > 0$, $n - 1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbb{N}$, and ${}^cD^{\beta_2}$ is the Caputo derivative of fractional order β_2, I^{β_1} is the Riemann-Liouville integral of fractional order β_1, $Z : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ and $\nu^{n-1} \neq \frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}$.

2. Preliminaries

Let us recall some notations, definitions and lemmas from multi-valued analysis [13, 19].

Let $W = C([0, 1], \mathbb{R})$ denote the standard Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm

$$\|w\| = \max\{|w(\xi)| : \xi \in [0, 1]\}.$$

A fixed point of a multi-valued map $Z : W \to \mathcal{P}(W)$ is $w \in W$ such that $w \in Z(w)$. Z is bounded on bounded sets if for any bounded subset D of W, $Z(D) = \bigcup_{w \in D} Z(w)$ is bounded in W. Z is said to be completely continuous if for every bounded subset D of W, $\overline{Z(D)}$ is compact. Z is closed (convex) valued if $Z(w)$ is closed (convex) for all $w \in W$. Z is called u.s.c. (upper semi-continuous) on W if the set $Z(w_0)$ is a nonempty closed subset of W for each $w_0 \in W$ and if there exists an open neighborhood E of w_0 such that $Z(E) \subseteq D$ for each open subset D of W containing $Z(w_0)$. Z has a closed graph if

$$w_n \to w^{\star}, z_n \to z^{\star}, w_n \in W, z_n \in Z(w_n) \;\Rightarrow\; z^{\star} \in Z(w^{\star}).$$

If Z has nonempty compact values and is completely continuous, then Z has a closed graph if and only if Z is u.s.c.

Throughout this paper, $BCC(W)$ is the set of all nonempty, convex, closed and bounded subsets of W. Let $L^1([0,1],\mathbb{R})$ be the standard Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|z\|_{L^1} = \int_0^1 |z(\xi)|\, d\xi.$$

The following definitions are well known [1, 11, 18].

Definition 2.1. The Caputo fractional derivative of order β for at least n-times differentiable function $w : [0,\infty) \to \mathbb{R}$ is defined as

$$^cD^\beta w(\xi) = \frac{1}{\Gamma(n-\beta)} \int_0^\xi (\xi - s)^{n-\beta-1} w^{(n)}(s)\, ds, \quad n-1 < \beta < n,\, n = \lceil \beta \rceil,$$

where $\lceil \beta \rceil$ denotes the least integer function of real number β.

Definition 2.2. The Riemann-Liouville integral of fractional order β is defined as

$$I^\beta w(\xi) = \frac{1}{\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} w(s)\, ds, \quad \beta > 0,$$

provided the integral exists.

Lemma 2.1 ([21]). *Let* $\nu^{n-1} \neq \frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}$, $\beta_1 > 0$, $n-1 < \beta_2 \leq n$, $0 < \nu < 1$. *Then for* $z \in C([0,1],\mathbb{R})$, *the fractional differential system*

$$\begin{cases} ^cD^{\beta_2} w(\xi) = z(\xi), \quad \xi \in [0,1], \\ w(\nu) = w'(0) = w''(0) = \cdots = w^{n-2}(0) = 0, \quad I^{\beta_1} w(1) = 0, \end{cases} \tag{2.1}$$

is equivalent to the integral equation

$$\begin{aligned} w(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} z(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} z(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z(s)\, ds, \end{aligned} \tag{2.2}$$

where

$$Q = \frac{\Gamma(\beta_1 + n)}{\Gamma(n) - \nu^{n-1}(\beta_1 + n - 1)(\beta_1 + n - 2)\cdots(\beta_1 + 1)}. \tag{2.3}$$

Lemma 2.2 ([20]). *A function* $w \in AC^n([0,1],\mathbb{R})$ *satisfying boundary conditions*

$$w(\nu) = w'(0) = w''(0) = \cdots = w^{n-2}(0) = 0, \quad I^{\beta_1} w(1) = 0,$$

is a solution of fractional differential inclusion (1.1) *if* $z(\xi) \in Z(\xi, w(\xi))$ *on* $[0,1]$ *for some function* $z \in L^1([0,1],\mathbb{R})$ *and*

$$w(\xi) = \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} z(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} z(s)\, ds$$

$$+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1 (1-s)^{\beta_1+\beta_2-1}z(s)\,ds$$
$$-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu (\nu-s)^{\beta_2-1}z(s)\,ds.$$

For the forthcoming analysis, we need the following assumptions.

(A) $Z:[0,1]\times\mathbb{R}\to BCC(\mathbb{R})$ for each $w\in\mathbb{R}$, $(\xi,w)\mapsto z(\xi,w)$ is u.s.c. with respect to w for a.e. $\xi\in[0,1]$ and is measurable with respect to ξ and the set $S_{Z,w}$ is non-empty for each fixed $w\in\mathbb{R}$.

(B) There exists a function $m_\epsilon\in L^1([0,1],\mathbb{R}_+)$ for each $\epsilon>0$ such that
$$\|Z(\xi,w)\|=\sup\{|v|:v(\xi)\in Z(\xi,w)\}\le m_\epsilon(\xi),$$
for each $(\xi,w)\in[0,1]\times\mathbb{R}$ with $|w|\le\epsilon$ and
$$\liminf_{\epsilon\to+\infty}\frac{\int_0^1 m_\epsilon(\xi)\,d\xi}{\epsilon}=\gamma<\infty.$$

Lemma 2.3 ([3])**.** *Let J be a compact real interval and Z be a multi-valued map satisfying assumption* (A) *and let ζ be a continuous and linear function from $L^1(J,\mathbb{R})$ into $C(J)$. Then the operator*
$$\zeta\circ S_Z:C(J)\to BCC(J),\quad y\mapsto(\zeta\circ S_Z)(y)=\zeta(S_{Z,y}),$$
is a closed graph operator in $C(J)\times C(J)$.

Lemma 2.4 ([7])**.** *Let W be a Banach space and D be a nonempty, convex, closed and bounded subset of W. Let $Z:D\to\mathcal{P}(W)\setminus\{\emptyset\}$ has convex, closed values and is u.s.c. with $Z(D)\subset D$ and $Z(\overline{D})$ is compact. Then Z has a fixed point.*

Let us define a multi-valued map $\psi:W\to\mathcal{P}(W)$ as
$$\psi(w)=\Bigg\{y\in W:y(\xi)=\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z(s)\,ds$$
$$+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z(s)\,ds$$
$$-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z(s)\,ds\Bigg\},$$
for $z\in S_{Z,w}=\{z(\xi)\in L^1([0,1],\mathbb{R}):z(\xi)\in Z(\xi,y)\text{ for a.e. }\xi\in[0,1]\}$.

Observe that a fixed point of ψ is a solution of (1.1). For convenience, we put
$$\Lambda=\frac{2}{\Gamma(\beta_2+1)}+\frac{|Q|}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}+\frac{|Q|}{\Gamma(\beta_1+\beta_2+1)}.$$

3. Main Results

Theorem 3.1. *Assume that* (A) *and* (B) *hold with* $\Lambda\gamma < 1$. *Then the fractional differential inclusion* (1.1) *has at least one solution.*

Proof. The proof is divided into four steps.

Step I. $\psi(w)$ is convex for each $w \in C[0,1]$.

Let $\lambda \in [0,1]$ and $y_1, y_2 \in \psi(w)$. Then there exist $z_1, z_2 \in S_{Z,w}$ such that for each $\xi \in [0,1]$, we have

$$\begin{aligned} y_i(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} z_i(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z_i(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)} \int_0^1 (1-s)^{\beta_1+\beta_2-1} z_i(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z_i(s)\, ds. \end{aligned}$$

Now,

$$\begin{aligned} (\lambda y_1 + (1-\lambda) y_2)(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2-1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds \\ &- \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)} \int_0^1 (1-s)^{\beta_1+\beta_2-1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} (\lambda z_1(s) + (1-\lambda) z_2(s))\, ds. \end{aligned}$$

Since Z has convex values, $S_{Z,w}$ is also convex. Thus, for $z_1, z_2 \in S_{Z,w}$ and $\lambda \in [0,1]$, we have $\lambda z_1 + (1-\lambda) z_2 \in S_{Z,w}$. Hence, $\lambda y_1 + (1-\lambda) y_2 \in \psi(w)$, i.e., $\psi(w)$ is convex.

Step II. Let $\epsilon > 0$ and $B_\epsilon = \{w \in C[0,1] : \|w\| \leq \epsilon\}$. Then B_ϵ is a closed, convex and bounded set in $C[0,1]$. We shall prove that there exists $\epsilon > 0$ such that $\psi(B_\epsilon) \subseteq B_\epsilon$. Suppose it is not true. Then for each $\epsilon > 0$, there exist $w_\epsilon \in B_\epsilon$ and $y_\epsilon \in \psi(w_\epsilon)$ with $\|\psi(w_\epsilon)\| > \epsilon$ and

$$\begin{aligned} y_\epsilon(\xi) =& \frac{1}{\Gamma(\beta_2)} \int_0^\xi (\xi - s)^{\beta_2 - 1} z_\epsilon(s)\, ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z_\epsilon(s)\, ds \\ &+ \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)} \int_0^1 (1-s)^{\beta_1+\beta_2-1} z_\epsilon(s)\, ds \\ &- \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2-1} z_\epsilon(s)\, ds, \end{aligned}$$

for some $z_\epsilon \in S_{Z,w_\epsilon}$.

Now,

$$\epsilon < \|\psi(w_\epsilon)\|$$

$$
\begin{aligned}
\leq & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2 - 1} |z_\epsilon(s)| \, ds + \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} |z_\epsilon(s)| \, ds \\
& + \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} |z_\epsilon(s)| \, ds \\
& + \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} |z_\epsilon(s)| \, ds \\
\leq & \frac{1}{\Gamma(\beta_2)} \int_0^1 m_\epsilon(s) \, ds + \frac{1}{\Gamma(\beta_2)} \int_0^1 m_\epsilon(s) \, ds \\
& + \frac{|Q|}{\Gamma(\beta_1 + \beta_2)} \int_0^1 m_\epsilon(s) \, ds + \frac{|Q|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^1 m_\epsilon(s) \, ds.
\end{aligned}
$$

Dividing both sides by ϵ and letting $\epsilon \to \infty$, we get

$$
\left[\frac{2}{\Gamma(\beta_2)} + \frac{|Q|}{\Gamma(\beta_1 + \beta_2)} + \frac{|Q|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \right] \gamma \geq 1,
$$

implying $\Lambda\gamma \geq 1$, which contradicts the given assumption. Therefore, there exists $\epsilon > 0$ such that $\psi(B_\epsilon) \subseteq B_\epsilon$.

Step III. $\psi(B_\epsilon)$ is equicontinuous.

Let $\xi_1, \xi_2 \in [0,1]$ with $\xi_1 < \xi_2$ and $w \in B_\epsilon,\ y \in \psi(w)$. Then there exists $z \in S_{Z,w}$ such that for each $\xi \in [0,1]$, we have

$$
\begin{aligned}
y(\xi) = & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi} (\xi - s)^{\beta_2 - 1} z(s) \, ds - \frac{1}{\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} z(s) \, ds \\
& + \frac{(\nu^{n-1} - \xi^{n-1})Q}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} z(s) \, ds \\
& - \frac{Q(\nu^{n-1} - \xi^{n-1})}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} z(s) \, ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|y(\xi_1) - y(\xi_2)| \leq & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi_1} |(\xi_2 - s)^{\beta_2 - 1} - (\xi_1 - s)^{\beta_2 - 1}| |z(s)| \, ds \\
& + \frac{1}{\Gamma(\beta_2)} \int_{\xi_1}^{\xi_2} |\xi_2 - s|^{\beta_2 - 1} |z(s)| \, ds \\
& + \frac{|Q||\xi_1^{n-1} - \xi_2^{n-1}|}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1 - s)^{\beta_1 + \beta_2 - 1} |z(s)| \, ds \\
& + \frac{|Q||\xi_1^{n-1} - \xi_2^{n-1}|}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^{\nu} (\nu - s)^{\beta_2 - 1} |z(s)| \, ds \\
\leq & \frac{1}{\Gamma(\beta_2)} \int_0^{\xi_1} |(\xi_2 - s)^{\beta_2 - 1} - (\xi_1 - s)^{\beta_2 - 1}| m_\epsilon(s) \, ds \\
& + \frac{1}{\Gamma(\beta_2)} \int_{\xi_1}^{\xi_2} |\xi_2 - s|^{\beta_2 - 1} m_\epsilon(s) \, ds
\end{aligned}
$$

$$+\frac{|Q||\xi_1^{n-1}-\xi_2^{n-1}|}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}m_\epsilon(s)\,ds$$
$$+\frac{|Q||\xi_1^{n-1}-\xi_2^{n-1}|}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}m_\epsilon(s)\,ds.$$

Now, the right-hand side approaches zero when ξ_1 approaches ξ_2, independently of $w\in B_\epsilon$. Hence, $\psi(B_\epsilon)$ is equicontinuous.

Combining Steps I to III and by a consequence of Arzelá-Ascoli theorem, we get that ψ is a compact valued map.

Step IV. ψ has a closed graph.

Let $w_n\to w^*$, $y_n\in\psi(w_n)$ and $y_n\to y^*$. We shall prove that $y^*\in\psi(w^*)$.

Now, $y_n\in\psi(w_n)$ implies that there exists $z_n\in S_{Z,w_n}$ such that for each $\xi\in[0,1]$, we have

$$\begin{aligned}y_n(\xi)=&\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z_n(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z_n(s)\,ds\\&+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z_n(s)\,ds\\&-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z_n(s)\,ds.\end{aligned}$$

We shall show that there exists $z^*\in S_{Z,w^*}$ such that for each $\xi\in[0,1]$, we have

$$\begin{aligned}y^*(\xi)=&\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z^*(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z^*(s)\,ds\\&+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z^*(s)\,ds\\&-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z^*(s)\,ds.\end{aligned}$$

Consider the continuous linear operator $\zeta:L^1([0,1],\mathbb{R})\to C[0,1]$ given by

$$\begin{aligned}\zeta(z)(\xi)=&\frac{1}{\Gamma(\beta_2)}\int_0^\xi(\xi-s)^{\beta_2-1}z(s)\,ds-\frac{1}{\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z(s)\,ds\\&+\frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1(1-s)^{\beta_1+\beta_2-1}z(s)\,ds\\&-\frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu(\nu-s)^{\beta_2-1}z(s)\,ds.\end{aligned}$$

Now, it is clear that $\|y_n(\xi)-y^*(\xi)\|\to 0$ as $n\to\infty$.

As a consequence of Lemma 2.3, we deduce that $\zeta\circ S_Z$ is a closed graph operator with $y_n(\xi)\in\zeta(S_{Z,w_n})$.

Since $w_n \to w^*$, we have from Lemma 2.3

$$\begin{aligned}y^*(\xi) =& \frac{1}{\Gamma(\beta_2)}\int_0^\xi (\xi-s)^{\beta_2-1}z^*(s)\,ds - \frac{1}{\Gamma(\beta_2)}\int_0^\nu (\nu-s)^{\beta_2-1}z^*(s)\,ds\\ &+ \frac{(\nu^{n-1}-\xi^{n-1})Q}{\Gamma(\beta_1+\beta_2)}\int_0^1 (1-s)^{\beta_1+\beta_2-1}z^*(s)\,ds\\ &- \frac{Q(\nu^{n-1}-\xi^{n-1})}{\Gamma(\beta_1+1)\Gamma(\beta_2)}\int_0^\nu (\nu-s)^{\beta_2-1}z^*(s)\,ds,\end{aligned}$$

for some $z^* \in S_{Z,w^*}$.

Thus, the compact operator ψ is u.s.c. with closed, convex values. From Lemma 2.4, we conclude that there exists a fixed point w of ψ, which is a solution of (1.1). □

Theorem 3.2. *Assume that* (A) *and the following condition hold.*

(C) *There exist functions* $k_1(\xi), k_2(\xi) \in L^1([0,1],\mathbb{R}^+)$ *such that*

$$\|Z(\xi,w)\| \le k_1(\xi)|w| + k_2(\xi),$$

for each $(\xi,w) \in [0,1]\times\mathbb{R}$, *with* $\Lambda\|k_1\|_{L^1} < 1$.

Then the BVP (1.1) *has at least one solution on* $[0,1]$.

Proof. The proof follows by taking $k_1(\xi)\epsilon + k_2(\xi)$ in place of $m_\epsilon(\xi)$ in the proof of Theorem 3.1. □

Theorem 3.3. *Assume that* (A) *and the following condition hold.*

(D) *There exist functions* $k_1(\xi), k_2(\xi) \in L^1([0,1],\mathbb{R}^+)$, $\sigma \in [0,1]$ *such that*

$$\|Z(\xi,w)\| \le k_1(\xi)|w|^\sigma + k_2(\xi),$$

for each $(\xi,w) \in [0,1]\times\mathbb{R}$.

Then the BVP (1.1) *has at least one solution on* $[0,1]$.

Proof. The proof is obvious. Here we have $k_1(\xi)\epsilon^\sigma + k_2(\xi)$ in place of $m_\epsilon(\xi)$. □

4. Examples

In this section, we give some examples in order to illustrate our results.

Example 4.1. As the first example, let us consider the following fractional differential inclusion

$$\begin{cases} {}^cD^{\frac{9}{2}}w(\xi) \in Z(\xi,w(\xi)), & \xi\in[0,1],\\ w(\frac{1}{10}) = 0, \quad w'(0)=0, & I^{\frac{7}{2}}w(1)=0,\end{cases} \tag{4.1}$$

where $Z(\xi,w(\xi))$ is such that $\|Z(\xi,w)\| \le \frac{1}{8(\xi+1)}|w| + e^{-\xi}$.

Here $\beta_2 = \frac{9}{2}$, implying $n=5$, $\nu = \frac{1}{10}$, $\beta_1 = \frac{7}{2}$,

$$\nu^{n-1} = \nu^4 = \frac{1}{10000} \neq \frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}$$

$$=\frac{4}{(\beta_1+1)(\beta_1+2)(\beta_1+3)(\beta_1+4)}=\frac{64}{19305}=0.003315.$$

As $\|Z(\xi,w)\|\leq\frac{1}{8(\xi+1)}|w|+e^{-\xi}$, therefore (C) is satisfied with $\|k_1\|_{L^1}=\frac{1}{8}\ln 2$. Further,

$$\begin{aligned}
&\Lambda\|k_1\|_{L^1}\\
=&\|k_1\|_{L^1}\Bigg[\frac{2}{\Gamma(\beta_2+1)}+\frac{\Gamma(\beta_1+5)}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)|\Gamma(5)-\nu^4(\beta_1+4)(\beta_1+3)(\beta_1+2)(\beta_1+1)|}\\
&+\frac{\Gamma(\beta_1+5)}{\Gamma(\beta_1+\beta_2+1)|\Gamma(5)-\nu^4(\beta_1+4)(\beta_1+3)(\beta_1+2)(\beta_1+1)|}\Bigg]\\
\approx&\frac{1}{8}\ln 2\left[\frac{64}{945\sqrt{\pi}}+\frac{286}{7\sqrt{\pi}\times 3.879344}+\frac{2027025\sqrt{\pi}}{2^8\times 7!\times 3.879344}\right]\\
\approx&\frac{1}{8}\ln 2[0.03821+5.942029+0.717803]\\
\approx&0.58034<1.
\end{aligned}$$

Thus, by Theorem 3.2, there exists at least one solution of the fractional differential inclusion (4.1).

Example 4.2. Now, consider the following fractional inclusion

$$\begin{cases}{}^cD^{\frac{5}{2}}w(\xi)\in Z(\xi,w(\xi)), & \xi\in[0,1],\\ w(\frac{1}{2})=0,\quad w'(0)=0, & I^{\frac{3}{2}}w(1)=0,\end{cases}\tag{4.2}$$

where $Z(\xi,w(\xi))$ is such that $\|Z(\xi,w)\|\leq\frac{1}{4(\xi+1)^2}|w|^{\frac{1}{3}}+e^{-\xi}$.

Here $\beta_2=\frac{5}{2}$ implies $n=3,\ \nu=\frac{1}{2},\ \beta_1=\frac{3}{2}$,

$$\nu^{n-1}=\nu^2=\frac{1}{4}\neq\frac{\Gamma(n)}{(\beta_1+n-1)(\beta_1+n-2)\cdots(\beta_1+1)}=\frac{2}{(\beta_1+2)(\beta_1+1)}=\frac{8}{35}.$$

Also, (D) is satisfied with $k_1(\xi)=\frac{1}{4(\xi+1)^2}$ and $k_2(\xi)=e^{-\xi}$ with $\sigma=\frac{1}{3}$. Therefore, it follows from Theorem 3.3 that there exists at least one solution of (4.2).

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