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HESITANT FUZZY SET THEORY APPLIED TO HILBERT ALGEBRAS

AIYARED IAMPAN[1,*], S. YAMUNADEVI[2], P. MARAGATHA MEENAKSHI[3], AND NEELAMEGARAJAN RAJESH,[4]

Abstract. The concept of hesitant fuzzy sets (HFSs) was first introduced by Torra (V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst. **25** (2010), 529–539). In this paper, the concept of HFSs to subalgebra, ideals, and deductive systems of Hilbert algebras is introduced. The relationships between hesitant fuzzy subalgebras (HF subalgebras), hesitant fuzzy ideals (HF ideals), and hesitant fuzzy deductive systems (HF deductive systems) and their level subsets are provided.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [21]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration of fuzzy sets and uncertainty theories, such as soft sets and rough sets, has been discussed in [1, 2, 5].

In 2009-2010, Torra and Narukawa [19, 20] introduced the notion of hesitant fuzzy sets, which is, a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion of fuzzy sets. The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the notion of hesitant fuzzy sets by Torra and Narukawa [19, 20], several researches were conducted on the generalizations of the notion of hesitant fuzzy sets and application

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to many logical algebras, such as: in 2012, Zhu et al. [23] introduced the notion of dual hesitant fuzzy sets, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin [15] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [16] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. In 2015, Jun and Song [17] introduced the notions of hesitant fuzzy prefilters (resp., filters) and positive implicative hesitant fuzzy prefilters (resp., filters) of EQ-algebras. In 2016, Jun and Ahn [14] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras. Iampan [11] introduced a new algebraic structure called a UP-algebra, and Mosrijai et al. [18] introduced the notion of hesitant fuzzy sets on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy ideals play an important role in studying the many logical algebras. Diego proved [7] that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [3, 4] and Jun [13], and some of their filters forming deductive systems were recognized. Dudek [8] considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras. In 2022, Iampan et al. [12] introduced the concepts of anti-hesitant fuzzy subalgebras, ideals, and deductive systems of Hilbert algebras.

This paper introduces the concept of HFSs to subalgebra, ideals, and deductive systems of Hilbert algebras, which provides a generalization of the concept of HF fuzzy subalgebras/ideals/deductive systems. The relationship between HF subalgebras/ideals/deductive systems and their level subsets is provided. In the future, our research team hopes to apply these concepts to solving decision-making problems.

2. Preliminaries

Before we begin, let us go through the concept of Hilbert algebras as described by Diego [7] in 1966.

Definition 2.1 ([7])**.** A *Hilbert algebra* is a triplet $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of X such that the following axioms hold:

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following result was proved in [8].

Lemma 2.1. *Let* $X = (X, \cdot, 1)$ *be a Hilbert algebra. Then,*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
(5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2 ([22])**.** A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *subalgebra* of X if $x \cdot y \in D$ for all $x, y \in D$.

Definition 2.3 ([6])**.** A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:

(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.4 ([10])**.** A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *deductive system* of X if

(1) $1 \in D$,
(2) $(\forall x, y \in X)(x, x \cdot y \in D \Rightarrow y \in D)$.

A *fuzzy set* [21] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.5 ([8])**.** A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.6 ([9])**.** A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy ideal* of X if the following conditions hold:

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.7 ([8])**.** A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy deductive system* of X if the following conditions hold:

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(y) \geq \min\{\mu(x \cdot y), \mu(x)\})$.

Definition 2.8 ([19])**.** Let X be a reference set. A *hesitant fuzzy set* (HFS) on X is a mapping $h : X \to \mathcal{P}([0, 1])$, where $\mathcal{P}([0, 1])$ means the power set of $[0, 1]$.

We will review the definition of the characteristic HFS, which is an important and convenient tool for investigating various properties of HF subalgebras/ideals/deductive systems of Hilbert algebras.

Let X be a reference set. If $Y \subseteq X$, the characteristic HFS h_Y on X is a function of X into $\mathcal{P}([0,1])$ defined as follows:

$$(\forall x \in X)\left(h_Y(x) = \begin{cases} [0,1], & \text{if } x \in Y, \\ \emptyset, & \text{otherwise} \end{cases}\right). \tag{2.1}$$

By the definition of characteristic HFSs, h_Y is a function of a nonempty set X into $\{\emptyset, [0,1]\}$. Hence, h_Y is an HFS on X.

Definition 2.9 ([19])**.** Let h be an HFS on a nonempty set X. The HFS h is defined by $\overline{h}(x) = [0,1] - h(x)$ for all $x \in X$ which is said to be the *complement* of h on X.

3. Hesitant Fuzzy Subalgebras/Ideals/Deductive System of Hilbert Algebras

In this section, we introduce the concepts of HF subalgebras/ideals/deductive systems of Hilbert algebras and investigate some related properties.

Definition 3.1. An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is called a *hesitant fuzzy subalgebra* (HF subalgebra) of X if it satisfies the following property:

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(x) \cap h(y)). \tag{3.1}$$

Example 3.1. Let $X = \{a, b, c, d, 1\}$ with the following Cayley table:

$\cdot$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

Then, X is a Hilbert algebra. We define an HFS h on X as follows:

$$h(1) = \{0.5, 0.2\}, \quad h(a) = h(b) = h(c) = h(d) = \{0.2\}.$$

Then, h is an HF subalgebra of X.

Proposition 3.1. *If h is an HF subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x \in X)(h(1) \supseteq h(x)). \tag{3.2}$$

Proof. For any $x \in X$, we have $h(1) = h(x \cdot x) \supseteq h(x) \cap h(x) = h(x)$. □

Lemma 3.1. *The constant 1 of a Hilbert algebra $X = (X, \cdot, 1)$ is in a nonempty subset S of X if and only if $h_S(1) \supseteq h_S(x)$ for all $x \in X$.*

Proof. If $1 \in S$, then $h_S(1) = [0,1]$. Thus, $h_S(1) = [0,1] \supseteq h_S(x)$ for all $x \in X$.

Conversely, assume that $h_S(1) \supseteq h_S(x)$ for all $x \in X$. Since S is a nonempty subset of X, we have $a \in S$ for some $a \in X$. Thus, $h_S(1) \supseteq h_S(a) = [0,1]$, so $h_S(1) = [0,1]$. Hence, $1 \in S$. □

Theorem 3.1. *A nonempty subset S of a Hilbert algebra $X = (X, \cdot, 1)$ is a subalgebra of X if and only if the characteristic HFS h_S is an HF subalgebra of X.*

Proof. Assume that S is a subalgebra of X. Let $x, y \in X$.

Case 1: $x, y \in S$. Then, $h_S(x) = [0, 1]$ and $h_S(y) = [0, 1]$. Thus, $h_S(x) \cap h_S(y) = [0, 1]$. Since S is a subalgebra of X, we have $x \cdot y \in S$ and so $h_S(x \cdot y) = [0, 1]$. Therefore, $h_S(x \cdot y) = [0, 1] \supseteq [0, 1] = h_S(x) \cap h_S(y)$.

Case 2: $x \in S$ and $y \notin S$. Then, $h_S(x) = [0, 1]$ and $h_S(y) = \emptyset$. Thus, $h_S(x) \cap h_S(y) = \emptyset$. Therefore, $h_S(x \cdot y) \supseteq \emptyset = h_S(x) \cap h_S(y)$.

Case 3: $x \notin S$ and $y \in S$. Then, $h_S(x) = \emptyset$ and $h_S(y) = [0, 1]$. Thus, $h_S(x) \cap h_S(y) = \emptyset$. Therefore, $h_S(x \cdot y) \supseteq \emptyset = h_S(x) \cap h_S(y)$.

Case 4: $x \notin S$ and $y \notin S$. Then, $h_S(x) = \emptyset$ and $h_S(y) = \emptyset$. Thus, $h_S(x) \cap h_S(y) = \emptyset$. Therefore, $h_S(x \cdot y) \supseteq \emptyset = h_S(x) \cap h_S(y)$.

Hence, h_S is an HF subalgebra of X.

Conversely, assume that h_S is an HF subalgebra of X. Since $h_S(1) \supseteq h_S(x)$ for all $x \in X$, it follows from Lemma 3.1 that $1 \in S$. Let $x, y \in S$. Then, $h_S(x) = [0, 1]$ and $h_S(y) = [0, 1]$. Thus, $h_S(x \cdot y) \supseteq h_S(x) \cap h_S(y) = [0, 1]$, so $h_S(x \cdot y) = [0, 1]$. Hence, $x \cdot y \in S$ and so S is a subalgebra of X. □

Definition 3.2. An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is called a *hesitant fuzzy ideal* (HF ideal) of X if it satisfies the following properties:

$$(\forall x \in X)(h(1) \supseteq h(x)), \tag{3.3}$$

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(y)), \tag{3.4}$$

$$(\forall x, y_1, y_2 \in X)(h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2)). \tag{3.5}$$

Example 3.2. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then, X is a Hilbert algebra. We define an HFS h on X as follows:

$$h(1) = \{0.4, 0.5, 0.7\}, \quad h(x) = \{0.4, 0.5\}, \quad h(y) = \{0.5\}, \quad h(z) = h(0) = \emptyset.$$

Then, h is an HF ideal of X.

Proposition 3.2. *If h is an HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(h((y \cdot x) \cdot x) \supseteq h(y)). \tag{3.6}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (3.5), we have $h((y \cdot x) \cdot x) \supseteq h(y) \cap h(1) = h(y)$ for all $x, y \in X$. □

Lemma 3.2. *If h is an HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(x \leq y \Rightarrow h(x) \subseteq h(y)). \tag{3.7}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then, $x \cdot y = 1$ and so

$$\begin{aligned} h(y) =& h(1 \cdot y) = h(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ \supseteq& h(x \cdot y) \cap h(x) \\ =& h(1) \cap h(x) = h(x). \end{aligned}$$

□

Theorem 3.2. *Every HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an HF subalgebra of X.*

Proof. Let h be an HF ideal of X and let $x, y \in X$. Since $y \leq x \cdot y$ and by Lemma 3.2, we have $h(y) \supseteq h(x \cdot y)$. It follows from (3.4) that

$$h(x \cdot y) \supseteq h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq h(x) \cap h(y).$$

Hence, h is an HF subalgebra of X. □

Definition 3.3. An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is called a *hesitant fuzzy deductive system* (HF deductive system) of X if it satisfies the following properties:

$$(\forall x \in X)(h(1) \supseteq h(x)), \tag{3.8}$$

$$(\forall x, y \in X)(h(y) \supseteq h(x \cdot y) \cap h(x)). \tag{3.9}$$

Proposition 3.3. *Every HF ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an HF deductive system of X.*

Proof. Let h be an HF ideal of X and let $x, y \in X$. If $y_1 = x \cdot y$ and $y_2 = x$, then by Lemma 2.1 and (3.5), we have

$$h(y) = h(1 \cdot y) = h(((x \cdot y) \cdot (x \cdot y)) \cdot y) \supseteq h(x \cdot y) \cap h(x).$$

Hence, h is an HF deductive system of X. □

Lemma 3.3. *If h is an HF deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y, z \in X)(z \leq x \cdot y \Rightarrow h(y) \supseteq h(x) \cap h(z)). \tag{3.10}$$

Proof. Let $x, y, z \in X$ be such that $z \leq x \cdot y$. Then, $z \cdot (x \cdot y) = 1$ and so

$$\begin{aligned} h(y) \supseteq& h(x \cdot y) \cap h(x) \supseteq h(z \cdot (x \cdot y)) \cap h(z) \cap h(x) \\ =& h(1) \cap h(z) \cap h(x) = h(x) \cap h(z). \end{aligned}$$

□

Lemma 3.4. *If h is an HF deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(x \leq y \Rightarrow h(y) \supseteq h(x)). \tag{3.11}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then, $x \cdot y = 1$ and so $h(y) \supseteq h(x \cdot y) \cap h(x) = h(1) \cap h(x) = h(x)$. □

Theorem 3.3. *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is a deductive system of X if and only if the characteristic HFS h_D is an HF deductive system of X.*

Proof. Assume that D is a deductive system of X. Since $1 \in D$, it follows from Lemma 3.1 that $h_D(1) \supseteq h_D(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1: $x, y \in D$. Then, $h_D(x) = [0, 1]$ and $h_D(y) = [0, 1]$. Thus, $h_D(y) = [0, 1] \supseteq h_D(x \cdot y) = h_D(x \cdot y) \cap h_D(x)$.

Case 2: $x \notin D$ and $y \in D$. Then, $h_D(x) = \emptyset$ and $h_D(y) = [0, 1]$. Thus, $h_D(y) = [0, 1] \supseteq \emptyset = h_D(x \cdot y) \cap h_D(x)$.

Case 3: $x \in D$ and $y \notin D$. Then, $h_D(x) = [0, 1]$ and $h_D(y) = \emptyset$. Since D is a deductive system of X, we have $x \cdot y \notin D$ or $x \notin D$. But $x \in D$, so $x \cdot y \notin D$. Then, $h_D(x \cdot y) = \emptyset$. Thus, $h_D(y) = \emptyset \supseteq \emptyset = h_D(x \cdot y) \cap h_D(x)$.

Case 4: $x \notin D$ and $y \notin D$. Then, $h_D(x) = \emptyset$ and $h_D(y) = \emptyset$. Thus, $h_D(y) = \emptyset \supseteq \emptyset = h_D(x \cdot y) \cap h_D(x)$.

Hence, h_D is an HF deductive system of X.

Conversely, assume that h_D is an HF deductive system of X. Since $h_D(1) \supseteq h_D(x)$ for all $x \in X$, it follows from Lemma 3.1 that $1 \in D$. Next, let $x, y \in X$ be such that $x \cdot y \in D$ and $x \in D$. Then, $h_D(x \cdot y) = [0, 1]$ and $h_D(x) = [0, 1]$. Thus, $h_D(y) \supseteq h_D(x \cdot y) \cap h_D(x) = [0, 1]$, so $h_D(y) = [0, 1]$. Therefore, $y \in D$ and so D is a deductive system of X. □

Theorem 3.4. *A nonempty subset I of a Hilbert algebra $X = (X, \cdot, 1)$ is an ideal of X if and only if the characteristic HFS h_I is an HF ideal of X.*

Proof. Assume that I is an ideal of X. Since $1 \in I$, it follows from Lemma 3.1 that $h_I(1) \supseteq h_I(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1: $x, y \in I$. Then $h_I(x) = [0, 1]$ and $h_I(y) = [0, 1]$. Thus, $h_I(x \cdot y) = [0, 1] \supseteq h_I(y)$.

Case 2: $x \notin I$ and $y \in I$. Then, $h_I(x) = \emptyset$ and $h_I(y) = [0, 1]$. Thus, $h_I(x \cdot y) = [0, 1] \supseteq h_I(y)$.

Case 3: $x \in I$ and $y \notin I$. Then, $h_I(x) = [0, 1]$ and $h_I(y) = \emptyset$. Since I is an ideal of X, we have $x \cdot y \notin I$ or $x \notin I$. But $x \in I$, so $x \cdot y \notin I$. Then, $h_I(x \cdot y) = \emptyset$. Thus, $h_I(x \cdot y) = \emptyset \supseteq \emptyset = h_I(y)$.

Case 4: $x \notin I$ and $y \notin I$. Then, $h_I(x) = \emptyset$ and $h_I(y) = \emptyset$. Thus, $h_I(x \cdot y) = \emptyset \supseteq \emptyset = h_I(y)$.

Now, let $x, y_1, y_2 \in X$.

Case 1: $x, y_1, y_2 \in I$. Then, $h_I(x) = [0, 1]$, $h_I(y_1) = [0, 1]$, and $h_I(y_2) = [0, 1]$. Since I is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = [0, 1] \supseteq h_I(y_1) \cap h_I(y_2)$.

Case 2: $x \notin I$ and $y_1, y_2 \in I$. Then, $h_I(x) = \emptyset$ and $h_I(y_1) = h_I(y_2) = [0, 1]$. Since I is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) = [0, 1] \supseteq h_I(y_1) \cap h_I(y_2)$.

Case 3: $x \in I$, and $y_1 \notin I$ or $y_2 \notin I$. Then, $h_I(x) = [0,1]$ and $h_I(y_1) = \emptyset$ or $h_I(y_2) = \emptyset$. Since I is an ideal of X, we have $y_1 \cdot (y_2 \cdot x) \cdot x \notin I$. Then, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset \supseteq \emptyset = h_I(y_1) \cap h_I(y_2)$.

Case 4: $x \notin I$, and $y_1 \notin I$ or $y_2 \notin I$. Then, $h_I(x) = \emptyset$, and $h_I(y_1) = \emptyset$ or $h_I(y_2) = \emptyset$. Since I is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \notin I$. Then, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = \emptyset \supseteq \emptyset = h_I(y_1) \cap h_I(y_2)$.

Hence, h_I is an HF ideal of X.

Conversely, assume that h_I is an HF ideal of X. Since $h_I(1) \supseteq h_I(x)$ for all $x \in X$, it follows from Lemma 3.1 that $1 \in I$. Let $x \in X$ and $y \in I$. Then, $h_I(y) = [0,1]$. Thus, $h_I(x \cdot y) \supseteq h_I(y) = [0,1]$, so $h_I(x \cdot y) = [0,1]$. Hence, $x \cdot y \in I$. Next, let $x, y_1, y_2 \in X$ be such that $y_1 \in I$ and $y_2 \in I$. Then, $h_I(y_1) = h_I(y_2) = [0,1]$. Thus, $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h_I(y_1) \cap h_I(y_2) = [0,1]$, so $h_I((y_1 \cdot (y_2 \cdot x)) \cdot x) = [0,1]$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$ and so I is an ideal of X. □

4. Level Subsets of an HFS on Hilbert Algebras

In this section, we provide the relationship between HF subalgebras, ideals, and deductive systems and their level subsets.

Definition 4.1. Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. For any $\pi \in \mathcal{P}([0,1])$, the sets $U(h,\pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h,\pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h,\pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h,\pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h,\pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then, $U(h,\pi) = U^+(h,\pi) \cup E(h,\pi)$ and $L(h,\pi) = L^-(h,\pi) \cup E(h,\pi)$.

Theorem 4.1. *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is an HF subalgebra of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a subalgebra of X.*

Proof. Assume that h is an HF subalgebra of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x \in U(h,\pi)$. Then, $h(x) \supseteq \pi$. Since h is an HF subalgebra of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h,\pi)$. Let $x, y \in U(h,\pi)$. Then, $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Since h is an HF subalgebra of X, we have $h(x \cdot y) \supseteq h(x) \cap h(y) \supseteq \pi$ and thus, $x \cdot y \in U(h,\pi)$. Hence, $U(h,\pi)$ is a subalgebra of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a subalgebra of X. Let $x, y \in X$. Choose $\pi = h(x) \cap h(y) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Thus, $x, y \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in U(h,\pi)$. So, $h(x \cdot y) \supseteq \pi = h(x) \cap h(y)$. Hence, h is an HF subalgebra of X. □

Theorem 4.2. *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is an HF ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is an ideal of X.*

Proof. Assume that h is an HF ideal of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x \in U(h,\pi)$. Then, $h(x) \supseteq \pi$. Since h is an HF ideal of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h,\pi)$. Next, let $x, y \in X$ be such that $y \in U(h,\pi)$. Then, $h(y) \supseteq \pi$. Since h is an HF ideal of X, we have $h(x \cdot y) \supseteq h(y) \supseteq \pi$. So, $x \cdot y \in U(h,\pi)$. Let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in U(h,\pi)$. Then, $h(y_1) \supseteq \pi$ and $h(y_2) \supseteq \pi$. Since h is an HF ideal of X, we have $h((y_1 \cdot (y_2 \cdot x)) \cdot x)) \supseteq h(y_1) \cap h(y_2) \supseteq \pi$. So, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(h,\pi)$. Hence, $U(h,\pi)$ is an ideal of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is an ideal of X. Let $x \in X$. Then $h(x) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then, $h(x) \supseteq \pi$. Thus, $x \in U(h,\pi) \neq \emptyset$. By assumption, we have $U(h,\pi)$ is an ideal of X and thus, $1 \in U(h,\pi)$. So, $h(1) \supseteq \pi = h(x)$. Next, let $x, y \in X$. Then, $h(x), h(y) \in \mathcal{P}([0,1])$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(y) \supseteq \pi$, so $x \in X$ and $y \in U(h,\pi) \neq \emptyset$. By assumption, we have $U(h,\pi)$ is an ideal of X and then $x \cdot y \in U(h,\pi)$. Thus, $h(x \cdot y) \supseteq \pi = h(y)$. Let $x, y_1, y_2 \in X$. Then, $h(x), h(y_1), h(y_2) \in \mathcal{P}([0,1])$. Choose $\pi = h(y_1) \cap h(y_2) \in \mathcal{P}([0,1])$. Then, $h(y_1) \supseteq \pi$ and $h(y_2) \supseteq \pi$, so $x \in X$ and $y_1, y_2 \in U(h,\pi) \neq \emptyset$. By assumption, we have $U(h,\pi)$ is an ideal of X and then $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(h,\pi)$. Thus, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq \pi = h(y_1) \cap h(y_2)$. Hence, h is an HF ideal of X. □

Theorem 4.3. *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is an HF deductive system of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a deductive system of X.*

Proof. Assume that h is an HF deductive system of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x \in U(h,\pi)$. Then, $h(x) \supseteq \pi$. Since h is an HF deductive system of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h,\pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in U(h,\pi)$. Then, $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Since h is an HF deductive system of X, we have $h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq \pi$. So, $y \in U(h,\pi)$. Hence, $U(h,\pi)$ is a deductive system of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U(h,\pi)$ of X is a deductive system of X. Let $x \in X$. Then $h(x) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$. Thus, $x \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a deductive system of X and thus $1 \in U(h,\pi)$. So, $h(1) \supseteq \pi = h(x)$. Let $x, y \in X$. Then, $h(x), h(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \cap h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$, so $x, x \cdot y \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a deductive system of X and then $y \in U(h,\pi)$. Thus, $h(y) \supseteq \pi = h(x) \cap h(x \cdot y)$. Hence, h is an HF deductive system of X. □

Theorem 4.4. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

(1) *If h is an HF subalgebra of X, then for all $\pi \in \mathcal{P}([0,1])$, $U^+(h,\pi)$ is a subalgebra of X if $U^+(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
(2) *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a subalgebra of X, then h is an HF subalgebra of X.*

Proof. (1) It is straightforward by Theorem 4.1.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a subalgebra of X. Assume that there exist $x, y \in X$ such that $h(x \cdot y) \not\supseteq h(x) \cap h(y)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(x \cdot y) \subset h(x) \cap h(y)$. Then, $h(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(x) \supset \pi$ and $h(y) \supset \pi$. Thus, $x, y \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in U^+(h,\pi)$. So, $h(x \cdot y) \supset \pi = h(x \cdot y)$, a contradiction. Hence, $h(x \cdot y) \supseteq h(x) \cap h(y)$ for all $x, y \in X$. Therefore, h is an HF subalgebra of X. □

Theorem 4.5. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

1. *If h is an HF ideal of X, then for all $\pi \in \mathcal{P}([0,1])$, $U^+(h,\pi)$ is an ideal of X if $U^+(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
2. *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is an ideal of X, then h is an HF ideal of X.*

Proof. (1) It is straightforward by Theorem 4.2.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is an ideal of X. Assume that there exists $x \in X$ such that $h(1) \not\supseteq h(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(1) \subset h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then $h(x) \supset h(1) = \pi$. Thus, $x \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is an ideal of X and thus $1 \in U^+(h,\pi)$. So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $h(1) \supseteq h(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $h(x \cdot y) \not\supseteq h(y)$. Since $\mathrm{Im}(h)$ is a chain, we have $h(x \cdot y) \subset h(y)$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(y) \supset \pi$. Thus $y \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is an ideal of X and thus $x \cdot y \in U^+(h,\pi)$. So, $h(x \cdot y) \supset \pi = h(x \cdot y)$, a contradiction. Hence, $h(x \cdot y) \supseteq h(y)$ for all $x, y \in X$. Assume that there exist $x, y_1, y_2 \in X$ such that $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \not\supseteq h(y_1) \cap h(y_2)$. Since $\mathrm{Im}(h)$ is a chain, we have $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subset h(y_1) \cap h(y_2)$. Choose $\pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x) \in \mathcal{P}([0,1])$. Then, $h(y_1) \supset \pi$ and $h(y_2) \supset \pi$. Thus, $y_1, y_2 \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is an ideal of X and thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U^+(h,\pi)$. So, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supset \pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x)$, a contradiction. Hence, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2)$ for all $x, y_1, y_2 \in X$. Therefore, h is an HF ideal of X. □

Theorem 4.6. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

1. *If h is an HF deductive system of X, then for all $\pi \in \mathcal{P}([0,1])$, $U^+(h,\pi)$ is a deductive system of X if $U^+(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
2. *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a deductive system of X, then h is an HF deductive system of X.*

Proof. (1) It is straightforward by Theorem 4.3.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $U^+(h,\pi)$ of X is a deductive system of X. Assume that there exists $x \in X$ such that

$h(1) \not\supseteq h(x)$. Since $\text{Im}(h)$ is a chain, we have $h(1) \subset h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then, $h(x) \supset h(1) = \pi$. Thus, $x \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is a deductive system of X and thus, $1 \in U^+(h,\pi)$. So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $h(1) \supseteq h(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $h(y) \not\supseteq h(x \cdot y) \cap h(x)$. Since $\text{Im}(h)$ is a chain, we have $h(y) \subset h(x \cdot y) \cap h(x)$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(x \cdot y) \supset \pi$ and $h(x) \supset \pi$. Thus, $x \cdot y, x \in U^+(h,\pi) \neq \emptyset$. By assumption, we have $U^+(h,\pi)$ is a deductive system of X and thus $y \in U^+(h,\pi)$. So, $h(y) \supset \pi = h(y)$, a contradiction. Hence, $h(y) \supseteq h(x \cdot y) \cap h(x)$ for all $x, y \in X$. Therefore, h is an HF deductive system of X. □

Theorem 4.7. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then, $\overline{h}$ is an HF subalgebra of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is a subalgebra of X.*

Proof. Assume that $\overline{h}$ is an HF subalgebra of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(h,\pi) \neq \emptyset$ and let $x, y \in L(h,\pi)$. Then $h(x) \subseteq \pi$ and $h(y) \subseteq \pi$. Since $\overline{h}$ is an HF subalgebra of X, we have $\overline{h}(x \cdot y) \supseteq \overline{h}(x) \cap \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \supseteq ([0,1] - h(x)) \cap ([0,1] - h(y)) = [0,1] - (h(x) \cup h(y))$. Thus, $h(x \cdot y) \subseteq h(x) \cup h(y) \subseteq \pi$. So, $x \cdot y \in L(h,\pi)$. Hence, $L(h,\pi)$ is a subalgebra of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is a subalgebra of X. Let $x, y \in X$. Choose $\pi = h(x) \cup h(y) \in \mathcal{P}([0,1])$. Then $h(x) \subseteq \pi$ and $h(y) \subseteq \pi$. Thus, $x, y \in L(h,\pi) \neq \emptyset$. By assumption, we have $L(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in L(h,\pi)$. So, $h(x \cdot y) \subseteq \pi = h(x) \cup h(y)$. Thus, $\overline{h}(x \cdot y) = [0,1] - h(x \cdot y) \supseteq [0,1] - (h(x) \cup h(y)) = ([0,1] - h(x)) \cap ([0,1] - h(y)) = \overline{h}(x) \cap \overline{h}(y)$. Hence, $\overline{h}$ is an HF subalgebra of X. □

Theorem 4.8. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then $\overline{h}$ is an HF ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is an ideal of X.*

Proof. Assume that $\overline{h}$ is an HF ideal of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(h,\pi) \neq \emptyset$ and let $a \in L(h,\pi)$. Then, $h(a) \subseteq \pi$. Since $\overline{h}$ is an HF ideal of X, we have $\overline{h}(1) \supseteq \overline{h}(a)$. Thus, $[0,1] - h(1) \supseteq [0,1] - h(a)$. So, $h(1) \subseteq h(a) \subseteq \pi$. Hence, $1 \in L(h,\pi)$. Next, let $x, y \in X$ be such that $y \in L(h,\pi)$. Then, $h(y) \subseteq \pi$. Since $\overline{h}$ is an HF ideal of X, we have $\overline{h}(x \cdot y) \supseteq \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \supseteq [0,1] - h(y)$. Thus, $h(x \cdot y) \subseteq h(y) \subseteq \pi$. So, $x \cdot y \in L(h,\pi)$. Also, let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in L(h,\pi)$. Then, $h(y_1) \subseteq \pi$ and $h(y_2) \subseteq \pi$. Since $\overline{h}$ is an HF ideal of X, we have $\overline{h}((y_1 \cdot (y_2 \cdot x) \cdot x)) \supseteq \overline{h}(y_1) \cap \overline{h}(y_2)$ and so $[0,1] - h(((y_1 \cdot (y_2 \cdot x)) \cdot x)) \supseteq ([0,1] - h(y_1)) \cap ([0,1] - h(y_2)) = [0,1] - (h(y_1) \cup h(y_2))$. Thus, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq h(x) \cup h(y) \subseteq \pi$. So, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(h,\pi)$. Hence, $L(h,\pi)$ is an ideal of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h,\pi)$ of X is an ideal of X. Let $x \in X$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then, $h(x) \subseteq \pi$, so $x \in L(h,\pi) \neq \emptyset$. By assumption, we have $L(h,\pi)$ is an ideal of X. Thus, $1 \in L(h,\pi)$. So, $h(1) \subseteq \pi = h(x)$. Hence, $\overline{h}(1) = [0,1] - h(1) \supseteq [0,1] - h(x) =$

$\overline{h}(x)$. Next, let $x, y \in X$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(y) \subseteq \pi$. Thus, $y \in L(h, \pi) \neq \emptyset$. By assumption, $L(h, \pi)$ is an ideal of X and thus, $x \cdot y \in L(h, \pi)$. So, $h(x \cdot y) \subseteq \pi = h(y)$. Thus, $\overline{h}(x \cdot y) = [0,1] - h(x \cdot y) \supseteq [0,1] - h(y) = \overline{h}(y)$. Also, let $x, y_1, y_2 \in X$. Choose $\pi = h(y_1) \cup h(y_2) \in \mathcal{P}([0,1])$. Then $h(y_1) \subseteq \pi$ and $h(y_2) \subseteq \pi$. Thus, $y_1, y_2 \in L(h, \pi) \neq \emptyset$. By assumption, $L(h, \pi)$ is an ideal of X and thus, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(h, \pi)$. So, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq \pi = h(y_1) \cup h(y_2)$. Thus, $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) = [0,1] - h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq [0,1] - (h(y_1) \cup h(y_2)) = ([0,1] - h(y_1)) \cap ([0,1] - h(y_2)) = \overline{h}(y_1) \cap \overline{h}(y_2)$. Hence, $\overline{h}$ is an HF ideal of X. □

Theorem 4.9. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then $\overline{h}$ is an HF deductive system of X if and only if for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h, \pi)$ of X is a deductive system of X.*

Proof. Assume that $\overline{h}$ is an HF deductive system of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(h, \pi) \neq \emptyset$ and let $a \in L(h, \pi)$. Then $h(a) \subseteq \pi$. Since $\overline{h}$ is an HF deductive system of X, we have $\overline{h}(1) \supseteq \overline{h}(a)$. Thus $[0,1] - h(1) \supseteq [0,1] - h(a)$. So, $h(1) \subseteq h(a) \subseteq \pi$. Hence, $1 \in L(h, \pi)$. Next, let $x, y \in X$ be such that $x \cdot y, x \in L(h, \pi)$. Then, $h(x \cdot y) \subseteq \pi$ and $h(x) \subseteq \pi$. Since $\overline{h}$ is an HF deductive system of X, we have $\overline{h}(y) \supseteq \overline{h}(x \cdot y) \cap \overline{h}(y)$ and so $[0,1] - h(y) \supseteq ([0,1] - h(x \cdot y)) \cap ([0,1] - h(x)) = [0,1] - (h(x \cdot y) \cup h(x))$. Thus, $h(y) \subseteq h(x \cdot y) \cup h(x) \subseteq \pi$. So, $y \in L(h, \pi)$. Hence, $L(h, \pi)$ is a deductive system of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L(h, \pi)$ of X is a deductive system of X. Let $x \in X$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then, $h(x) \subseteq \pi$, so $x \in L(h, \pi) \neq \emptyset$. By assumption, $L(h, \pi)$ is a deductive system of X. Thus, $1 \in L(h, \pi)$. So, $h(1) \subseteq \pi = h(x)$. Hence, $\overline{h}(1) = [0,1] - h(1) \supseteq [0,1] - h(x) = \overline{h}(x)$. Next, let $x, y \in X$. Choose $\pi = h(x \cdot y) \cup h(x) \in \mathcal{P}([0,1])$. Then, $h(x \cdot y) \subseteq \pi$ and $h(x) \subseteq \pi$. Thus, $x \cdot y, x \in L(h, \pi) \neq \emptyset$. By assumption, $L(h, \pi)$ is a deductive system of X and thus $y \in L(h, \pi)$. So, $h(y) \subseteq \pi = h(x \cdot y) \cup h(x)$. Thus, $\overline{h}(y) = [0,1] - h(y) \supseteq [0,1] - (h(x \cdot y) \cup h(x)) = ([0,1] - h(x \cdot y)) \cap ([0,1] - h(x)) = \overline{h}(x \cdot y) \cap \overline{h}(x)$. Hence, $\overline{h}$ is an HF deductive system of X. □

Theorem 4.10. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then, the following statements hold.*

1. *If $\overline{h}$ is an HF subalgebra of X, then for all $\pi \in \mathcal{P}([0,1])$, $L^-(h, \pi)$ is a subalgebra of X if $L^-(h, \pi)$ is nonempty and $E(h, \pi)$ is empty.*
2. *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h, \pi)$ of X is a subalgebra of X, then $\overline{h}$ is an HF subalgebra of X.*

Proof. (1) It is straightforward by Theorem 4.7.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h, \pi)$ of X is a subalgebra of X. Assume that there exist $x, y \in X$ such that $\overline{h}(x \cdot y) \not\supseteq \overline{h}(x) \cap \overline{h}(y)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(x \cdot y) \subset \overline{h}(x) \cap \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \subset ([0,1] - h(x)) \cap ([0,1] - h(y)) = [0,1] - (h(x) \cup h(y))$. Then, $h(x \cdot y) \supset h(x) \cup h(y)$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then $h(x) \subset \pi$ and $h(y) \subset \pi$.

Thus, $x, y \in L^-(h,\pi) \neq \emptyset$. By assumption, we have $L^-(h,\pi)$ is a subalgebra of X and thus $x \cdot y \in L^-(h,\pi)$. So, $h(x \cdot y) \subset \pi = h(x \cdot y)$, a contradiction. Hence, $\overline{h}(x \cdot y) \supseteq \overline{h}(x) \cap \overline{h}(y)$ for all $x, y \in X$. Therefore, $\overline{h}$ is an HF subalgebra of X. □

Theorem 4.11. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

(1) *If $\overline{h}$ is an HF ideal of X, then for all $\pi \in \mathcal{P}([0,1])$, $L^-(h,\pi)$ is an ideal of X if $L^-(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
(2) *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h,\pi)$ of X is an ideal of X, then $\overline{h}$ is an HF ideal of X.*

Proof. (1) It is straightforward by Theorem 4.8.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h,\pi)$ of X is an ideal of X. Assume that there exists $x \in X$ such that $\overline{h}(1) \not\supseteq \overline{h}(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(1) \subset \overline{h}(x)$. Then, $[0,1] - h(1) \subset [0,1] - h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then, $h(x) \subset h(1) = \pi$. Thus, $x \in L^-(h,\pi) \neq \emptyset$. By assumption, $L^-(h,\pi)$ is an ideal of X and thus $1 \in L^-(h,\pi)$. So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $\overline{h}(1) \supseteq \overline{h}(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $\overline{h}(x \cdot y) \not\supseteq \overline{h}(y)$. Since $\mathrm{Im}(h)$ is a chain, $\overline{h}(x \cdot y) \subset \overline{h}(y)$ and so $[0,1] - h(x \cdot y) \subset [0,1] - h(y)$. Then, $h(x \cdot y) \supset h(y)$. Choose $\pi = h(x \cdot y) \in \mathcal{P}([0,1])$. Then, $h(y) \subset \pi$. Thus, $y \in L^-(h,\pi) \neq \emptyset$. By assumption, $L^-(h,\pi)$ is an ideal of X and thus $x \cdot y \in L^-(h,\pi)$. So, $h(x \cdot y) \supset \pi = h(x \cdot y)$, a contradiction. Hence, $\overline{h}(x \cdot y) \supseteq \overline{h}(y)$ for all $x, y \in X$. Assume that there exist $x, y_1, y_2 \in X$ such that $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \not\supseteq \overline{h}(y_1) \cap \overline{h}(y_2)$. Since $\mathrm{Im}(h)$ is a chain, $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \subset \overline{h}(y_1) \cap \overline{h}(y_2)$ and so $[0,1] - h((y_1 \cdot (y_2 \cdot x)) \cdot x) \subset ([0,1] - h(y_1)) \cap ([0,1] - h(y_2)) = [0,1] - (h(y_1) \cup h(y_2))$. Then. $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supset h(y_1) \cup h(y_2)$. Choose $\pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x) \in \mathcal{P}([0,1])$. Then, $h(y_1) \subset \pi$ and $h(y_2) \subset \pi$. Thus, $y_1, y_2 \in L^-(h,\pi) \neq \emptyset$. By assumption, $L^-(h,\pi)$ is an ideal of X and thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L^-(h,\pi)$. So, $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supset \pi = h((y_1 \cdot (y_2 \cdot x)) \cdot x)$, a contradiction. Hence, $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq \overline{h}(y_1) \cap \overline{h}(y_2)$ for all $x, y_1, y_2 \in X$. Therefore, $\overline{h}$ is an HF ideal of X. □

Theorem 4.12. *Let h be an HFS on a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold.*

(1) *If h is an HF deductive system of X, then for all $\pi \in \mathcal{P}([0,1])$, $L^-(h,\pi)$ is a deductive system of X if $L^-(h,\pi)$ is nonempty and $E(h,\pi)$ is empty.*
(2) *If $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h,\pi)$ of X is a deductive system of X, then $\overline{h}$ is an HF deductive system of X.*

Proof. (1) It is straightforward by Theorem 4.9.

(2) Assume that $\mathrm{Im}(h)$ is a chain and for all $\pi \in \mathcal{P}([0,1])$, a nonempty subset $L^-(h,\pi)$ of X is a deductive system of X. Assume that there exists $x \in X$ such that $\overline{h}(1) \not\supseteq \overline{h}(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(1) \subset \overline{h}(x)$. Then $[0,1] - h(1) \subset [0,1] - h(x)$. Choose $\pi = h(1) \in \mathcal{P}([0,1])$. Then $h(x) \subset h(1) = \pi$. Thus, $x \in L^-(h,\pi) \neq \emptyset$. By assumption, we have $L^-(h,\pi)$ is a deductive system of X and thus, $1 \in L^-(h,\pi)$.

So, $h(1) \supset \pi = h(1)$, a contradiction. Hence, $\overline{h}(1) \supseteq \overline{h}(x)$ for all $x \in X$. Assume that there exist $x, y \in X$ such that $\overline{h}(y) \not\supseteq \overline{h}(x \cdot y) \cap \overline{h}(x)$. Since $\mathrm{Im}(h)$ is a chain, we have $\overline{h}(y) \subset \overline{h}(x \cdot y) \cap \overline{h}(x)$ and so $[0,1] - h(y) \subset ([0,1] - h(x \cdot y)) \cap ([0,1] - h(x)) = [0,1] - (h(x \cdot y) \cup h(x))$. Then, $h(y) \supset h(x \cdot y) \cup h(x)$. Choose $\pi = h(y) \in \mathcal{P}([0,1])$. Then, $h(x \cdot y) \subset \pi$ and $h(x) \subset \pi$. Thus, $x \cdot y, x \in L^-(h, \pi) \neq \emptyset$. By assumption, $L^-(h, \pi)$ is a deductive system of X and thus, $y \in L^-(h, \pi)$. So, $h(y) \supset \pi = h(y)$, a contradiction. Hence, $\overline{h}(y) \supseteq \overline{h}(x \cdot y) \cap \overline{h}(x)$ for all $x, y \in X$. Therefore, $\overline{h}$ is an HF deductive system of X. □

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[1]Department of Mathematics,
School of Science,
University of Phayao,
Mae Ka, Mueang, Phayao 56000, Thailand
Email address: aiyared.ia@up.ac.th
ORCID iD: https://orcid.org/0000-0002-0475-3320

[2]Department of Mathematics,
Vivekanandha College of Arts and Science for Women,
Tiruchengode 637205, Tamilnadu, India
Email address: yamunawin10@gmail.com

[3]Department of Mathematics,
Thanthai Periyar Government Arts and Science College
(affiliated to Bharathidasan University),
Tiruchirappalli 624024, Tamilnadu, India
Email address: maragathameenakship@gmail.com
ORCID iD: https://orcid.org/0009-0003-3558-3875

[4]Department of Mathematics,
Rajah Serfoji Government College,
Thanjavur 613005, Tamilnadu, India
Email address: nrajesh_topology@yahoo.co.in

[*]Corresponding Author