

## THE NEW INEQUALITIES FOR $tgs$ -CONVEX FUNCTIONS

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**ABSTRACT.** In this paper, we establish some Hadamard-Hadamard type inequalities for  $tgs$ -convex functions. Our results are the generalizations of some known results. The new generalized estimate of the midpoints product of two  $tgs$ -convex functions is also considered.

### 1. INTRODUCTION

**Definition 1.1.** A function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is said to be convex on  $I$  if the inequality

$$(1.1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ . We say that  $f$  is concave if  $(-f)$  is convex.

For convex functions, we have the following inequality which is known in the literature as Hermite-Hadamard inequality.

**Theorem 1.1.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and  $a, b \in I$  with  $a < b$ . Then

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

If  $f$  is a positive concave function, then the inequality is reversed.

In 1906, Fejér [1] showed the following weighted generalization of inequality (1.2).

**Theorem 1.2.** If  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is a convex function, then the following inequality holds:

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \int_a^b q(t)dt \leq \frac{1}{b-a} \int_a^b f(t)q(t)dt \leq \frac{f(a) + f(b)}{2} \int_a^b q(t)dt,$$

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where  $q : [a, b] \rightarrow \mathbb{R}$  is positive, integrable, and symmetric with respect to  $\frac{a+b}{2}$ .

Some refinements, variations, generalizations and improvements of inequalities (1.2) and (1.3) can be seen [2, 3] and [4].

**Definition 1.2** ([5]). Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a nonnegative function.  $f$  is called a *tgs-convex* function on  $I$  if the inequality

$$(1.4) \quad f(tx + (1-t)y) \leq t(1-t)(f(x) + f(y))$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ . We say that  $f$  is *tgs-concave* if  $(-f)$  is *tgs-convex*.

For *tgs-convex* functions, the following results hold [5].

**Theorem 1.3.** Assume that  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is a *tgs-convex* function and  $a, b \in I$  with  $a < b$ , then we have

$$(1.5) \quad 2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{6}.$$

**Theorem 1.4.** Assume that  $f$  and  $g$  are real valued, nonnegative *tgs-convex* functions on  $[a, b]$ , then we have

$$(1.6) \quad 8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)g(t)dt + \frac{1}{30} [M(a, b) + N(a, b)],$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$  and  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

The recent results on *tgs-convex* functions can be seen in [5, 6] and [7].

In this paper, we give the improvements of (1.5) and (1.6). The weighted generalization of inequality (1.5) are also established.

## 2. MAIN RESULTS

The following result is an improvement of (1.5).

**Theorem 2.1.** Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is a *tgs-convex* function, then we have

$$(2.1) \quad \begin{aligned} 4f\left(\frac{a+b}{2}\right) &\leq f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \\ &\leq \frac{1}{b-a} \int_a^b f(t)dt \\ &\leq \frac{f(a) + f(b)}{12} + \frac{f\left(\frac{a+b}{2}\right)}{6} \\ &\leq \frac{f(a) + f(b)}{8}. \end{aligned}$$

*Proof.* Using (1.5) in  $[a, \frac{a+b}{2}]$  and  $[\frac{a+b}{2}, b]$ , we have

$$\begin{aligned} 2f\left(\frac{3a+b}{4}\right) &\leq \frac{2}{b-a} \int_a^{\frac{a+b}{2}} f(t)dt \leq \frac{f(a) + f(\frac{a+b}{2})}{6}, \\ 2f\left(\frac{a+3b}{4}\right) &\leq \frac{2}{b-a} \int_{\frac{a+b}{2}}^b f(t)dt \leq \frac{f(b) + f(\frac{a+b}{2})}{6}. \end{aligned}$$

Form the above inequalities, we have

$$f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b) + 2f(\frac{a+b}{2})}{12}.$$

A combination of the above inequality and the following results

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{(3a+b)/4 + (a+3b)/4}{2}\right) \leq \frac{1}{4} \left( f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right), \\ f\left(\frac{a+b}{2}\right) &\leq \frac{f(a) + f(b)}{4}, \end{aligned}$$

deduces the desired inequality (2.1).  $\square$

The following Hadamard-Hadamard-Fejér type inequality for  $tgs$ -convex function holds.

**Theorem 2.2.** Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is a  $tgs$ -convex function, then we have

$$\begin{aligned} (2.2) \quad 2f\left(\frac{a+b}{2}\right) \int_a^b q(x)dx &\leq \int_a^b f(x)q(x)dx \\ &\leq (f(a) + f(b)) \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} q(x)dx, \end{aligned}$$

where  $q : [a, b] \rightarrow \mathbb{R}$  is positive, integrable, and symmetric with respect to  $\frac{a+b}{2}$ .

*Proof.* Since  $q(x) = q(a+b-x)$ , we have

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) \int_a^b q(x)dx &\leq 2 \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right) q(x)dx, \\ &\leq \frac{1}{2} \int_a^b f(x)q(x)dx + \frac{1}{2} \int_a^b f(a+b-x)q(a+b-x)dx \\ &= \int_a^b f(x)q(x)dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_a^b f(x)q(x)dx &= (b-a) \int_0^1 f(tb + (1-t)a)q(tb + (1-t)a)dt \\ &\leq (b-a)(f(a) + f(b)) \int_0^1 t(1-t)q(tb + (1-t)a)dt \end{aligned}$$

$$=(f(a) + f(b)) \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} q(x) dx. \quad \square$$

*Remark 2.1.* We get (1.5) by putting  $q(x) = 1$  in (2.2).

The following inequalities are improvements of (1.6).

**Theorem 2.3.** *Assume that  $f$  and  $g$  are real valued, nonnegative tgs-convex functions on  $[a, b]$ , then we have*

$$\begin{aligned} 8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(t)q(t)dt \\ &\quad + \frac{1}{60}(N((a+b)/2, (a+b)/2) + N(a, b) \\ &\quad + N(a, (a+b)/2) + N((a+b)/2, b)] \\ &\leq \frac{1}{b-a} \int_a^b f(t)dt \\ &\quad + \frac{1}{480}[5M(a, b) + 13N(a, b)], \end{aligned}$$

where  $M(a, b)$  and  $N(a, b)$  are defined in Theorem 1.4.

*Proof.* For  $\lambda \in [0, 1]$ , we have

$$\begin{aligned} &8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ &= 8f\left(\frac{(1-\lambda)b + \lambda(a+b)/2}{2} + \frac{(1-\lambda)a + \lambda(a+b)/2}{2}\right) \\ &\quad \times g\left(\frac{(1-\lambda)b + \lambda(a+b)/2}{2} + \frac{(1-\lambda)a + \lambda(a+b)/2}{2}\right) \\ &\leq \frac{1}{2}f((1-\lambda)b + \lambda(a+b)/2)g((1-\lambda)b + \lambda(a+b)/2) \\ &\quad + \frac{1}{2}f((1-\lambda)a + \lambda(a+b)/2)g((1-\lambda)a + \lambda(a+b)/2) \\ &\quad + \frac{1}{2}f((1-\lambda)b + \lambda(a+b)/2)g((1-\lambda)a + \lambda(a+b)/2) \\ &\quad + \frac{1}{2}f((1-\lambda)a + \lambda(a+b)/2)g((1-\lambda)b + \lambda(a+b)/2) \\ &\leq \frac{1}{2}f((1-\lambda)b + \lambda(a+b)/2)g((1-\lambda)b + \lambda(a+b)/2) \\ &\quad + \frac{1}{2}f((1-\lambda)a + \lambda(a+b)/2)g((1-\lambda)a + \lambda(a+b)/2) \\ &\quad + \frac{1}{2}(1-\lambda)^2\lambda^2[f(a) + f((a+b)/2)](g((a+b)/2) + g(b)) \\ &\quad + (f((a+b)/2) + f(b))(g((a+b)/2) + g(a)) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} f((1-\lambda)b + \lambda(a+b)/2) g((1-\lambda)b + \lambda(a+b)/2) \\
&\quad + \frac{1}{2} f((1-\lambda)a + \lambda(a+b)/2) g((1-\lambda)a + \lambda(a+b)/2) \\
&\quad + \frac{1}{2} (1-\lambda)^2 \lambda^2 [N((a+b)/2, (a+b)/2)] \\
&\quad + N(a, b) + N(a, (a+b)/2) + N((a+b)/2, b)].
\end{aligned}$$

Integrating both sides of the above inequality with respect to  $\lambda$  over  $[0, 1]$ , we have

$$\begin{aligned}
&8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{2} \int_0^1 f((1-\lambda)b + \lambda(a+b)/2) g((1-\lambda)b + \lambda(a+b)/2) d\lambda \\
&\quad + \frac{1}{2} \int_0^1 f((1-\lambda)a + \lambda(a+b)/2) g((1-\lambda)a + \lambda(a+b)/2) d\lambda \\
&\quad + \frac{1}{2} \int_0^1 (1-\lambda)^2 \lambda^2 [N((a+b)/2, (a+b)/2) + N(a, b) \\
&\quad + N(a, (a+b)/2) + N((a+b)/2, b)] d\lambda \\
&= \frac{1}{b-a} \left[ \int_{\frac{a+b}{2}}^b f(x)g(x)dx + \int_a^{\frac{a+b}{2}} f(x)g(x)dx \right] \\
&\quad + \frac{1}{60} [N((a+b)/2, (a+b)/2) + N(a, b) \\
&\quad + N(a, (a+b)/2) + N((a+b)/2, b)] \\
&= \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{60} [N((a+b)/2, (a+b)/2) + N(a, b) \\
&\quad + N(a, (a+b)/2) + N((a+b)/2, b)].
\end{aligned}$$

On the other hand, since

$$(2.3) \quad N((a+b)/2, (a+b)/2) \leq \frac{1}{8} [M(a, b) + N(a, b)]$$

and

$$(2.4) \quad N(a, (a+b)/2) + N((a+b)/2, b) \leq \frac{1}{2} [M(a, b) + N(a, b)],$$

we have

$$\begin{aligned}
8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x)g(x)dx \\
&\quad + \frac{1}{60} (N((a+b)/2, (a+b)/2) + N(a, b) \\
&\quad + N(a, (a+b)/2) + N((a+b)/2, b)] \\
&\leq \frac{1}{b-a} \int_a^b f(t)dt
\end{aligned}$$

$$+ \frac{1}{480}[5M(a, b) + 13N(a, b)]. \quad \square$$

### 3. APPLICATIONS TO PROBABILITY DENSITY FUNCTION

Let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with the probability density function  $f : [a, b] \rightarrow [0, 1]$  with the cumulative distribution function  $F(x) = Pr(X \leq x) = \int_a^x f(t)dt$ .

**Theorem 3.1.** *With the assumptions of Theorem 2.1, we have the inequality*

$$(3.1) \quad \begin{aligned} 4F\left(\frac{a+b}{2}\right) &\leq F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) \\ &\leq \frac{b - E(X)}{b - a} \\ &\leq \frac{F(a) + F(b)}{12} + \frac{F((a+b)/2)}{6} \\ &\leq \frac{F(a) + F(b)}{8}. \end{aligned}$$

*Proof.* In the proof of Theorem 2.1, letting  $f = F$ , and taking into account that

$$E(x) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (3.1). □

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